Math 51 - Autumn 2014 - Midterm Exam II
Solutions

1. (15 points) Compute the eigenvalues and the corresponding eigenvectors of the matrix

\[ A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \]

After a short computation, we see that the characteristic polynomial of \( A \) equals

\[ p(\lambda) = \lambda(\lambda - 1)(\lambda - 2). \]

So the eigenvalues are \( \lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 2 \). We use row reduction to compute the corresponding eigenvectors. These are:

\[ v_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}. \]

2. (a) (3 points) State the definition of a symmetric matrix, and give an example of a 2 \( \times \) 2 symmetric matrix with three distinct entries.

The matrix \( A \) which has entries \( a_{ij} \) is called symmetric if its entries satisfy \( a_{ij} = a_{ji} \) for every \( i, j \). This is equivalent to the statement that \( A \) equals its transpose, \( A = A^T \). The matrix

\[ A = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \]

is symmetric.

(b) (3 points) State precisely the main result discussed in lecture and the book about the eigenvalues and eigenvectors of symmetric matrices (this is known as the spectral theorem).
The spectral theorem states that if $A$ is an $n$-by-$n$ matrix which is symmetric $A = A^T$, then it has a complete basis of eigenvectors $v_1, \ldots, v_n$, with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. Moreover, we can choose these eigenvectors to be mutually orthonormal, i.e. $v_i \cdot v_j = 0$ if $i \neq j$ and $||v_i|| = 1$ for each $i$, and the eigenvalues $\lambda_i$ are necessarily all real numbers.

(c) (4 points) Define a quadratic form on $\mathbb{R}^2$ by

$$Q(x) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

where $A$ is a $2 \times 2$ matrix with

$$\text{Tr}(A) = -\pi, \quad \text{rref}(A) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$ 

(Recall that the trace of $A$, $\text{Tr}(A)$, is the sum of the eigenvalues.) What can we say about the definiteness or semi-definiteness of $Q$?

We know that $\text{Tr}(A) = \lambda_1 + \lambda_2 = -\pi$, where $\lambda_1, \lambda_2$ are the eigenvalues of $A$; furthermore, since $\text{rref}(A)$ has its second row identically zero, there is one free variable and one pivot variable, and we know that the nullspace of $A$ is 1-dimensional, which means that one of the eigenvalues, say $\lambda_1$, must equal 0. Therefore the other eigenvalue equals $-\pi$. This means that if we write any vector $x$ in terms of the corresponding eigenbasis, i.e., $x = y_1v_1 + y_2v_2$, where $Av_j = \lambda_jv_j$, then

$$Q(x) = \lambda_1y_1^2 + \lambda_2y_2^2 = -\pi y_2^2.$$ 

In other words, $Q$ is negative semidefinite.

3. (15 points altogether – 3 points each)

(a) Write down an example of a positive definite quadratic form $Q_1$. Describe all of the level sets of the particular quadratic form $Q_1 : \mathbb{R}^2 \to \mathbb{R}$ that you have chosen (as the height $c$ varies).

$Q_1(x) = x_1^2 + \frac{x_2^2}{4}$ is positive definite. Its level set $Q_1(x) = c$ are either empty if $c < 0$, the single point $(0,0)$ if $c = 0$, or ellipses if $c > 0$. 

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(b) Write down an example of a positive semi-definite (but not positive definite) quadratic form $Q_2$. Describe all of the level sets of this new $Q_2 : \mathbb{R}^2 \to \mathbb{R}$ (as the height $c$ varies).

$Q_2(x) = x_1^2$ is positive semi-definite. Its level sets $Q_2(x) = c$ are empty if $c < 0$, the $x_2$ axis (i.e., $x_1 = 0$) if $c = 0$, and two parallel lines $x_1 = \pm \sqrt{c}$ if $c > 0$.

(c) Write down an example of a negative definite quadratic form $Q_3$. Describe all of the level sets of $Q_3 : \mathbb{R}^2 \to \mathbb{R}$ that you have chosen (as the height $c$ varies).

