Solutions to Math 51 First Exam — January 29, 2015

1. (10 points)
(a) Complete the following sentence: A set of vectors \( \{v_1, \ldots, v_k\} \) is defined to be linearly dependent if

(2 points) there exist \( c_1, \ldots, c_k \in \mathbb{R} \), not all zero, such that

\[
c_1v_1 + \ldots + c_kv_k = 0.
\]

(b) For this and part (c), suppose the set of vectors \( \{u, v, w\} \) in \( \mathbb{R}^7 \) is linearly independent. Is the set \( \{u, v\} \) necessarily linearly independent as well? Explain why, or give a counterexample illustrating why not.

(3 points) Yes. Suppose \( \{u, v\} \) is linearly dependent. Then there exists \( c_1, c_2 \in \mathbb{R} \), not both zero, such that

\[
c_1v_1 + c_2v_2 = 0.
\]

This implies

\[
c_1v_1 + c_2v_2 + 0 \cdot v_3 = 0,
\]

which in turn implies that \( \{u, v, w\} \) is linearly dependent, contradicting our assumption.

(c) With \( \{u, v, w\} \) as in part (b), is the set \( \{3u + 2v, v, u + v + w\} \) necessarily linearly independent? Explain why, or give a counterexample to illustrate why not.

(5 points) Yes. We will show that

\[
c_1(3u + 2v) + c_2v + c_3(u + v + w) = 0
\]

holds only if \( c_1 = c_2 = c_3 = 0 \). We can rewrite this equation as

\[
(3c_1 + c_3)u + (2c_1 + c_2 + c_3)v + c_3w = 0.
\]

Since \( \{u, v, w\} \) is a linearly independent set, we obtain

\[
3c_1 + c_3 = 0
\]

\[
2c_1 + c_2 + c_3 = 0
\]

\[
c_3 = 0.
\]

Solving this system of equation, we see that \( c_1 = c_2 = c_3 = 0 \) is the only solution, as desired.
2. (10 points) We are given three points: (0, 1, 1), (2, -1, 1), and (1, 0, 2) in \( \mathbb{R}^3 \). Let \( P \) denote the plane that contains these three points.

(a) Find, showing all steps, a nonzero vector \( \mathbf{n} \) that is perpendicular (normal) to the plane \( P \).

(3 points) The vector from (0,1,1) to (2,-1,1) is \[
\begin{bmatrix}
2 \\
-2 \\
0
\end{bmatrix}
\]. The vector from (0,1,1) to (1,0,2) is \[
\begin{bmatrix}
1 \\
-1 \\
1
\end{bmatrix}
\]. To find the normal vector we take the cross-product of these to obtain

\[
\begin{bmatrix}
-2 \\
1 \\
2
\end{bmatrix} \times \begin{bmatrix}
1 \\
-1 \\
1
\end{bmatrix} = \begin{bmatrix}
-2 \\
1 \\
-2
\end{bmatrix}.
\]

(b) Find an equation for the plane \( P \); your answer should be in the form \( ax + by + cz = d \).

(3 points) The formula for this is \( \mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) = 0 \), where \( \mathbf{n} \) is the normal vector to the plane and \( \mathbf{x}_0 \) is the base-point for our plane. Here \( \mathbf{n} \) is given by part (a) and \( \mathbf{x}_0 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \). Computing this gives us

\[
0 = \mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0)
= \begin{bmatrix}
-2 \\
-2 \\
0
\end{bmatrix} \cdot \begin{bmatrix}
x - 0 \\
y - 1 \\
z - 1
\end{bmatrix}
= -2x + (-2)(y - 1).
\]

Putting this in the appropriate form yields

\[-2x - 2y = -2\]

(c) Let \( P_2 = \text{span} \left( \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \). Determine, with reasoning, whether the intersection of \( P \) and \( P_2 \) is empty, or a point, or a line, or a plane.

