1. (8 points) Let \( f(x, y) = \sqrt{y - x^2} \).

(a) On the axes for \( \mathbb{R}^2 \) provided below, sketch and label the sets \( f^{-1}(0), f^{-1}(1), \) and \( f^{-1}(2) \); that is, the level sets of \( f \) at heights \( 0, 1, \) and \( 2 \). Be sure to label the scales on your axes for full credit.

(4 points) The level set \( f^{-1}(c) \) corresponds to the points \( (x, y) \) such that \( f(x, y) = \sqrt{y - x^2} = c \), i.e., \( y = x^2 + c^2 \). Where are interested in the cases \( c = 0, 1, 2 \). These correspond to the parabolas \( y = x^2, y = x^2 + 1, y = x^2 + 4 \), respectively.

(b) Find an equation for the plane in \( \mathbb{R}^3 \) tangent to the graph of \( f \) (i.e., the surface with equation \( z = f(x, y) \)) at the point \( (1, 2, 1) \).

(4 points) There are two ways for computing a tangent plane to a surface.

First method: using graphs. We recall the tangent plane to a graph \( z = f(x, y) \) at a point \( (x, y) = (a, b) \) is given by:

\[
z = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b).
\]

Now \( \frac{\partial f}{\partial x} = \frac{-x}{\sqrt{y - x^2}} \) and \( \frac{\partial f}{\partial y} = \frac{1}{2\sqrt{y - x^2}} \). We have that at the point \( (x, y, f(x, y)) = (1, 2, 1) \) the tangent plane is given by:

\[
z = 1 - 1(x - 1) + \frac{1}{2}(y - 2) \quad \iff \quad x - \frac{1}{2}y + z = 1.
\]

Second method: level sets. The surface we are interested can be thought of as the 0-level set of the function \( F(x, y, z) = z - \sqrt{y - x^2} \). Therefore, the normal to the tangent plane is given by the gradient of \( F \):

\[
\nabla F(x, y, z) = \begin{bmatrix}
\frac{x}{\sqrt{y - x^2}} \\
\frac{1}{2\sqrt{y - x^2}} \\
1
\end{bmatrix}.
\]

Hence, \( \nabla F(1, 2, 1) = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ 1 \end{bmatrix} \) is the normal vector to the tangent plane we are looking for, so its equation is given by:

\[
1(x - 1) - \frac{1}{2}(y - 2) + 1(z - 1) = 0 \quad \iff \quad x - \frac{1}{2}y + z = 1.
\]
2. (10 points) Let \( f : \mathbb{R}^3 \to \mathbb{R}^2 \) be defined by \( f(x, y, z) = (y^3, -x^2 + 4yz) \).

(a) Find the matrix \( Df(x, y, z) \).

(5 points) The matrix is

\[
Df(x, y, z) = \begin{bmatrix}
0 & 3y^2 & 0 \\
-2x & 0 & 4y \\
4z & 4y & 0
\end{bmatrix}.
\]

(b) Suppose the differentiable function \( g : \mathbb{R}^2 \to \mathbb{R} \) has \( Dg(-1, -1) = \begin{bmatrix} 5 & -2 \end{bmatrix} \). Find the matrix \( D(g \circ f)(1, -1, 0) \).

(5 points) First notice that \( f(1, -1, 0) = (-1, -1) \). By the chain rule

\[
D(g \circ f)(1, -1, 0) = Dg(-1, -1) \ Df(1, -1, 0)
\]

\[
= \begin{bmatrix} 5 & -2 \end{bmatrix} \begin{bmatrix} 0 & 3 & 0 \\
-2 & 0 & -4 \\
4 & 15 & 8
\end{bmatrix}
\]

\[
= \begin{bmatrix} 4 & 15 & 8 \end{bmatrix}.
\]
3. (8 points)

(a) Let \( C \) be the image in \( \mathbb{R}^3 \) of the parameterized curve \( f(t) = \begin{bmatrix} \cos 3t \\ \sin t \\ t^2 \end{bmatrix} \). Find, in parametric form, the line tangent to \( C \) at the point \( (0, 1, \frac{\pi^2}{4}) \).