$Q_3(x) = -Q_1(x)$ is negative definite. Its level sets are ellipses if $c < 0$, the origin if $c = 0$ and empty if $c > 0$.

(d) Write down an example of a negative semi-definite (but not negative definite) quadratic form $Q_4$. Describe all of the level sets of this $Q_4 : \mathbb{R}^2 \to \mathbb{R}$ (as the height $c$ varies).

$Q_4(x) = -x_1^2$ is negative semi-definite. Its level sets are a pair of parallel lines when $c < 0$, a single line when $c = 0$ and empty when $c > 0$.

(e) Write down an example of an indefinite (but not positive definite, positive semi-definite, negative definite or negative semi-definite) quadratic form $Q_5$. Describe all of the level sets of this $Q_5 : \mathbb{R}^2 \to \mathbb{R}$ (as the height $c$ varies).

Finally, $Q_5(x) = x_1^2 - 16x_2^2$ is indefinite. Each of its level sets $Q_5(x) = c$, for $c \neq 0$, consist of the two branches of a hyperbola. When $c = 0$, the level set is a pair of lines which intersect at the origin.
(a) (5 points) Let $\beta = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$. Transform the following vectors (shown in standard coordinates) into coordinates with respect to the basis $\beta$.

We have

\[
\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 4 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ -1 \end{bmatrix}.
\]

For simplicity below we write $v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

(b) (5 points) Let $A$ be a $2 \times 2$ matrix such that $v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ are eigenvectors with corresponding eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 2$. Using part a), compute the following:

\[
\begin{align*}
A \begin{bmatrix} 2 \\ 1 \end{bmatrix} &= 4Av_1 + 3Av_2 = 4(-1)v_1 + 3(2)v_2 = -4v_1 + 6v_2 = \begin{bmatrix} 16 \\ -10 \end{bmatrix} \\
A \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= 2Av_1 + 1Av_2 = 2(-1)v_1 + 1(2)v_2 = -2v_1 + 2v_2 = \begin{bmatrix} 6 \\ -4 \end{bmatrix} \\
A \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= 1Av_1 + 1Av_2 = 1(-1)v_1 + 1(2)v_2 = -1v_1 + 2v_2 = \begin{bmatrix} 5 \\ -3 \end{bmatrix}
\end{align*}
\]

(c) (5 points) The matrix for $A$ in the standard basis has columns $A \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $A \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, hence

\[
A = \begin{bmatrix} 5 & 6 \\ -3 & -4 \end{bmatrix}
\]
5. (12 points altogether – 3 points each) Below are four linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$. For each transformations, determine a basis of eigenvectors (if one exists) and indicate your chosen basis by drawing arrows in the $xy$-plane. Label each arrow with the corresponding eigenvalue. If the transformation does not have a basis of eigenvectors, write underneath the plane "No eigenbasis exists". No justification necessary.

1. Scalar multiplication by 3
   \[ T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \]
   \[ T(v) = 3v \]

2. Projection onto the line $L$
   \[ T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \]
   \[ T(v) = \text{Proj}_L(v) \]

3. Reflection in the line $L$
   \[ T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \]
   \[ T(v) = \text{Refl}_L(v) \]

4. Rotation by $\theta$
   \[ T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \]
   \[ T(v) = \text{Rot}_\theta(v) \]
Solution: The objective is to identify which of the linear maps $T$ has a basis of eigenvectors. Geometrically, this means identifying a vector $v$ that is mapped by $T$ to a vector in $\text{span}(v)$, and then identifying a linearly independent vector $w$ that is mapped by $T$ to a vector in $\text{span}(w)$. Equivalently, it means identifying two distinct lines through the origin in $\mathbb{R}^2$ that are each (as a set) mapped by $T$ back to itself.

Part 1.

1. Scalar multiplication by 3
$T : \mathbb{R}^2 \rightarrow \mathbb{R}^2,$
$T(v) = 3v$

Since this matrix acts by multiplication by 3, any nonzero vector in $\mathbb{R}^2$ is an eigenvector with eigenvalue 3. (The whole domain $\mathbb{R}^2$ is the eigenspace $E_3$.) Thus, any choice of two linearly independent vectors is an eigenbasis for this map. Both vectors should be labelled by 3. Here is a sample solution:
Part 2.