(4 points) Any vector in this span is given by

\[
\begin{bmatrix}
s \\
-s \\
t
\end{bmatrix}
\]

for some \( s, t \in \mathbb{R} \) (this is the definition of the span). Plugging this vector into our equation from (b) yields:

\[-2 = (-2)(s) - 2(-s) = -2s + 2s = 0.\]

Since this is always inconsistent, it must be that \( P \) and \( P_2 \) never intersect.
3. (10 points) Let \( u \) and \( v \) be two vectors in \( \mathbb{R}^n \). Suppose that \( \| u \| = 1 \) and \( \| v \| = \frac{1}{2} \).

(a) Are \( u + 2v \) and \( u - 2v \) orthogonal? Justify completely.

\[
\begin{align*}
(5 \text{ points}) \quad u + 2v \text{ and } u - 2v \text{ are orthogonal if and only if their dot product is zero. We compute:} \\
(u + 2v) \cdot (u - 2v) &= u \cdot u - u \cdot 2v + u \cdot 2v - 2 \cdot 2 \cdot v \cdot v = \| u \|^2 - 4\| v \|^2 = 1 - 4 \cdot \frac{1}{4} = 0 \\
\text{So it follows that } u + 2v \text{ and } u - 2v \text{ are orthogonal.}
\end{align*}
\]

(b) Show that \( u + v \) is a unit vector if and only if \( u \cdot v = -\frac{1}{8} \).

\[
\begin{align*}
(5 \text{ points}) \quad u + v \text{ is a unit vector if and only if } \| u + v \| = 1. \\
\| u + v \| = 1 & \iff ||u + v||^2 = 1 \\
\quad & \iff u \cdot u + u \cdot v + v \cdot v + 1 = 1 \\
\quad & \iff \| u \|^2 + 2u \cdot v + \| v \|^2 = 1 \\
\quad & \iff 1 + 2u \cdot v + \frac{1}{4} = 1 \\
\quad & \iff 2u \cdot v = -\frac{1}{4} \\
\quad & \iff u \cdot v = -\frac{1}{8}
\end{align*}
\]
4. (10 points) Suppose \(a\) is a fixed real number, and consider the matrix \(A = \begin{bmatrix} 1 & -1 & a \\ 1 & 2 & 0 \end{bmatrix}\).

(a) Find, with reasoning, a basis for the column space \(C(A)\). (Your answer might need to be expressed in terms of \(a\).)

(3 points) Our general algorithm for finding a basis for the column space is to row-reduce, find the pivot columns, and take precisely those columns in the original matrix that correspond to the pivot columns (this works because row reduction preserves linear dependence relations between the columns, and the pivot columns are always a basis for the column space of a row-reduced matrix). So let’s get to work:

\[
\begin{bmatrix} 1 & -1 & a \\ 1 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & a \\ 0 & 3 & -a \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & a \\ 0 & 1 & -a/3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2a/3 \\ 0 & 1 & -a/3 \end{bmatrix}.
\]

Now our matrix is in reduced row echelon form, and the pivot columns are clearly the first two. Therefore a basis for \(C(A)\) is given by

\[
\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}.
\]

Of course, this is far from the only possible basis we could have found. Some people solved this problem by noticing right away that \(C(A)\) had to be all of \(\mathbb{R}^2\), because the first two columns were already linearly independent, and therefore saved themselves the trouble of row reducing (but we have to do that later anyway, so we might as well do it now). Thus any basis of \(\mathbb{R}^2\) would work, in particular the standard basis.

The most common errors were row-reduction errors. Several people put the vector \(\begin{bmatrix} a \\ 0 \end{bmatrix}\) as one of their basis vectors, which only works when \(a \neq 0\)!

(b) Find, with reasoning, a basis for the null space \(N(A)\). (Your answer might need to be expressed in terms of \(a\).)

(3 points) If we row-reduced in the previous part, we’ve done all the hard work. We get the equations

\[
\begin{align*}
x_1 + \frac{2a}{3} x_3 &= 0, \\
x_2 - \frac{a}{3} x_3 &= 0.
\end{align*}
\]

The variable \(x_3\) is free, and once it is picked \(x_1\) and \(x_2\) are determined by \(x_1 = -\frac{2a}{3} x_3\) and \(x_1 = \frac{a}{3} x_3\). Therefore a basis for the null space might be the set containing the single vector

\[
\begin{bmatrix} -\frac{2a}{3} \\ \frac{a}{3} \\ 1 \end{bmatrix}
\]

or any nonzero multiple thereof (for example, you could multiply everything by three and it would still be correct, of course).