(4 points) First, we solve for \( t_0 \) corresponding to the point \( (0, 1, \frac{\pi^2}{4}) \) by setting
\[
\begin{bmatrix} \cos 3t \\ \sin t \\ t^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \frac{\pi^2}{4} \end{bmatrix} \implies t = \frac{\pi}{2}
\]
Thus, the tangent vector at \( f\left(\frac{\pi}{2}\right) = (0, 1, \frac{\pi^2}{4}) \) is given by
\[
f'(\frac{\pi}{2}) = \begin{bmatrix} -3 \sin \frac{3\pi}{2} \\ \cos \frac{\pi}{2} \\ 2(\frac{\pi}{2}) \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ \pi \end{bmatrix}
\]
Therefore, the tangent line in parametric form is given by
\[
\left\{ \begin{bmatrix} 0 \\ 1 \\ \frac{\pi^2}{4} \end{bmatrix} + s \begin{bmatrix} 3 \\ 0 \\ \pi \end{bmatrix} : s \in \mathbb{R} \right\}
\]

(b) Suppose \( g(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix} \) is a parameterized curve in \( \mathbb{R}^2 \) with the property that for all \( t \), the speed \( ||g'(t)|| \) is constant (does not depend on \( t \)). Explain why the velocity and acceleration vectors (that is, \( g'(t) \) and \( g''(t) \)) must be orthogonal to each other at every time \( t \). (Hint: differentiate an expression for the square of speed.)

(4 points) Since the speed \( ||g'(t)|| \) is constant, we have
\[
(g_1'(t))^2 + (g_2'(t))^2 = ||g'(t)||^2 = C^2
\]
for some constant \( C \). Differentiating both sides with respect to \( t \) yields
\[
2g_1'(t)g_1''(t) + 2g_2'(t)g_2''(t) = 0 \iff \begin{bmatrix} g_1'(t) \\ g_2'(t) \end{bmatrix} \cdot \begin{bmatrix} g_1''(t) \\ g_2''(t) \end{bmatrix} = 0
\]
That is \( g'(t), g''(t) \) are orthogonal as desired.

**Common mistake:** many students argue that the constant speed implies that the acceleration is zero \( (g''(t) = 0) \), so that \( g'(t) \cdot g''(t) = 0 \). This is not correct. Even though the speed is constant, the velocity vector might be changing its direction and hence the acceleration might not be zero. For example, take \( g(t) = (\sin t, \cos t) \), then \( g'(t) = (\cos t, -\sin t) \), so the speed \( ||g'(t)|| = 1 \) is a constant, but the acceleration is \( g''(t) = (-\sin t, -\cos t) \).
4. (10 points) The temperature at point \((x, y)\) of \(\mathbb{R}^2\) is given by \(T(x, y) = x^2 - y^4\).

(a) A spider is at the point \((-2, 1)\) and its velocity vector is \(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\). Find the rate of change of the spider’s temperature at this moment.

(3 points) The gradient of \(T\) is \(\nabla T = \begin{bmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x \\ -4y^3 \end{bmatrix}\). The desired rate of change is the directional derivative of \(T\) in the direction \(v\).

\[
D_v T(-2, 1) = \nabla T(-2, 1) \cdot v = \begin{bmatrix} 2(-2) \\ -4(1) \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (-4) + (-4) = -8
\]

(b) An ant is at the point \((1, -1)\) and can move with speed equal to 3. What should the ant’s velocity vector be in order to warm up as quickly as possible?

(3 points) In order for the ant to move as quickly as possible, it should head in the direction \(w\), where \(w\) is a positive scalar times the gradient of \(T\):

\[
w = c \nabla T(1, -1) = \begin{bmatrix} 2c \\ 4c \end{bmatrix}
\]

Meanwhile, the speed is the magnitude of the velocity, so we need \(\|w\| = 3\); thus

\[
4c^2 + 16c^2 = 9 \implies c^2 = \frac{9}{20} \implies c = \frac{3}{2\sqrt{5}} \text{ (since } c > 0)\]

So the velocity vector is \(w = \begin{bmatrix} \frac{3}{2\sqrt{5}} \\ \frac{6}{\sqrt{5}} \end{bmatrix}\).

