Unless otherwise indicated, you can use results covered in lecture and homework, provided they are clearly stated.

If necessary, continue solutions on backs of pages
Note: work sheets are provided for your convenience, but will not be graded

Name (Print Clearly): ____________________________

I understand and accept the provisions of the honor code (Signed) ____________________________
1 (a) (3 points): Give the definition of “lim \(a_n = \ell\),” where \(\{a_n\}_{n=1,2,...}\) is a given sequence in \(\mathbb{R}\) and \(\ell \in \mathbb{R}\), and use your definition to prove \(\ell \geq 0\), assuming that the limit \(\ell\) exists and that \(a_n \geq 0 \ \forall n\).

**Solution:** \(\lim a_n = \ell\) means that for each \(\varepsilon > 0\) there is \(N\) such that \(|a_n - \ell| < \varepsilon\) for all \(n \geq N\). This says \(\ell - \varepsilon < a_n < \ell + \varepsilon\) for all \(n \geq N\). Now if \(\ell < 0\) then we can take \(\varepsilon = -\ell\), in which case the above implies \(\exists N\) such that \(a_n < \ell - \ell = 0\) for all \(n \geq N\), contradicting the fact that \(a_n \geq 0 \ \forall n\).

(b) (3 points): Suppose that \(S\) is a non-empty subset of \(\mathbb{R}\) which is bounded above, and let \(\alpha = \sup S\).

(i) Prove that for each \(\varepsilon > 0\) there is \(x \in S\) with \(x > \alpha - \varepsilon\).

(ii) Prove that there is a sequence \(\{x_n\}_{n=1,2,...}\) with \(x_n \in S\) for each \(n\) and \(\lim x_n = \alpha\).

**Solution (i):** If this fails for any \(\varepsilon > 0\), then \(\alpha - \varepsilon\) would be an upper bound for \(S\), contradicting the fact that \(\alpha\) is the least upper bound.

**Solution (ii):** For each \(n = 1,2,\ldots\) we can use (i) with \(\varepsilon = 1/n\), thus showing that there is \(x_n \in S\) with \(x_n > \alpha - 1/n\). Then \(\alpha - 1/n \leq x_n \leq \alpha\) and so the Sandwich Theorem gives \(\lim x_n = \alpha\).
2 (a) (3 points): Suppose $a, b$ are distinct vectors in $\mathbb{R}^n$.

(i) Give the definition of “the line $\ell$ through $a$ parallel to $b - a$,” and find the vector $v \in \ell$ which is equi-distant from $a, b$ (i.e. $\|v - a\| = \|v - b\|$).

(ii) If $v$ is as in (i) and $\|a\| = \|b\|$, prove $v \cdot (b - a) = 0$.

Solution (i): The line $\ell$ through $a$ parallel to $b - a$ is defined by $\ell = \{a + t(b - a) : t \in \mathbb{R}\}$. We want the mid-point of the part of the line joining $a$ to $b$ and this intuitively should be given by taking $t = \frac{1}{2}$, i.e. $v = a + \frac{1}{2}(b - a) = \frac{1}{2}(a + b)$. To check that this works, we calculate $v - a = \frac{1}{2}(b - a)$, whereas $v - b = \frac{1}{2}(a - b) = -\frac{1}{2}(b - a)$, so indeed $\|v - a\| = \|v - b\|$. 

Solution (ii): $v \cdot (b - a) = \frac{1}{2}(b + a) \cdot (b - a) = \frac{1}{2}(b \cdot b - a \cdot a + a \cdot b - b \cdot a) = \frac{1}{2}(\|b\|^2 - \|a\|^2) = 0$.

(b) (3 points): Prove that $2|x \cdot y| \|x\|^2 \leq \|x\|^6 + \|y\|^2$ for all vectors $x, y \in \mathbb{R}^n$.