2. Projection onto the line L

\[ T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \]

\[ T(v) = \text{Proj}_L(v) \]

Any vector along \( L \) will be fixed by the projection onto \( L \), and so all nonzero vectors in the direction of \( L \) are eigenvectors with eigenvalue 1. Any vector that is orthogonal to \( L \) will be mapped to 0 by the projection, and so any nonzero vector perpendicular to \( L \) is an eigenvector with eigenvalue 0. A correct solution would be any nonzero vector along \( L \) labelled by 1, and any nonzero vector perpendicular to \( L \) labelled by 0. Here is a sample solution:
3. Reflection in the line $L$

$T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$T(v) = \text{Refl}_L(v)$

There are two lines that are mapped back to themselves by $T$. The first is the line $L$, which is fixed by reflection in $L$. Every nonzero vector in the direction of $L$ is therefore an eigenvector with eigenvalue 1. The second line is the line perpendicular to $L$, since reflection in $L$ negates the vectors in this line. Thus any nonzero vector perpendicular to $L$ is an eigenvector with eigenvalue $-1$. A correct solution would be any nonzero vector along $L$ labelled by 1, and any nonzero perpendicular to $L$ labelled by $-1$. Here is a sample solution:
4. Rotation by $\theta$

$T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$T(v) = \text{Rot}_\theta(v)$

Since every line through the origin in $\mathbb{R}^2$ is rotated by $\theta$, no line is mapped back to itself by this map. Thus, there are no eigenvectors, and the solution is no eigenbasis exists. In general, rotation in the plane by an angle $\theta$ will have no real eigenvalues unless $\theta$ is a multiple of $\pi$ (in which case the map is $\pm I_2$).

6. (1 point each)
(a): TRUE.

Since $p(\lambda) = (\lambda - 4)(\lambda + 4)$, $p(4) = 0$ and 4 is an eigenvalue of $A$. Any eigenvector $v$ of $A$ with eigenvalue 4 will be a nonzero element in the nullspace of the matrix $4I_2 - A$. It follows that this matrix is noninvertible.

(b): TRUE.

An $n \times n$ matrix $A$ has $C(A) = \mathbb{R}^n$ if and only if its nullspace has no nonzero vectors, that is, $N(A) = \{0\}$. A nonzero element in $N(A)$ is exactly an eigenvector with eigenvalue 0. Since $A$ is a $3 \times 3$ matrix, it can have at most 3 distinct eigenvalues, and so given that we know 1, 2, and $-3$ are eigenvalues, $A$ cannot possibly have zero as an eigenvalue. The matrix $A$ must therefore be surjective.

(c): TRUE.
Since the characteristic polynomial has three distinct roots 0, 1, and $-1$, the $3 \times 3$ matrix $A$ has three distinct eigenvalues. By Proposition 23.3, $A$ must be diagonalizable.

(d): TRUE.

This function is continuous on $\mathbb{R}^3$, so we can evaluate the limit by plugging in $x = 1$, $y = 1$ and $z = 1$.

(e): FALSE.

We can disprove this statement by checking the products of some $2 \times 2$ symmetric matrices. Consider for example the matrices $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Even though both are symmetric, their product $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ is not symmetric.

(f): TRUE.

The level set at $c$ is by definition the set of solutions to the equation $5x + 2y = c$. Solving for $y$, we see that this is the graph of the function $y = -\frac{5}{2}x + \frac{c}{2}$, a line in $\mathbb{R}^2$.

(g): TRUE.

The vectors $\{v_1, v_2, v_3, v_4\}$ are a basis of eigenvectors for the eigenspace associated to $\lambda_1$. Since this space is four dimensional and $A$ is a $4 \times 4$ matrix, this must in fact be a basis for the domain of $A$, $\mathbb{R}^4$. By definition, it is an eigenbasis, and $A$ is a diagonalizable matrix.