(c) A potato bug is at the point \((2, 2)\) and is moving in order to keep its temperature constant. The \(x\)-component of its velocity is \(-1\); find the \(y\)-component of its velocity at this instant.

(4 points) The direction in which the bug keeps its temperature constant \(\begin{bmatrix} -1 \\ y \end{bmatrix}\) is orthogonal to the gradient of \(T\) at \((2, 2)\). We have

\[
\nabla T(2, 2) \cdot \begin{bmatrix} -1 \\ y \end{bmatrix} = 0 \iff \begin{bmatrix} 2 \cdot 2 \\ -4 \cdot 2^3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ y \end{bmatrix} = 0
\]

\[
\iff -4 - 32y = 0
\]

\[
\iff y = \frac{1}{8}
\]
5. (10 points) Let \( S \) be the surface in \( \mathbb{R}^3 \) given by the equation \( e^{(x^2-y^2)} - yz^3 = 28 \).

Let \( f(x, y, z) = e^{(x^2-y^2)} - yz^3 \), so that \( S \) is the level set of \( f \) at height 28.

(a) Find an equation for the tangent plane to \( S \) at the point \( a = (1, 1, -3) \).

(4 points) We first compute the gradient of \( f \) at the point \( a = (1, 1, -3) \).

\[
\nabla f(x, y, z) = \begin{bmatrix} 2xe^{(x^2-y^2)} \\ -2ye^{(x^2-y^2)} - z^3 \\ -3yz^2 \end{bmatrix} \quad \Rightarrow \quad \nabla f(1, 1, -3) = \begin{bmatrix} 2 \cdot 1 \cdot e^{(1^2-1^2)} \\ -2 \cdot 1 \cdot e^{(1^2-1^2)} - (-3)^3 \\ -3 \cdot 1 \cdot (-3)^2 \end{bmatrix} = \begin{bmatrix} 2 \\ -27 \end{bmatrix}
\]

Since the gradient of \( f \) is perpendicular to the level sets of \( f \), the vector \( \mathbf{n} = \nabla f(a) = \begin{bmatrix} 2 \\ -27 \end{bmatrix} \) is perpendicular to \( S \) at \( a = (1, 1, -3) \). Therefore the tangent plane \( P \) is the plane perpendicular to \( \mathbf{n} \) and passing through \( a \), so its equation is

\[
2(x - 1) + 25(y - 1) - 27(z - (-3)) = 0 \iff 2(x - 1) + 25(y - 1) - 27(z + 3) = 0 \iff 2x + 25y - 27z = 108
\]

(b) There is only one point on \( S \) with \( y = 1.03 \) and \( z = -3.01 \) (you do not have to prove this). Use linear approximations to estimate the \( x \) coordinate of this point; simplify your answer as much as possible.

(3 points) The surface \( S \) is approximated near \( a \) by the plane \( P \) from part (a), so we are looking for the \( x \)-coordinate of the point \((x, y, z)\) on \( P \) with \( y = 1.03 \) and \( z = -3.01 \). Plugging these into the equation \( 2(x - 1) + 25(y - 1) - 27(z - (-3)) = 0 \) from part (a), we obtain

\[
2(x - 1) + 25(1.03 - 1) - 27((-3.01) - (-3)) = 0 \iff 0 = 2(x - 1) + 25(0.03) - 27(-0.01)
\iff 0 = 2(x - 1) + 0.75 + 0.27
\iff 0 = 2(x - 1) + 1.02
\iff 0 = (x - 1) + 0.51
\iff 0 = x - 0.49
\iff x = 0.49
\]

(c) Use linear approximations to estimate \( e^{(1.1)^2-(1.01)^2}} - (1.01)(-2.99)^3 \); simplify your answer as much as possible.