One cannot divide by \(a\), because \(a\) might equal zero. Another common error was to forget that \(N(A)\) is here a subspace of \(\mathbb{R}^3\): it is the solution space to a matrix equation with a \(2 \times 3\) matrix (two equations in three unknowns).
(c) Find, with reasoning, all solutions to the equation $A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, or say why no solutions exist. (As in parts (a) and (b), be sure to express any dependence on $a$.)

(4 points) There are two ways to do this problem. In the first method, one can row reduce the augmented matrix

$$
\begin{bmatrix}
1 & -1 & a & 1 \\
1 & 2 & 0 & 1
\end{bmatrix},
$$

which (check!) yields

$$
\begin{bmatrix}
1 & 0 & 2a/3 & 1 \\
0 & 1 & -a/3 & 0
\end{bmatrix},
$$

and thus the two equations

$$
x_1 + \frac{2a}{3}x_3 = 1, \\
x_2 - \frac{a}{3}x_3 = 0.
$$

Using $x_3$ as a free variable, we find that $x_1 = 1 - \frac{2a}{3}x_3$ and $x_2 = \frac{a}{3}x_3$. Writing this in parametric form, the answer is

$$
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} + x_3 \begin{bmatrix}
-\frac{2a}{3} \\
\frac{a}{3} \\
1
\end{bmatrix} \quad \text{for } x_3 \in \mathbb{R}.
$$

There are other ways of writing the same answer.

The second method uses Proposition 8.2 in the text to note that if we can find any particular solution, the general solution will be a translation of $N(A)$ by that particular solution. And in fact, a particular solution is guaranteed to exist, as we have already noted that $C(A)$ is all of $\mathbb{R}^2$. As it so happens one possible particular solution is staring us in the face: since $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the first column of $A$, the vector

$$
\mathbf{x}_p = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
$$

is a solution. The solution to the problem is the set

$$
\mathbf{x}_p + N(A),
$$

which we can write as above.

One caution about notation: it is not correct to write the solution as

$$
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} + x_3 \begin{bmatrix}
-\frac{2a}{3} \\
\frac{a}{3} \\
1
\end{bmatrix} \quad \text{for } x_3, a \in \mathbb{R}.
$$

In this problem $a$ is given to us, fixed for all time, even though we don’t know what it is. No description of a solution set could involve varying over $a$. 
5. (10 points) Consider the matrix \( A = \begin{bmatrix}
1 & 1 & 9 \\
1 & -1 & -1 \\
-1 & 1 & 1 \\
1 & 1 & 9
\end{bmatrix} \)

(a) Find one or more conditions on \( b \in \mathbb{R}^4 \) that determine precisely whether \( b \) lies in the column space of \( A \), or alternatively show that every such \( b \) lies in \( C(A) \). (If you give conditions, they should be in the form of one or more equations involving the components \( b_1, \ldots, b_4 \) of \( b \).)

(6 points) Consider the augmented matrix:

\[
\begin{bmatrix}
1 & 1 & 9 & b_1 \\
1 & -1 & -1 & b_2 \\
-1 & 1 & 1 & b_3 \\
1 & 1 & 9 & b_4
\end{bmatrix}
\]

Then its RREF is:

\[
\begin{bmatrix}
1 & 0 & 4 & \frac{1}{2}b_1 + \frac{1}{2}b_2 \\
0 & 1 & 5 & \frac{1}{2}b_1 - \frac{1}{2}b_2 \\
0 & 0 & 0 & b_3 + b_2 \\
0 & 0 & 0 & b_4 - b_1
\end{bmatrix}
\]

In order to make the vector \( b \) in the column space, it is enough to set the last two components in the augmented part as 0.

\[b_2 + b_3 = 0, b_4 - b_1 = 0.\]

(Notice there is always solution to the first two components in the augmented part)

(b) Find a nonzero vector \( w \in \mathbb{R}^3 \) so that the product \( Aw = 0 \), or explain why no such \( w \) exists.

