EE359 Discussion Session 3
Capacity of Flat and Frequency Selective Channels

February 5, 2020
Note about scattering functions
For deterministic response

\[ h(t, \tau) = \sum_i \alpha_i(t) e^{-j2\pi(t-\tau_i(t))} \delta(t - \tau_i(t)) \]

With no movement (Doppler), \( h(t, \tau) \rightarrow h(\tau) \) becomes time-invariant response (LTI system)
For deterministic response

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\[ t \text{ (Time)} \leftrightarrow \rho \text{ (Doppler)} \]

\[ \tau \text{ (Delay)} \leftrightarrow f \text{ (Frequency)} \]
What is Capacity?

- Maximum achievable rate with no errors
- Maximum mutual information over all input distributions
- Usually found by proving matching lower (achievability) and upper (converse) bounds
- Often easy to bound, but hard to prove
Notions of Capacity in Wireless Systems

- AWGN Capacity

- Only CSI distribution known at TX and RX

- CSI at RX only
  - With or without outage

- CSI at TX and RX
  - With or without adaptation
  - With or without outage
Capacity of fixed channel with no fading.

\[ C = B \log_2(1 + \gamma) \]

Can be used to bound other settings
CSI Distribution Known

Really complicated
Channel capacity with CSIR

Two notions of capacity

Ergodic capacity

\[ C = B \int \log_2 (1 + \gamma) p(\gamma) d\gamma \]

- Achieved by coding over fading states \( \gamma \)
Channel capacity with CSIR

Two notions of capacity

### Ergodic capacity

\[ C = B \int \log_2 (1 + \gamma) p(\gamma) d\gamma \]

- Achieved by coding over fading states \( \gamma \)

### Outage capacity

- Find out minimum SNR \( \gamma_{\text{min}} \) needed to achieve outage prob \( P_{out} \). If it violates power constraint then \( C = 0 \)
- Transmit at that SNR thereby achieving

\[ C = (1 - P_{out}) B \log_2 (1 + \gamma_{\text{min}}) \]
Outage capacity (without CSIT)

**Outage capacity**

- Find out minimum SNR $\gamma_0$ needed to achieve outage prob $P_{out}$. If it violates power constraint then $C = 0$
- Transmit at that SNR thereby achieving

$$C = (1 - P_{out})B \log_2(1 + \gamma_0)$$
Capacity with CSIR and CSIT

System model

\[ y[i] = \sqrt{g[i]} x[i] + n[i] \]

\( \sqrt{g[i]} \sim \text{fading distribution} \)

\[ E[|x[i]|^2] \leq \bar{P} \]

- \( g[i] \) known at transmitter and receiver
Capacity with CSIR and CSIT

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- \( g[i] \) known at transmitter and receiver
- What is capacity with fixed TX power?
Capacity with CSIR and CSIT

System model

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\[ \sqrt{g[i]} \sim \text{fading distribution} \]
\[ E[|x[i]|^2] \leq P \]

- \( g[i] \) known at transmitter and receiver

- What is capacity with fixed TX power?

- Can capacity increase with rate and power adaptation?
Towards optimally exploiting the CSI $g[i]$

**Idea**

Vary transmit power $P$ and rate $R$ as a function of $g[i]$ or equivalently, of

$$\gamma = \frac{g[i] \bar{P}}{N_0 B}$$

- Power $P(\gamma)$
- Rate $R = B \log \left(1 + \gamma \frac{P(\gamma)}{\bar{P}}\right)$
Towards optimally exploiting the CSI $g[i]$

**Idea**

Vary transmit power $P$ and rate $R$ as a function of $g[i]$ or equivalently, of

$$\gamma = \frac{g[i]\bar{P}}{N_0B}$$

- **Power** $P(\gamma)$
- **Rate** $R = B \log \left(1 + \gamma \frac{P(\gamma)}{P} \right)$