(3 points) We are asked to approximate \( f(1.1, 1.01, -2.99) \). This is very close to the point \( a = (1, 1, -3) \); if we let \( \mathbf{h} \) be the vector \( \begin{bmatrix} 0.1 \\ 0.01 \end{bmatrix} \), then what we’re asked to approximate is \( f(a + \mathbf{h}) \). The linearization \( L \) of \( f \) near the point \( a \) is defined by

\[
L(x) = f(a) + Df(a)(x - a) \iff L(a + \mathbf{v}) = f(a) + Df(a)\mathbf{v},
\]

which can also be written as

\[
L(x) = f(a) + \nabla f(a) \cdot (x - a) \iff L(a + \mathbf{v}) = f(a) + \nabla f(a) \cdot \mathbf{v}.
\]

The approximation of \( f(a + \mathbf{h}) \) is therefore given by

\[
f(a + \mathbf{h}) \approx L(a + \mathbf{h}) = f(a) + \nabla f(a) \cdot \mathbf{h} = 28 + \begin{bmatrix} 2 \\ 25 \\ -27 \end{bmatrix} \cdot \begin{bmatrix} 0.1 \\ 0.01 \end{bmatrix} = 28 + (0.2 + 0.25 - 0.27) = 28 + 0.18 = 28.18.
\]
6. (10 points) Let \( S : \mathbb{R}^3 \to \mathbb{R}^2 \) be the linear transformation defined by \( S(v) = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 0 \end{bmatrix} v \).

In addition, suppose \( c \) is a fixed real number; let \( T : \mathbb{R}^2 \to \mathbb{R}^3 \) be defined by \( T(w) = \begin{bmatrix} 0 & 2 \\ -1 & c \\ 1 & 0 \end{bmatrix} w \).

(a) For precisely what value(s) of \( c \), if any, is the linear transformation \( S \circ T \) invertible? Justify your answer completely.

(5 points) The linear map \( S \circ T \) is invertible if and only if the determinant of its matrix is nonzero; meanwhile, the matrix associated to \( S \circ T \) is the product of the matrices associated to \( S \) and \( T \). So first we must find the matrix associated to \( S \circ T \) and then find the values of \( c \) for which its determinant is nonzero.

\[
\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 2 \\ -1 & c \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 4 + c \\ -2 & 2c \end{bmatrix}.
\]

The determinant of this \( 2 \times 2 \) matrix is \(-2c + 8\), which is nonzero precisely when \( c \neq 4 \). Therefore \( S \circ T \) is invertible if and only if \( c \neq 4 \).

(b) For precisely what value(s) of \( c \), if any, is the linear transformation \( T \circ S \) invertible? Justify your answer completely.

(5 points) As in part (a) we first find the matrix associated to \( T \circ S \):

\[
\begin{bmatrix} 0 & 2 \\ -1 & c \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 0 \\ -2 & 2c - 1 & 1 \\ 2 & 1 & -1 \end{bmatrix}.
\]

The determinant of this matrix is 0 (row expand by the first row). This implies that \( T \circ S \) is not invertible for any \( c \).

Alternate solution: Another way to argue that \( T \circ S \) is not invertible is as follows. First, note that \( T \circ S : \mathbb{R}^3 \to \mathbb{R}^3 \), so the condition of this linear map being invertible is equivalent to its having an associated matrix that has rank 3. Next, since \((T \circ S)(x) = T(S(x))\), any vector in the image (i.e., range) of \( T \circ S \) is contained in the image (range) of \( T \). But the image of \( T \) is the span of \( T([1,0]) \) and \( T([0,1]) \), so it is at most 2-dimensional; thus the image of \( T \circ S \) is at most 2-dimensional as well. This means the matrix associated to \( T \circ S \) has rank (dimension of column space) at most 2, which is less than 3. Thus, \( T \circ S \) cannot be invertible for any \( c \).
7. (10 points) Consider the matrix 

\[
A = \begin{bmatrix}
1 & 2 & 7 & 1 \\
0 & 1 & 5 & 1 \\
0 & 0 & -1 & -1 \\
0 & 0 & 2 & 3
\end{bmatrix}
\]

(a) Is \(A\) invertible? If so, find \(A^{-1}\); if not, explain your reasoning.