(4 points) It is enough to get the null space of the RREF, since row operations do not change null space.

\[
\begin{bmatrix}
1 & 0 & 4 & x \\
0 & 1 & 5 & y \\
0 & 0 & 0 & z
\end{bmatrix}
= 0.
\]

Then we get the relation:

\[x = -4z, y = -5z;\]

So one such vector is

\[
\begin{bmatrix}
4 \\
5 \\
-1
\end{bmatrix}.
\]

(Any nonzero parallel vector will be fine)
6. (10 points)

(a) Complete the sentence: A set $V$ of vectors in $\mathbb{R}^n$ is a (linear) subspace if the following three properties hold...

(3 points)
(i) $0 \in V$.

(ii) $V$ is closed under scalar multiplication; that is, if $x \in V$ and $c \in \mathbb{R}$, then $cx \in V$.

(iii) $V$ is closed under addition; that is, if $x, y \in V$, then $x + y \in V$.

(b) Let $W$ be the set of vectors in $\mathbb{R}^4$ that are orthogonal to both $a = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

Is $W$ a subspace of $\mathbb{R}^4$? Justify your answer completely.

(7 points) Yes, $W$ is a subspace of $\mathbb{R}^4$. To check this, we need to verify that $W$ satisfies the three properties (i)-(iii) in (a). Recall first that two vectors $x, y \in \mathbb{R}^n$ are orthogonal if and only if $x \cdot y = 0$.

(i) $0$ is orthogonal to every vector in $\mathbb{R}^4$, so in particular, $0 \cdot a = 0 \cdot b = 0$, hence $0 \in W$. (Note that some people got confused on: $0 \in \mathbb{R}^4$ (on the left side of the equation) is a vector, but $0$ (on the right side) is a scalar.)

(ii) Suppose that $x \in W, c \in \mathbb{R}$. We must show that $cx \in W$. Well,

$$(cx) \cdot a = c(x \cdot a) = c(0) \quad \text{(since } x \in W)$$

$$= 0.$$

A similar calculation gives $(cx) \cdot b = 0$. Thus, $cx \in W$.

(iii) Suppose that $x, y \in W$. We need to show that $x + y \in W$. Well,

$$(x + y) \cdot a = x \cdot a + y \cdot a = 0 + 0 \quad \text{(since } x, y \in W)$$

$$= 0.$$

A similar calculation shows that $(x + y) \cdot b = 0$. Thus $x + y \in W$. 

7. (10 points) In each of the following parts, some information is specified about an $m \times n$ matrix $A$; we wish to draw conclusions about linear systems of the form

$$Ax = b$$

for choices of $b$ in $\mathbb{R}^m$. By circling the appropriate response in each sub-part, indicate

i. whether $Ax = b$ has a solution for any choice of $b \in \mathbb{R}^m$ (“all $b$”), or has a solution for only some choices of $b \in \mathbb{R}^m$ (“some $b$”), or has a solution only in the single case $b = 0$; or whether there is insufficient information specified about $A$ to know; and

ii. given any choice of $b$ where a solution to $Ax = b$ does exist, whether that solution is unique, or is among infinitely many solutions; or whether there is insufficient information (about $A$, $b$, or both) to know.

Note: No justification is necessary.

(a) $A$ is a $3 \times 5$ matrix, with first two columns linearly independent, and each of the remaining three columns expressible as a combination of the first two columns.

i. Circle one: (all $b$) (some $b$) (only $b = 0$) (insufficient info)

ii. Circle one: (unique) (infinitely many) (insufficient info)

The rank of the matrix is 2. dim $N(A) = 3$.

(b) $A$ is a $3 \times 3$ matrix, and $x = 0$ is the only solution to $Ax = 0$.

i. Circle one: (all $b$) (some $b$) (only $b = 0$) (insufficient info)

ii. Circle one: (unique) (infinitely many) (insufficient info)

The assumption gives $N(A) = \{0\}$; it implies that $C(A) = \mathbb{R}^3$ in this case.