(h): TRUE.

If we approach $(0,0)$ along the path $x = 0$, we have $\lim_{y \to 0} \frac{0}{y^4} = 0$. However, if we approach $(0,0)$ along the path $x = y$, we have $\lim_{y \to 0} \frac{y^4}{y^4} = 1$. We conclude that the limit does not exist.

7. (a) (5 points) Find the parametric equation for the tangent line $L$ to the curve $r(t) = (t - 1, \sqrt{t}, \cos(\pi t))$

(which is defined for $t > 0$) at the point $(3, 2, 1)$. 

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At the point $(3,2,1)$ we have $t = 4$. The velocity vector at time $t$ is

$$v(t) = \frac{dr(t)}{dt} = \left(1, \frac{1}{2}t^{-1/2}, -\pi \sin(\pi t)\right)$$

So at $t = 4$ this is $v(4) = (1, 1/4, 0)$. Hence the tangent line has parametric form

$$L = \{r(4) + s v(4), \quad s \in \mathbb{R}\} = \{(3,2,1) + s(1,1/4,0), \quad s \in \mathbb{R}\}$$

(b) (5 points) This tangent line $L$ intersects the curve $r(t)$ at $(3,2,1)$. Does it intersect this curve at any other point? If so, where?

If the tangent line intersects the curve at some point, then

$$(t - 1, \sqrt{t}, \cos(\pi t)) = (3 + s, 2 + s/4, 1)$$

This gives $t = 4 + s$ and $\sqrt{t} = 2 + s/4$, so

$$4 + s = t = \left(2 + \frac{s}{4}\right)^2 = 4 + s + \frac{s^2}{16}$$

Giving $s = 0$, which shows that the only intersection point is the original $(3,2,1)$.

8. (a) (5 points) Let $A$ be a $3 \times 3$ matrix with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$. Let

$$B = \begin{bmatrix} 4 & 6 & 8 \\ 0 & 5 & 7 \\ 0 & 0 & 6 \end{bmatrix}.$$ 

Calculate $\det(A^{-1}B)$.

We know the determinant of $A$ is the product of its eigenvalues, so $\det(A) = \lambda_1 \lambda_2 \lambda_3 = 6$. Also, $B$ is upper-triangular, hence its determinant is the product of the diagonal entries, that is $\det(B) = 4 \cdot 5 \cdot 6 = 120$. These give

$$\det(A^{-1}B) = \det(A^{-1})\det(B) = \frac{1}{\det(A)}\det(B) = \frac{120}{6} = 20$$

(b) (5 points) What are the eigenvalues of $A^{2014}$? (You may leave your answer in the form of an exponent.)

If $v_1$ is an eigenvector of $A$ with eigenvalue $\lambda_1$, then $A v_1 = \lambda_1 v_1$, which then implies that $A^{2014} v_1 = \lambda_1^{2014} v_1$. So $v_1$ is an eigenvector of $A^{2014}$ with eigenvalue $\lambda_1^{2014}$. The same argument for $\lambda_2$ and $\lambda_3$ shows that the eigenvalues of $A^{2014}$ are exactly $\lambda_1^{2014} = 1^{2014} = 1$, $\lambda_2^{2014} = 2^{2014}$ and $\lambda_3^{2014} = 3^{2014}$. 

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(c) (5 points) Let $R$ be the matrix which represents rotation in $\mathbb{R}^3$ by $\pi$ around the $x$-axis. What are the eigenvalues of $R$? For each eigenvalue give a basis for its eigenspace.

If $e_1$, $e_2$ and $e_3$ are the standard basis vectors, then we see that the rotation $R$ fixes $e_1$, and maps $e_2$ and $e_3$ to their negatives, that is

$$R(e_1) = e_1$$

$$R(e_2) = -e_2$$

$$R(e_3) = -e_3$$

So the matrix of $R$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$  Hence the eigenvalues are 1 and -1, with eigenspaces $E_1 = \text{Span}(e_1)$ and $E_{-1} = \text{Span}(e_2, e_3)$