Solution: The Cauchy-Schwarz inequality says $|x \cdot y| \leq \|x\| \|y\|$, so $\|x\|^6 + \|y\|^2 - 2|x \cdot y| \|x\|^2 \geq \|x\|^6 + \|y\|^2 - 2\|x\|^3\|y\| = (\|x\|^3 - \|y\|)^2 \geq 0$. 


3 (a) (4 points): Suppose

\[ A = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 2 & 1 & 1 & 3 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 1 & 1 & 2 & 0 & 2 \end{pmatrix} \]

Find (i) a basis for the null space \( N(A) \) of \( A \) (show all row operations!), and (ii) a basis for the column space \( C(A) \).

Make sure you justify your results by referring to the appropriate results from lecture.

Solution: We compute the reduced row echelon form of \( A \) as follows:

\[
\begin{align*}
\begin{pmatrix}
1 & 0 & 1 & 1 & 1 \\
2 & 1 & 1 & 3 & 1 \\
1 & 1 & 2 & 0 & 2 \\
0 & 0 & 1 & -1 & 1 
\end{pmatrix}
& \xrightarrow{r_3 \leftrightarrow r_4} \begin{pmatrix}
1 & 0 & 1 & 1 & 1 \\
0 & 1 & -1 & 1 & -1 \\
0 & 1 & -1 & 1 & -1 \\
0 & 0 & 1 & -1 & 1 
\end{pmatrix} \\
& \xrightarrow{r_3 \mapsto r_3/2} \begin{pmatrix}
1 & 0 & 1 & 1 & 1 \\
0 & 1 & -1 & 1 & -1 \\
0 & 0 & 1 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 
\end{pmatrix} \\
& \xrightarrow{r_4 \mapsto r_4 - 3/2 r_3} \begin{pmatrix}
1 & 0 & 1 & 1 & 1 \\
0 & 1 & -1 & 1 & -1 \\
0 & 0 & 1 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 
\end{pmatrix} \\
& \xrightarrow{r_2 \mapsto r_2 - r_1} \begin{pmatrix}
1 & 0 & 0 & 0 & 2 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 
\end{pmatrix}
\end{align*}
\]

Thus \( \vec{x} \) is a solution of \( A \vec{x} = 0 \iff x_3 = x_4 - x_5, x_2 = 0, x_1 = -2x_4 \iff \vec{x} = (-2x_4, 0, x_4 - x_5, x_4, x_5)^T = x_4(-2, 0, 1, 1, 0)^T + x_5(0, 0, -1, 0, 1)^T \), where \( x_4, x_5 \) are arbitrary reals, so the null space is the subspace spanned by \((-2, 0, 1, 1, 0)^T \) and \((0, 0, -1, 0, 1)^T \). Since \((-2, 0, 1, 1, 0)^T \) and \((0, 0, -1, 0, 1)^T \) are l.i. (which can be justified either by a direct check or by the fact that we are following the general method of lecture, which was shown always to yield l.i. vectors and hence a basis for the null space), this is a 2-dimensional space and \((-2, 0, 1, 1, 0)^T \) and \((0, 0, -1, 0, 1)^T \) are a basis.

(ii) In lecture we proved that if \( j_1, \ldots, j_Q \) are the column numbers of the pivot columns of \( \text{rref}(A) \) then the columns \( \alpha_{j_1}, \ldots, \alpha_{j_Q} \) of \( A \) are a basis for \( C(A) \). In this case we have \( Q = 3 \) and \( j_1, j_2, j_3 = 1, 2, 3 \) respectively, so the first 3 cols. of \( A \) are a basis for \( C(A) \).

3 (b) (3 points): Suppose \( V \subset \mathbb{R}^n \) is a non-trivial subspace of dimension \( k \). Give the proof that any \( k \) vectors \( \vec{v}_1, \ldots, \vec{v}_k \in V \) which span \( V \) (i.e. \( V = \text{span}\{\vec{v}_1, \ldots, \vec{v}_k\} \)) must automatically be a basis for \( V \).

