Math 51 - Autumn 2014 - Midterm Exam II

Name: ________________________________

Student SUNet address: ______________________

Circle your section:

<table>
<thead>
<tr>
<th>Section</th>
<th>Instructor 1</th>
<th>Instructor 2</th>
<th>Instructor 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACE (1:15–3:05pm)</td>
<td>Khoa Nguyen</td>
<td>Christos Mantoulidis</td>
<td>Gergely Szucs</td>
</tr>
<tr>
<td>08 (9:00–9:50am)</td>
<td>09 (10:00–10:50am)</td>
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<td></td>
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<tr>
<td>10 (10:00–10:50am)</td>
<td>Seungki Kim</td>
<td>Jacek Skryzalin</td>
<td>Christos Mantoulidis</td>
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<tr>
<td>11 (10:00–10:50am)</td>
<td>12 (10:00–10:50am)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>13 (11:00–11:50am)</td>
<td>Chao Li</td>
<td>Graham White</td>
<td>Jacek Skryzalin</td>
</tr>
<tr>
<td>14 (11:00–11:50am)</td>
<td>15 (11:00–11:50am)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16 (11:00–11:50am)</td>
<td>Ho Chung Siu</td>
<td>Chao Li</td>
<td>Graham White</td>
</tr>
<tr>
<td>17 (1:15–2:05pm)</td>
<td>18 (1:15–2:05pm)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>19 (1:15–2:05pm)</td>
<td>Ho Chung Siu</td>
<td>Seungki Kim</td>
<td>Gergely Szucs</td>
</tr>
<tr>
<td>20 (1:15–2:05pm)</td>
<td>21 (2:15–3:05pm)</td>
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<td></td>
</tr>
</tbody>
</table>

“I accept the Honor Code”. Signature: ________________________________

- Only correct answers showing all work/reasoning will be given full credit.
- There are 8 problems and 12 pages to the exam (including this title page), for a total of 100 points. The last page is blank. Point values are given in parentheses. Please check that the version of the exam you have is complete, and correctly stapled.
- Only a scanned copy of the exam will be seen by graders, so they will only see the fronts of each page. If you need extra space to do calculations, you may use extra sheets, but your exam pages must contain your complete answers – the extra sheets cannot be handed in. The extra page at the end can be used for calculations but will not be seen by the graders! Do not unstaple or detach pages from this exam.
- Read each question carefully. In order to receive full credit, please show all of your work and justify your answers unless you are explicitly instructed not to.
- You have 90 minutes. This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted. If you finish early, you must hand your exam paper to a member of teaching staff.
Problem 1. (15 points) Compute the eigenvalues and the corresponding eigenvectors of the matrix

\[
A = \begin{bmatrix}
1 & -1 & 1 \\
-1 & 1 & -1 \\
-1 & -1 & 1 \\
\end{bmatrix}
\]
Problem 2. (a) (3 points) State the definition of a symmetric matrix, and give an example of a $2 \times 2$ symmetric matrix with three distinct entries.

(b) (3 points) State precisely the main result discussed in lecture and the book about the eigenvalues and eigenvectors of symmetric matrices (this is known as the spectral theorem).

(c) (4 points) Define a quadratic form on $\mathbb{R}^2$ by

$$Q(x) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

where $A$ is a $2 \times 2$ matrix with

$$\text{Tr}(A) = -\pi, \quad \text{rref}(A) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}. $$

(Recall that the trace of $A$, $\text{Tr}(A)$, is the sum of the eigenvalues.) What can we say about the definiteness or semi-definiteness of $Q$?
Problem 3. (15 points altogether – 3 points each)  
Justification is not necessary.

(a) Write down an example of a positive definite quadratic form $Q_1$. Describe all of the level sets of the particular quadratic form $Q_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$ that you have chosen (as the height $c$ varies).

(b) Write down an example of a positive semi-definite (but not positive definite) quadratic form $Q_2$. Describe all of the level sets of this new $Q_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$ (as the height $c$ varies).

(c) Write down an example of a negative definite quadratic form $Q_3$. Describe all of the level sets of $Q_3 : \mathbb{R}^2 \rightarrow \mathbb{R}$ that you have chosen (as the height $c$ varies).
(d) Write down an example of a negative semi-definite (but not negative definite) quadratic form $Q_4$. Describe all of the level sets of this $Q_4: \mathbb{R}^2 \to \mathbb{R}$ (as the height $c$ varies).