**Notation**

- Fading distribution given by $p(\gamma)$
- Assuming that Tx uses fixed average power $\bar{P}$. 
Optimization problems for optimal CSIT and CSIR use

Optimization problem

\[
\max_{P(\gamma)} \int B \log \left( 1 + \frac{P(\gamma)}{\bar{P}} \gamma \right) p(\gamma) d\gamma \\
\text{s.t. } E[P(\gamma)] \leq \bar{P} \\
P(\gamma) \geq 0 \ \forall \ \gamma
\]

Optimization problem (discrete \( \gamma \))

\[
\max_{P(\gamma)} \sum_i B \log \left( 1 + \frac{P(\gamma_i)}{\bar{P}} \gamma_i \right) p(\gamma_i) \\
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P(\gamma) \geq 0 \ \forall \ \gamma
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Optimization problem (discrete \( \gamma \))

\[
\text{min}_{P(\gamma)} \sum_i -B \log \left(1 + \frac{P(\gamma_i)}{\bar{P}} \gamma_i\right) p(\gamma_i) \\
\text{s.t. } E[P(\gamma)] \leq \bar{P} \\
P(\gamma_i) \geq 0 \ \forall \ i
\]
Lagrangian Methods

Optimization problem

\[
\begin{align*}
\min_{x \in S} f_0(x), \\
\text{s.t. } f_i(x) &\leq 0, \forall i = 0, \ldots, m, \\
\text{s.t. } h_i(x) &= 0, \forall i = 0, \ldots, p,
\end{align*}
\]

Lagrangian

\[
L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)
\]

- \(\nu, \lambda\) are called Lagrange multipliers or dual variables
- Process sometimes referred to as “regularizing constraints”.
Lagrangian Duality

Dual problem

$$g(\lambda, \nu) = \inf_{x \in S} L(x, \nu, \lambda)$$

$$= \inf_{x \in S} \left( f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right)$$

Solving the dual problem

- Find closed form expression for $g(\lambda, \nu)$: $\nabla_x L(x, \lambda, \nu) = 0$
- $g(\lambda, \nu)$ is now a convex optimization problem (in our case of one variable)!
Why consider dual problem?

- Let $p^*$ be the optimal value of the original optimization problem:
  \[ f_0(\tilde{x}) = p^*, \tilde{x} \text{ is feasible}. \]

- Let $d^*$ be the optimal value of the dual problem:
  \[ g(\tilde{\lambda}, \tilde{\nu}) = d^* \]

**Lower Bound Property**

$p^* \geq d^*$ as long as $\lambda \geq 0$.

**Strong Duality**

$p^* = d^*$ for many problems! **True in our case**

Take EE364a/b to understand when and why this is true.
Example: Least squares

\[
\begin{align*}
\min & \quad x^\top x \\
\text{s.t} & \quad Ax = b
\end{align*}
\]

Dual Function

\[
L(x, \nu) = x^\top x + \nu^\top (Ax - b) \\
\nabla_x L(x, \nu) = 2x + A^\top \nu = 0 \Rightarrow x = -(1/2)A^\top \nu.
\]
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Plug into \( L \) to get \( g \):

\[
g(\nu) = L((-1/2))A^\top \nu, \nu) = -(1/4)\nu^\top AA^\top \nu - b^\top \nu
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Example: Least squares

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Plug into \( L \) to get \( g \):

\[ g(\nu) = L((-1/2))A^\top \nu, \nu) = -(1/4)\nu^\top AA^\top \nu - b^\top \nu \]

Lower Bound Property

\[ p^* \geq -(1/4)\nu^\top AA^\top \nu - b^\top \nu \quad \forall \nu \]
Form Lagrangian by relaxing $P_j > 0$:

$$\min_{P_j} \sum_j -B \log \left( 1 + \frac{P_j}{\bar{P}} \gamma_j \right) p(\gamma_j) \to f_0(P_j)$$

$$\sum_j \frac{P_j}{\bar{P}} \leq 1 \Rightarrow \sum_j P_j - \bar{P} = 0 \to f_j(P_j)$$
Deriving Waterfilling Expression