Solution: Since \( \vec{v}_1, \ldots, \vec{v}_k \) are given to span \( V \), we just have to show they are l.i. Suppose on the contrary that they are l.d. Then from lecture at least one of them, say \( \vec{v}_j \), is a linear combination of the others. Thus \( \vec{v}_j = \sum_{i \neq j} c_i \vec{v}_i \) for some constants \( c_i, i \neq j \). But then any linear combination of \( \vec{v}_1, \ldots, \vec{v}_k \) can be rewritten as a linear combination of \( \vec{v}_i, i \neq j \). Then \( V = \text{span}\{\vec{v}_1, \ldots, \vec{v}_k\} = \text{span}\{\vec{v}_j : i \neq j\} \). But then a basis \( \vec{w}_1, \ldots, \vec{w}_k \) for \( V \) would consist of \( k \) l.i. vectors in the span of the \( k - 1 \) vectors \( \vec{w}_i, i \neq j \), contradicting the linear dependence lemma.
4 (a) (3 points): Suppose $A$ is an $m \times n$ matrix and $b \in \mathbb{R}^m$. (i) Give the proof that $Ax = b$ has at least one solution $x \in \mathbb{R}^n \iff b \in C(A)$, and (ii) In case $m = n$ and $N(A) = \{0\}$, prove that $Ax = b$ has a solution for each $b \in \mathbb{R}^n$.

Hint for (ii): Use the rank/nullity theorem.

Solution: (i) As we checked in lecture, $Ax = \sum_{j=1}^{n} x_j \alpha_j$, where $\alpha_j$ is the $j$'th column of $A$, so $\exists x = (x_1, \ldots, x_n)^T$ with $Ax = b \iff \sum_{j=1}^{n} x_j \alpha_j = b$. Thus there is a solution of $Ax = b$ if and only if some linear combination of the $\alpha_j$ is equal to $b$, i.e. if and only if $b \in \text{span}\{\alpha_1, \ldots, \alpha_n\} = C(A)$.

(ii) If $N(A) = \{0\}$ then the rank/nullity theorem tells us that the dimension of $C(A) = n$. That is $C(A)$ is a subspace of $\mathbb{R}^n$ of dimension $n$ and hence it must be all of $\mathbb{R}^n$ because by a theorem of lecture any $k$ l.i. vectors in a $k$-dimensional subspace of $\mathbb{R}^n$ must be a basis for that subspace. Thus $C(A) = \mathbb{R}^n$ and hence by part (i) $Ax = b$ has a solution for all $b \in \mathbb{R}^n$.

4 (b) (2 points): If $V$ is a subspace of $\mathbb{R}^n$, give the definition of $V^\perp$. Prove (i) that $V^\perp$ is a subspace, and (ii) that $V \cap V^\perp = \{0\}$.

Solution: $V^\perp$ is the set of all vectors $u \in \mathbb{R}^n$ such that $u \cdot v = 0$ for every $v \in V$.

(i) First note that (a) trivially $0 \in V^\perp$, and (b) $x, y \in V^\perp \Rightarrow x \cdot (x + y) = x \cdot x + x \cdot y = 0 + 0 = 0$ for each $x \in V$, so $x + y \in V^\perp$. Finally (c) $\lambda \in \mathbb{R}$ and $y \in V^\perp \Rightarrow (\lambda y) \cdot v = \lambda (y \cdot v) = \lambda.0 = 0$ for each $v \in V$, so $\lambda y \in V^\perp$. Thus $V^\perp$ has the required 3 properties, hence is a subspace.

(ii) $w \in V \cap V^\perp \Rightarrow w \in V^\perp \Rightarrow w \cdot v = 0 \forall v \in V$. But $w \in V$, so then $w \cdot w = \|w\|^2 = 0$, so $w = 0$. 