Lecture 12
Modulation and Sampling

- The Fourier transform of the product of two signals
- Modulation of a signal with a sinusoid
- Sampling with an impulse train
- The sampling theorem
Convolution and the Fourier transform

suppose \( f(t), g(t) \) have Fourier transforms \( F(\omega), G(\omega) \)

the convolution \( y = f * g \) of \( f \) and \( g \) is given by

\[
y(t) = \int_{-\infty}^{\infty} f(\tau)g(t - \tau) \, d\tau
\]

(we integrate from \(-\infty\) to \(\infty\) because \(f(t)\) and \(g(t)\) are not necessarily zero for negative \(t\))

from the table of Fourier transform properties:

\[
Y(\omega) = F(\omega)G(\omega)
\]

\(i.e.,\) convolution in the time domain corresponds to multiplication in the frequency domain
Multiplication and the Fourier transform

the Fourier transform of the product

\[ y(t) = f(t)g(t) \]

is given by

\[
Y(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda)G(\omega - \lambda) d\lambda
\]

\[ Y = \frac{1}{2\pi} (F \ast G) \]

i.e., multiplication in the time domain corresponds to convolution in the frequency domain
example:

\[ f(t) = e^{-|t|}, \quad F(\omega) = \frac{2}{1 + \omega^2} \]
\[ g(t) = \cos 20t, \quad G(\omega) = \pi \delta(\omega - 20) + \pi \delta(\omega + 20) \]

the Fourier transform of \( y(t) = e^{-|t|} \cos 20t \) is given by

\[
Y(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda)G(\omega - \lambda) \, d\lambda
\]
\[
= \frac{1}{2} \int_{-\infty}^{\infty} F(\lambda)\delta(\omega - \lambda - 20) \, d\lambda + \frac{1}{2} \int_{-\infty}^{\infty} F(\lambda)\delta(\omega - \lambda + 20) \, d\lambda
\]
\[
= \frac{1}{2} F(\omega - 20) + \frac{1}{2} F(\omega + 20)
\]
\[
= \frac{1}{1 + (\omega - 20)^2} + \frac{1}{1 + (\omega + 20)^2}
\]
Modulation and Sampling
**Sinusoidal amplitude modulation (AM)**

\[ y(t) = u(t) \cos \omega_0 t \]

\[ \cos \omega_0 t \]

Fourier transform of \( y \)

\[ Y(\omega) = \frac{1}{2\pi} U(\omega) \ast (\pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0)) \]

\[ = \frac{1}{2} U(\omega - \omega_0) + \frac{1}{2} U(\omega + \omega_0) \]

- \( \cos \omega_0 t \) is the *carrier signal*
- \( y(t) \) is the modulated signal
- the Fourier transform of the modulated signal is the Fourier transform of the input signal, shifted by \( \pm \omega_0 \)
Sinusoidal amplitude modulation

Baseband Signal

\[ U(\omega) \]

\[ \omega \]

\[ 0 \]

Carrier

\[ \pi \delta(\omega + \omega_0) \]

\[ \pi \delta(\omega - \omega_0) \]

Modulated Signal

\[ \frac{1}{2} U(\omega + \omega_0) \]

\[ \frac{1}{2} U(\omega - \omega_0) \]

\[ \omega \]
example: \( u(t) = 2 + \cos t, \omega_0 = 20 \)

\[ u(t) = 2 + \cos t, \omega_0 = 20 \]

\[ U(\omega) = 4\pi \delta(\omega) + \pi \delta(\omega - 1) + \pi \delta(\omega + 1) \]

\[ Y(\omega) = \frac{1}{2} U(\omega - 20) + \frac{1}{2} U(\omega + 20) \]
demodulation

\[ y(t) = u(t) \cos \omega_0 t \]

\[ z(t) \]

\[ \text{lowpass filter} \]

\[ u(t) \]

\[ \cos \omega_0 t \]

Fourier transform of \( y \) and \( z \):

\[
Y(\omega) = \frac{1}{2} U(\omega - \omega_0) + \frac{1}{2} U(\omega + \omega_0)
\]

\[
Z(\omega) = \frac{1}{2\pi} Y(\omega) \ast (\pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0))
\]

\[
= \frac{1}{2} Y(\omega - \omega_0) + \frac{1}{2} Y(\omega + \omega_0)
\]

\[
= \frac{1}{4} U(\omega - 2\omega_0) + \frac{1}{2} U(\omega) + \frac{1}{4} U(\omega + 2\omega_0)
\]

if \( U \) is bandlimited, we can eliminate the 1st and 3rd term by lowpass filtering
Sinusoidal amplitude demodulation

Modulated Signal

\[ (1/2)U(\omega+\omega_0) \quad (1/2)U(\omega-\omega_0) \]