1. Form Lagrangian by relaxing $P_j > 0$:

$$
\min_{P_j} \sum_j -B \log \left( 1 + \frac{P_j}{\bar{P}} \gamma_j \right) p(\gamma_j) \rightarrow f_0(P_j)
$$

$$
\frac{\sum_j P_j}{\bar{P}} \leq 1 \Rightarrow \sum_j P_j - \bar{P} = 0 \rightarrow f_j(P_j)
$$

2. Solve $\partial L / \partial P_j = 0$ for $P_j / P$. Let $\gamma_0 = \lambda P$. 
Deriving Waterfilling Expression

1. Form Lagrangian by relaxing $P_j > 0$:

$$\min_{P_j} \sum_j -B \log \left(1 + \frac{P_j}{\bar{P}} \gamma_j\right) p(\gamma_j) \rightarrow f_0(P_j)$$

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2. Solve $\partial L / \partial P_j = 0$ for $P_j / P$. Let $\gamma_0 = \lambda P$.

Hint: Write Lagrangian in terms of $P_j, \bar{P}, \gamma$ before taking derivative.
Extensions of waterfilling idea

Can be applied to any system where the sum of logarithms need to be optimized with a sum power and positivity constraints

Examples

- Continuous fading states
- Time-invariant frequency selective fading channel - waterfilling over frequency
- Time-varying frequency selective fading channel - waterfilling over time and frequency (may not be optimal)
- MIMO channels - waterfilling over spatial diversity
Block Fading:

$$\sum_{\gamma_i \geq \gamma_0} \left( \frac{1}{\gamma_0} - \frac{1}{\gamma_i} \right) p(\gamma_i) = 1$$

Frequency-selective Fading:

$$\sum_{\gamma_i \geq \gamma_0} \left( \frac{1}{\gamma_0} - \frac{1}{\gamma_i} \right) = 1$$
Finding the optimal power allocation

1. Assume $P(\gamma_i) > 0 \ \forall i$

2. Solve for $\gamma_0$:

\[
\frac{1}{\gamma_0} = 1 + \sum_i \frac{P(\gamma_i)}{\gamma_i} \quad \text{or} \quad \frac{N}{\gamma_0} = 1 + \sum_i \frac{1}{\gamma_i}
\]

3. If any $P(\gamma_i) < 0$ assume the $P(\gamma_i)$ for lowest $\gamma_i$ is zero and repeat previous step

4. Given $\gamma_0$, can compute $P(\gamma_i)$ and $C$. 
“Waterfilling” interpretation of the solution $P(\gamma_i)$
Suboptimal power adaptation schemes

Power adaptation buys you little in practice.

Other Ideas

- Fix TX power but vary rate
- Channel inversion: Received SNR is constant
- Truncated channel inversion: Received SNR is constant, and do not use channel if gain is too low
Channel inversion in more details

Channel inversion

- If target SNR is $\sigma$, transmit at $\frac{\sigma}{\gamma}$
- Expected power constraint gives $\sigma = \frac{\bar{P}}{E[1/\gamma]}$
- What happens for Rayleigh fading?
Channel inversion in more details

Channel inversion
- If target SNR is $\sigma$, transmit at $\frac{\sigma}{\gamma}$
- Expected power constraint gives $\sigma = \frac{\bar{P}}{E[1/\gamma]}$
- What happens for Rayleigh fading? $E[1/\gamma] = \infty$

Truncated channel inversion
- Do not use the channel if $\gamma < \gamma_1$ (outage)
- If target SNR is $\sigma$, transmit at $\frac{\sigma}{\gamma}$
- Expected power constraint gives $\sigma = \frac{\bar{P}}{E_{\gamma>\gamma_1}[1/\gamma]}$
Question

How does capacity under truncated inversion behave with increasing outage probability?