(e) Write down an example of an indefinite (but not positive definite, positive semi-definite, negative definite or negative semi-definite) quadratic form $Q_5$. Describe all of the level sets of this $Q_5 : \mathbb{R}^2 \to \mathbb{R}$ (as the height $c$ varies).
Problem 4. For the following question, please show your work.

(a) (5 points) Let \( \beta = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\} \). Transform the following vectors (shown in standard coordinates) into coordinates with respect to the basis \( \beta \).

\[ v = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad [v]_\beta = \]

\[ v = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad [v]_\beta = \]

\[ v = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad [v]_\beta = \]
(b) (5 points) Let $A$ be a $2 \times 2$ matrix such that $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ are eigenvectors with corresponding eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 2$. Using part a), compute the following:

\[
A \begin{bmatrix} 2 \\ 1 \end{bmatrix} =
\]

\[
A \begin{bmatrix} 0 \\ 1 \end{bmatrix} =
\]

\[
A \begin{bmatrix} 1 \\ 0 \end{bmatrix} =
\]

(c) (5 points) Write down the matrix for $A$ (in the standard basis).
Problem 5. (12 points altogether – 3 points each) Below are four linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$. For each transformations, determine a basis of eigenvectors (if one exists) and indicate your chosen basis by drawing arrows in the $xy$–plane. Label each arrow with the corresponding eigenvalue. If the transformation does not have a basis of eigenvectors, write underneath the plane "No eigenbasis exists". No justification necessary.

1. Scalar multiplication by 3
\[
T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \\
T(v) = 3v
\]

2. Projection onto the line $L$
\[
T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \\
T(v) = \text{Proj}_L(v)
\]

3. Reflection in the line $L$
\[
T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \\
T(v) = \text{Refl}_L(v)
\]

4. Rotation by $\theta$
\[
T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \\
T(v) = \text{Rot}_\theta(v)
\]
Problem 6. (1 point each) Indicate which of the following statements are always true by writing TRUE or FALSE under the statement. No justification is necessary.

(a) Suppose that $A$ is a $2 \times 2$ matrix which has characteristic polynomial $p(\lambda) = \lambda^2 - 16$. Then the matrix $(4I_2 - A)$ is not invertible.

(b) If $A$ is a $3 \times 3$ matrix with eigenvalues 1, 2, and $-3$, then $A$ must have $C(A) = \mathbb{R}^3$.

(c) If $A$ is a $3 \times 3$ matrix with characteristic polynomial $p(\lambda) = \lambda(\lambda - 1)(\lambda + 1)$, then $A$ is diagonalizable.

(d) 
\[ \lim_{(x,y,z) \to (1,1,1)} (xy + z^2 + \cos(\pi xyz)) = 1 \]

(e) If $A$ and $B$ are symmetric $n \times n$ matrices, then $AB$ is always symmetric.

(f) The level sets of $f(x,y) = 5x + 2y$ are lines.

(g) Suppose that $A$ is a $4 \times 4$ matrix with precisely one eigenvalue, $\lambda_1$. Suppose that \{ $v_1$, $v_2$, $v_3$, $v_4$ \} is a linearly independent set of vectors which span the eigenspace associated to $\lambda_1$. Then $A$ is diagonalizable.

(h) $\lim_{(x,y) \to (0,0)} x^4 / y^4$ does not exist.
Problem 7.
(a) (5 points) Find the parametric equation for the tangent line $L$ to the curve

$$\mathbf{r}(t) = (t - 1, \sqrt{t}, \cos(\pi t))$$

(which is defined for $t > 0$) at the point $(3, 2, 1)$.

(b) (5 points) This tangent line $L$ intersects the curve $\mathbf{r}(t)$ at $(3, 2, 1)$. Does it intersect this curve at any other point? If so, where?
Problem 8.
(a) (5 points) Let $A$ be a $3 \times 3$ matrix with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$. Let

$$B = \begin{bmatrix} 4 & 6 & 8 \\ 0 & 5 & 7 \\ 0 & 0 & 6 \end{bmatrix}.$$ 

Calculate $\det(A^{-1}B)$.

(b) (5 points) What are the eigenvalues of $A^{2014}$? (You may leave your answer in the form of an exponent.)

(c) (5 points) Let $R$ be the matrix which represents rotation in $\mathbb{R}^3$ by $\pi$ around the $x$-axis. What are the eigenvalues of $R$? For each eigenvalue give a basis for its eigenspace.