Demodulation Cosine

\[ \pi\delta(\omega+\omega_0) \quad \pi\delta(\omega-\omega_0) \]

Demodulated Signal

\[ (1/4)U(\omega+2\omega_0) \quad (1/2)U(\omega) \quad \text{Lowpass Filter} \quad (1/4)U(\omega-2\omega_0) \]
Suppose for example that $u(t)$ is an audio signal (frequency range 10Hz – 20kHz)

We rather not transmit $u$ directly using electromagnetic waves:

- the wavelength is several 100 km, so we’d need very large antennas
- we’d be able to transmit only one signal at a time
- the Navy communicates with submerged submarines in this band

Modulating the signal with a carrier signal with frequency 500 kHz to 5 GHz:

- allows us to transmit and receive the signal
- allows us to transmit many signals simultaneously (frequency division multiplexing)
Sampling with an impulse train

Multiply a signal \( x(t) \) with a unit impulse train with period \( T \)

\[
p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)
\]

Sampled signal: \( y(t) = x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT) = \sum_{k=-\infty}^{\infty} x(kT)\delta(t - kT) \)

(a train of impulses with magnitude \( \ldots, x(-T), x(0), x(T), x(2T), \ldots \))
The Fourier transform of an impulse train

train of unit impulses with period $T$: $p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$

Fourier transform (from table): $P(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi k}{T})$
Consequences of Sampling

- Frequencies well below the sampling rate ($\omega << 2\pi/T$) are “sampled” in the sense we expect.
- Frequencies at multiples of the sampling rate ($\omega = 2\pi n/T$) look like they are constant. We can’t tell them from DC. These frequencies “alias” as DC.
Frequency domain interpretation of sampling

The Fourier transform of the sampled signal is

\[ Y = \frac{1}{2\pi} (X \ast P), \]

i.e., the convolution of \( X \) with \( P(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi k}{T}) \)

\[
Y(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda) P(\omega - \lambda) \, d\lambda
\]

\[
= \frac{1}{T} \int_{-\infty}^{\infty} X(\lambda) \left( \sum_{k=-\infty}^{\infty} \delta(\omega - \lambda - \frac{2\pi k}{T}) \right) \, d\lambda
\]

\[
= \frac{1}{T} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} X(\lambda) \delta(\omega - \lambda - \frac{2\pi k}{T}) \, d\lambda
\]

\[
= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega - \frac{2\pi k}{T})
\]
example: sample $x(t) = e^{-|t|}$ at different rates

$x$ sampled with $T = 1$ ($2\pi/T = 6.3$)
$x$ sampled with $T = 0.5$ ($2\pi/T = 12.6$)

$x$ sampled with $T = 0.2$ ($2\pi/T = 31.4$)
The sampling theorem

can we recover the original signal \( x \) from the sampled signal \( y \)?

**example:** a *band-limited* signal \( x \) (with bandwidth \( W \))

\[
X(\omega)
\]

\[
\omega
\]

\[
-\omega \quad \omega
\]

\[
\omega \quad W \quad -W
\]

Fourier transform of \( y(t) = \sum_k x(kT)\delta(t-kT) \):

\[
Y(\omega)
\]

\[
\omega \quad -\omega \quad W \quad 2\pi/T \quad 4\pi/T \quad 0
\]
suppose we filter $y$ through an ideal lowpass filter with cutoff frequency $\omega_c$, i.e., we multiply $Y(\omega)$ with

$$H(\omega) = \begin{cases} 1 & -\omega_c \leq \omega \leq \omega_c \\ 0 & |\omega| \geq \omega_c \end{cases}$$

if $W \leq \omega_c \leq 2\pi/T - W$, then the result is $H(\omega)Y(\omega) = X(\omega)/T$, i.e., we recover $X$ exactly.

\[\text{Diagram:}\]

\[\text{Diagram:}\]
same signal, sampled with $T = \pi/W$

we can still recover $X(\omega)$ perfectly by lowpass filtering with $\omega_c = W$

sample with $T > \pi/W$

$X(\omega)$ cannot be recovered from $Y(\omega)$ by lowpass filtering
the sampling theorem

suppose \( x \) is a \textit{band-limited} signal with bandwidth \( W \), i.e.,

\[
X(\omega) = 0 \text{ for } |\omega| > W
\]

and we sample at a rate \( 1/T \)

\[
y(t) = \sum_{k=-\infty}^{\infty} x(kT) \delta(t - kT)
\]

then we can recover \( x \) from \( y \) if \( T \leq \pi/W \)

- the sampling rate must be at least \( 1/T = W/\pi \) samples per second
  (\( W/\pi \) is called the \textit{Nyquist rate})

- the distortion introduced by sampling below the Nyquist rate is called \textit{aliasing}