(c) $A$ is a $5 \times 3$ matrix, and the entry in the (second row, second column) is 17.

i. Circle one: (all $b$) (some $b$) (only $b = 0$) (insufficient info)

ii. Circle one: (unique) (infinitely many) (insufficient info)

We know $1 \leq \dim C(A) \leq 3$. We don’t know if the columns of $A$ are independent or not.

(d) $A$ is a $3 \times 5$ matrix, with null space containing 5 linearly independent vectors.

i. Circle one: (all $b$) (some $b$) (only $b = 0$) (insufficient info)

ii. Circle one: (unique) (infinitely many) (insufficient info)

The assumption implies dim $N(A) = 5$; Rank-Nullity Theorem implies dim $C(A) = 0$, that is $C(A) = \{0\}$.

(e) $A$ is a $4 \times 8$ matrix, and the reduced row echelon form of $A$ contains 4 pivots.

i. Circle one: (all $b$) (some $b$) (only $b = 0$) (insufficient info)

ii. Circle one: (unique) (infinitely many) (insufficient info)

$rref(A)$ has 4 pivots, so dim $C(A) = 4$, therefore $C(A) = \mathbb{R}^4$. There are $8 - 4 = 4$ free variables.
8. (8 points) Each of the statements below is either always true ("T"), or always false ("F"), or sometimes true and sometimes false, depending on the situation ("MAYBE"). For each part, decide which and circle the appropriate choice; you do not need to justify your answers.

(1 point each)

(a) A set of three vectors in \( \mathbb{R}^5 \) is linearly independent. T F MAYBE

Any three vectors taken from a basis for \( \mathbb{R}^5 \) will be linearly independent, but three colinear vectors will be linearly dependent.

(b) A set of six vectors in \( \mathbb{R}^5 \) is linearly independent. T F MAYBE

If we write the six vectors as the columns of a \( 5 \times 6 \) matrix, this matrix will have at least one non-zero vector \( v \) in its nullspace (think about the number of free variables in the reduced row echelon form, or alternatively use the Rank-Nullity Theorem). Then the components of \( v \) give a non-trivial linear dependence between the columns.

(c) A subspace \( V \) of \( \mathbb{R}^3 \) that is spanned by a linearly dependent set \( \{v_1, v_2, v_3, v_4\} \) satisfies \( V = \mathbb{R}^3 \). T F MAYBE

Notice that \( \{v_1, v_2, v_3, v_4\} \) is automatically a linearly dependent set, being four vectors in \( \mathbb{R}^3 \). If for example \( \{v_1, v_2, v_3\} \) is a basis for \( \mathbb{R}^3 \), the result will be true. However, \( v_1, v_2, v_3, v_4 \) could also for example be colinear, in which case they span only a line in \( \mathbb{R}^3 \).

(d) Given a \( 3 \times 5 \) matrix \( A \) and four vectors \( w_1, w_2, w_3, w_4 \) in \( \mathbb{R}^5 \), the set \( \{Aw_1, Aw_2, Aw_3, Aw_4\} \) is linearly dependent. T F MAYBE

Any four vectors in \( \mathbb{R}^3 \) are linearly dependent.

(e) Given a matrix \( A \), the column space of \( A \) is equal to the column space of the reduced row echelon form of \( A \). T F MAYBE

On the one hand, consider any \( A \) already in RREF, for which certainly \( C(A) = C(\text{rref}(A)) \); on the other, consider \( A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \), for which \( C(A) = \text{span}(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) \) but \( C(\text{rref}(A)) = C(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}) = \text{span}(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) \).

(f) Given a matrix \( A \), the null space of \( A \) is equal to the null space of the reduced row echelon form of \( A \). T F MAYBE

In fact, any two row-equivalent matrices have the same null space (because row operations on a system of linear equations do not alter the solution set; this is one of the key reasons we care about row operations).

(g) Given a \( 1 \times 3 \) matrix \( A \), then \( \dim N(A) \geq 1 \). T F MAYBE

The rank is at most one, so the Rank-Nullity Theorem says that the nullity is at least two. Notice that two is greater than one!

(h) Given a \( 3 \times 2 \) matrix \( A \), then \( \dim N(A) \leq 2 \). T F MAYBE

The null space is a subspace of \( \mathbb{R}^2 \), so it can be at most two dimensional.