Unless otherwise indicated, you can use results covered in lecture and homework, provided they are clearly stated.

If necessary, continue solutions on backs of pages
Note: work sheets are provided for your convenience, but will not be graded

Q.1
Q.2
Q.3
Q.4
Q.5
Q.6
Q.7
Q.8
T/40

Name (Print Clearly): ______________________________

I understand and accept the provisions of the honor code (Signed) ________________________
1(a) (2 points): Calculate the determinant of

\[
\begin{pmatrix}
11 & 12 & 13 & 426 \\
2001 & 2002 & 2003 & 421 \\
2 & 1 & 0 & -419 \\
101 & 101 & 102 & 2000
\end{pmatrix}
\]

No calculators: Clearly state all column/row operations.

(b) (3 points) Find the matrix of the orthogonal projection onto the plane \( V = \{(x, y, z) \in \mathbb{R}^3 : 2x + y - z = 0\} \).

Hint: Start by finding the orthogonal projection onto the (1-dimensional) normal space \( V^\perp \).
2(a) (2 points): If $u : \mathbb{R}^n \rightarrow \mathbb{R}$ is $C^1$ and if $\gamma : \mathbb{R} \rightarrow \mathbb{R}^n$ is also $C^1$, prove that the velocity vector $\Gamma'(t)$ of the curve $\Gamma(t) = \begin{pmatrix} \gamma(t) \\ u(\gamma(t)) \end{pmatrix}$ is orthogonal to the vector $\begin{pmatrix} \nabla u(\gamma(t)) \\ -1 \end{pmatrix}$ for each $t \in \mathbb{R}$.

(b) (3 points): Let $e^x$ be defined as usual by $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for $x \in \mathbb{R}$. Prove:

(i) $\lim_{x \to 0} |x|^p e^{-1/x^2} = 0$ for each $p > 0$.

Note: You can of course assume, without giving the proof, the standard property $e^{u+v} = e^u e^v$ (so in particular $e^{-u} = 1/e^u$).

(ii) If $f(x) = e^{-1/x^2}$ for $x \neq 0$ and $f(0) = 0$, find the Taylor series $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ of $f$.

Hint for (ii): Start by checking (by induction on $n$) that for $x \neq 0$ each derivative $f^{(n)}(x)$ has the form $p_n(1/x)e^{-1/x^2}$, where $p_n$ is a polynomial.
3(a) (2 points): Define the term “open set” in $\mathbb{R}^n$, and prove that the intersection $U \cap V$ of 2 open sets $U, V$ is again an open set.

3(b) (3 points): If $\varphi : \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R}^n \to \mathbb{R}$ are both continuous, and if $S = \{x \in \mathbb{R}^n : \varphi(x) = 0\}$ is bounded, prove there is a point $a \in S$ such that $f(x) \leq f(a) \forall x \in S$. 
4(a) (3 points): State (without proof) the Spectral Theorem for a real symmetric $n \times n$ matrix $A$, and use it to prove that for a given quadratic form $H(x) = \sum_{i,j=1}^{n} a_{ij} x_i x_j$ ($a_{ij} = a_{ji}$ real) there is a change of coordinates $y = Q^T x$ with $Q$ orthogonal (i.e. $Q^T Q = QQ^T = I$) such that the quadratic form $H(x)$ is transformed to an expression of the form $\sum_{j=1}^{n} \lambda_j y_j^2$ for suitable real $\lambda_1, \ldots, \lambda_n$.

(b) (2 points): Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$
5(a) (2 points): Give the “(ε, δ) definition” of continuity of a function \( f : (a, b) \to \mathbb{R} \) at a point \( c \in (a, b) \), and using the definition prove that if \( f : (0, 1) \to \mathbb{R} \) is continuous at a point \( c \in (0, 1) \) and if \( f(c) = 1 \) then there is \( \delta > 0 \) such that \( f(x) > \frac{1}{2} \) for all \( x \in (c - \delta, c + \delta) \).

5(b) (3 points): Prove that the function \( f(x, y) = 1 - 2x - y + 4x^2 + 4xy + 2y^2 + 3xy \sin xy \) has a critical point at \( (x, y) = \left( \frac{1}{4}, 0 \right) \) and that \( f \) has a strict local minimum there.
6(a) (2 points): Find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by the vectors $v_1 = (1, 1, 0, 0)^T$, $v_2 = (0, 1, 1, 0)^T$, $v_3 = (0, 0, 1, 1)^T$.

(b) (3 points): Find the set of all solutions of the inhomogeneous system $Ax = y$ where

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 3 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 \\ 4 \\ 1 \\ -1 \end{pmatrix}$$

(Give your answer as an affine space.)
7(a) (2 points): Find all eigenvalues and corresponding eigenvectors for the matrix

\[ A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}. \]

7(b) (3 points): Show that the system of two non-linear equations

\[
\begin{align*}
(x^2 - y^2)y + 7x &= 1 \\
(x^2 - y^2)x + 5y &= 1
\end{align*}
\]

has a solution \((x, y)\) with \(x^2 + y^2 < 1\).

Hint: Define \(f(x, y) = \left( \frac{1}{4}(1 - (x^2 - y^2)y), \frac{1}{5}(1 - (x^2 - y^2)x) \right)\) and start by proving that \(f\) is a contraction mapping \(D \to D\), where \(D = \{(x, y) : x^2 + y^2 \leq 1\}\).
8(a) (2 points): Let $A$ be an $n \times n$ real matrix $(a_{ij})$. Define the adjoint matrix $\text{adj} A$ and give the proof that $A \text{adj} A = (\det A) I$.

8(b) (3 points): Show that $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + 4y^2 + z^2 = 1\}$ is a 2-dimensional $C^1$ manifold and find a point $a \in S$ at which the function $f(x, y, z) = xyz$ takes its maximum.

Note: You should begin by discussing the existence of such a point $a \in S$. 