(5 points) We can determine whether \(A\) is invertible, and find \(A^{-1}\) if it is, by row-reducing \([A|I_4]\):

\[
\begin{bmatrix}
1 & 2 & 7 & 1 \\
0 & 1 & 5 & 1 \\
0 & 0 & -1 & -1 \\
0 & 0 & 2 & 3
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 2 & 7 & 1 \\
0 & 1 & 5 & 1 \\
0 & 0 & -1 & -1 \\
0 & 0 & 0 & 1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & -3 & -1 \\
0 & 1 & 5 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 0 & -4 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Since \(\text{rref}[A|I_4] = [I_4|\text{something}]\), we know \(A\) is invertible and \(A^{-1} = \begin{bmatrix}
1 & -2 & -7 & -2 \\
0 & 1 & 13 & 4 \\
0 & 0 & -3 & -1 \\
0 & 0 & 2 & 1
\end{bmatrix}\).

(b) What is the determinant of the matrix \(A^{2015}\)? Simplify your answer, showing your reasoning.

(5 points) The first step of the row-reduction above, namely \(IV \rightarrow IV + 2 \times III\), does not affect the determinant, so

\[
\det(A) = \begin{vmatrix}
1 & 2 & 7 & 1 \\
0 & 1 & 5 & 1 \\
0 & 0 & -1 & -1 \\
0 & 0 & 2 & 3
\end{vmatrix} = \begin{vmatrix}
1 & 2 & 7 & 1 \\
0 & 1 & 5 & 1 \\
0 & 0 & -1 & -1 \\
0 & 0 & 0 & 1
\end{vmatrix} = 1 \cdot 1 \cdot (-1) \cdot 1 = -1.
\]

Just as \(\det(AB) = \det(A)\det(B)\), we have \(\det(A^{2015}) = (\det(A))^{2015}\). Therefore

\[
\det(A^{2015}) = (\det(A))^{2015} = (-1)^{2015} = -1.
\]
8. (8 points) For both parts of this problem, let \( \mathbf{w}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \) and \( \mathbf{w}_2 = \begin{bmatrix} 7 \\ 3 \end{bmatrix} \).

(a) Suppose \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) is the linear map given by \( T(\mathbf{e}_1) = \mathbf{w}_1 \) and \( T(\mathbf{e}_2) = \mathbf{w}_2 \). If \( D \) is a region in \( \mathbb{R}^2 \) with area equal to 4, find the area of the region \( T(D) \).

\[(4 \text{ points}) \text{ From the given information, the matrix of } T \text{ is given by} \]
\[A = \begin{bmatrix} 3 & 7 \\ 1 & 3 \end{bmatrix}\]

Hence,
\[
(Area \ of \ T(D)) = |\det(A)|(Area \ of \ D) = |9 - 7|(4) = 8
\]

(b) Suppose \( \mathbf{w} = \begin{bmatrix} x \\ y \end{bmatrix} \). If \( \mathbf{w} = u\mathbf{w}_1 + v\mathbf{w}_2 \), find simplified expressions for \( u \) and \( v \) in terms of \( x \) and \( y \).

\[(4 \text{ points}) \text{ We set} \]
\[
\begin{bmatrix} x \\ y \end{bmatrix} = u \begin{bmatrix} 3 \\ 1 \end{bmatrix} + v \begin{bmatrix} 7 \\ 3 \end{bmatrix} = A \begin{bmatrix} u \\ v \end{bmatrix} \Rightarrow \begin{bmatrix} u \\ v \end{bmatrix} = A^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3 & -7 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3x - 7y}{2} \\ -x + 3y \end{bmatrix}
\]

That is,
\[u = \frac{3x - 7y}{2}, \quad v = \frac{-x + 3y}{2}\]