Lecture 9
Time-domain properties of convolution systems

• impulse response
• step response
• fading memory
• DC gain
• peak gain
• stability
Impulse response

if \( u = \delta \) we have

\[
y(t) = \int_{0-}^{t} h(t - \tau)u(\tau) \, d\tau = h(t)
\]

so \( h \) is the output (response) when \( u = \delta \) (hence the name impulse response)

impulse response testing:
- apply impulse input and record resulting output \((h)\)
- now you can predict output for any input signal
- practical problem: linear model often fails for very large input signals
Step response

the (unit) step response is the output when the input is a unit step:

\[ s(t) = \int_{0}^{t} h(\tau) \, d\tau \]

(symbol \( s \) clashes with frequency variable, but usually this doesn't cause any harm)

relation with impulse response: \( s(t) \) is the integral of \( h \), so

\[ h(t) = s'(t) \]

step response testing:

- apply unit step to input and record output (\( s \))
- the impulse response is \( h(t) = s'(t) \), so now you can predict output for any input signal
- widely used

Time-domain properties of convolution systems
Fading memory

we say the convolution system has *fading memory* if \( h(\tau) \to 0 \) as \( \tau \to \infty \)

• means current output \( y(t) \) depends less and less on \( u(t - \tau) \) as \( \tau \) gets large (\( i.e., \) the remote past input)

• if \( h(\tau) = 0 \) for \( \tau > T \), then system has *finite memory*: \( y(t) \) depends only on \( u(\tau) \) for \( t - T \leq \tau \leq t \)

if \( H \) is rational, fading memory means poles of \( H \) are in left halfplane

(poles in right halfplane or on the imaginary axis give terms in \( h \) that don’t decay to zero)
DC gain

the DC (direct current) or static gain of a convolution system is

$$H(0) = \int_{0}^{\infty} h(\tau) \ d\tau$$

(if finite, i.e., if $s = 0$ is in ROC of $H$)

in terms of step response:

$$H(0) = \lim_{t \to \infty} s(t)$$

**interpretation:** if $u$ is constant, then for large $t$,

$$y(t) = u \int_{0}^{t} h(\tau) \ d\tau \approx H(0)u$$

so $H(0)$ gives the gain for static (constant) signals
Vehicle suspension example

transfer function from road to vehicle height (page 7-7):

\[ H(s) = \frac{bs + k}{ms^2 + bs + k} \]

- for \( m > 0, b > 0, k > 0 \) poles are in LHP, hence system has fading memory
- DC gain: \( H(0) = 1 \) (obvious!)

step response gives vehicle height after going over unit high curb at \( t = 0 \)
impulse response and step response for $k = 1$, $b = 0.5$, $m = 1$

- poles are $-0.25 \pm j 0.97$ (underdamped)
- step response ‘overshoots’ about 50%; settles at one in about 20sec
impulse response and step response for $k = 1, b = 2, m = 1$

- repeated pole at $-1$ (critical damping)
- about 15% overshoot; step response settles in about 5 sec
Example

wire modeled as 3 RC segments:

(Except for values, could model interconnect wire in IC)

(after a lot of algebra) we find

\[ H(s) = \frac{1}{s^3 + 5s^2 + 6s + 1} \]

• poles are \(-3.247, -1.555, -0.198\)

• DC gain is \(H(0) = 1\) (again, obvious)
step response gives $v_{\text{out}}$ when $v_{\text{in}}$ is unit step (as in $0 \rightarrow 1$ logic transition)

wire delays transition about 20sec or so
(Peak) gain

\[ y(t) = \int_0^t h(\tau) u(t - \tau) \, d\tau \]

the peak values of the input & output signals as

\[ \text{peak}(y) = \max_{t \geq 0} |y(t)|, \quad \text{peak}(u) = \max_{t \geq 0} |u(t)| \]

**question:** how large can \( \frac{\text{peak}(y)}{\text{peak}(u)} \) be?

answer is given by the *peak gain* of the system, defined as

\[ \alpha = \max_{u \neq 0} \frac{\text{peak}(y)}{\text{peak}(u)} = \int_0^\infty |h(\tau)| \, d\tau \]

\( i.e., \) for any signal \( u \) we have \( \text{peak}(y) \leq \alpha \text{ peak}(u) \) and there are signals where equality holds
for any $t$ we have

$$|y(t)| = \left| \int_0^t h(\tau)u(t - \tau) \, d\tau \right|$$

$$\leq \int_0^t |h(\tau)| \, |u(t - \tau)| \, d\tau$$

$$\leq \text{peak}(u) \int_0^t |h(\tau)| \, d\tau$$

$$\leq \text{peak}(u) \int_0^\infty |h(\tau)| \, d\tau$$

which shows that $\text{peak}(y) \leq \alpha \text{peak}(u)$
conversely, we can find an input signal with

$$\frac{\text{peak}(y)}{\text{peak}(u)} \approx \int_0^\infty |h(\tau)| \, d\tau$$

choose $T$ large and define

$$u(t) = \begin{cases} \text{sign}(h(T - t)) & t \leq T \\ 0 & t > T \end{cases}$$

then $\text{peak}(u) = 1$ and

$$y(T) = \int_0^T h(\tau)\text{sign}(h(\tau)) \, d\tau = \int_0^T |h(\tau)| \, d\tau,$$

for large $T$ this signal satisfies

$$\frac{\text{peak}(y)}{\text{peak}(u)} \approx \int_0^\infty |h(\tau)| \, d\tau$$
**example:** $H(s) = 1/(s + 1)$, so $h(t) = e^{-t}$

- DC gain is one, *i.e.*, constant signals are amplified by one
- peak gain is $\int_{0}^{\infty} |e^{-t}| \, d\tau = 1$ which is the same as the DC gain

so for this system, peak of the output is no more than the peak of the input

more generally,

- peak gain always at least as big as DC gain since

\[
\int_{0}^{\infty} |h(\tau)| \, d\tau \geq \left| \int_{0}^{\infty} h(\tau) \, d\tau \right| = |H(0)|
\]

- they are equal only when impulse response is always nonnegative (or nonpositive), *i.e.*, step response is monotonic
Stability

A system is *stable* if its peak gain is finite.

**Interpretation:** bounded inputs give bounded outputs

\[ \text{peak}(y) \leq \alpha \text{peak}(u) \]

Also called *bounded-input bounded-output stability* (to distinguish from other definitions of stability).

If \( H \) is rational, stability means poles of \( H \) are in left half-plane.