Unless otherwise indicated, you can use results covered in lecture and homework, provided they are clearly stated.

If necessary, continue solutions on backs of pages

Note: work sheets are provided for your convenience, but will not be graded

\[
\begin{array}{c|c}
\text{Q.1} & \\
\text{Q.2} & \\
\text{Q.3} & \\
\text{Q.4} & \\
\text{T/25} & \\
\end{array}
\]

Name (Print Clearly): ______________________________

I understand and accept the provisions of the honor code (Signed) ______________________________
1 (a) (3 points): (i) Give the $\varepsilon, N$ definition of “$\lim a_n = \ell$,” where $\{a_n\}_{n=1,2,...}$ is a given sequence in $\mathbb{R}$ and $\ell \in \mathbb{R}$, and (ii) use your definition to prove that if $\{a_n\}_{n=1,2,...}, \{b_n\}_{n=1,2,...}$ satisfy $\lim a_n = \ell$, $\lim b_n = m$, then $\lim(a_n - b_n) = \ell - m$.

Note for (ii): You may not use any of our limit theorems to prove (ii), only the definition of the limit, and properties of the reals.

Solution: $\lim a_n = \ell$ means that for each $\varepsilon > 0$ there is $N$ such that $|a_n - \ell| < \varepsilon$ for all $n \geq N$.

Now, suppose $\varepsilon > 0$. Since $\lim a_n = \ell$, applying the definition with $\varepsilon/2 > 0$ means that there is $N_1$ such that $n \geq N_1$ implies $|a_n - \ell| < \varepsilon/2$. Similarly, $\lim b_n = m$ means that there is $N_2$ such that $|b_n - m| < \varepsilon/2$ for $n \geq N_2$. Thus, for $n \geq N = \max(N_1, N_2),$

$$|(a_n - b_n) - (\ell - m)| = |(a_n - \ell) + (m - b_n)| \leq |a_n - \ell| + |m - b_n| < \varepsilon,$$

proving the conclusion.

1(b) (3 points): Suppose that $S$ is a bounded non-empty subset of $\mathbb{R}$ with the property that $x, y \in S, x < z < y$ imply that $z \in S$. Let $a = \inf S, b = \sup S$. Show that $S$ must be one of the intervals $(a, b), (a, b], [a, b), [a, b]$ (with only the last possibility if $a = b$).

Hint for (b): The conclusion is equivalent to $a < z < b$ implying that $z \in S$, together with $z \notin [a, b]$ implying $z \notin S$.

Solution: Suppose $a < z < b$. As $z > a$, $z$ is not a lower bound for $S$, i.e. there exists some $x \in S$ such that $x < z$. Similarly, as $z < b$, $z$ is not an upper bound for $S$ so there exists some $y \in S$ such that $z < y$. Thus, $x < z < y, x, y \in S$, so $z \in S$. Thus, $(a, b) \subset S$.

Now, if $z < a$ then $z \notin S$ since $a$ is a lower bound for $S$, and similarly if $z > b$ then $z \notin S$ since $b$ is an upper bound for $S$. Thus $(-\infty, a) \cup (b, +\infty) \subset S^c = \mathbb{R} \setminus S$.

As $\mathbb{R} = (-\infty, a) \cup \{a\} \cup (a, b) \cup \{b\} \cup (b, +\infty)$, the only question is whether $a$ and $b$ are in $S$; listing the four possibilities (only two if $a = b$) gives the four intervals (only one if $a = b$ as we assumed that $S$ was non-empty).
2(a) (3 points): (i) Give the definition of a collection \( v_1, \ldots, v_k \) of vectors in \( \mathbb{R}^n \) being linearly independent, and (ii) if \( v_1, \ldots, v_k \) are non-zero mutually orthogonal (i.e. \( v_i \cdot v_j = 0 \forall i \neq j \)) vectors in \( \mathbb{R}^n \), prove that \( v_1, \ldots, v_k \) are linearly independent.

Solution: A collection \( v_1, \ldots, v_k \) of vectors in \( \mathbb{R}^n \) is linearly independent if there is no non-trivial linear combination of them which is 0, i.e. \( \sum_{j=1}^{k} c_j v_j = 0 \) implies \( c_j = 0 \) for all \( j \). (ii) \( \sum_{j=1}^{k} c_j v_j = 0 \Rightarrow 0 = v_i \cdot (\sum_{j=1}^{k} c_j v_j) = \sum_{j=1}^{k} c_j v_i \cdot v_j = c_i \|v_i\|^2 \Rightarrow c_i = 0 \) for each \( i = 1, \ldots, k \), where we used the fact that \( v_i \cdot v_j = 0 \) if \( j \neq i \) and \( \|v_i\|^2 \neq 0 \) if \( j = i \).

2(b) (4 points): Suppose that \( V \) is a non-trivial subspace of \( \mathbb{R}^n \). Show that there is an orthogonal basis of \( V \), i.e. that there is a basis \( \{v_1, \ldots, v_k\} \) for \( V \) with \( v_i \cdot v_j = 0 \) if \( i \neq j \). (You may assume the result of part (a) even if you have not proved it.)

Hint for (b): As in the proof of the basis theorem, consider a maximum size set of non-zero mutually orthogonal vectors; you need to show along the way that this exists. Orthocomplements may be useful in proving the spanning property.

Solution: If \( \{v_1, \ldots, v_k\} \) is any collection of mutually orthogonal non-zero vectors, it is linearly independent by part (a), so by the linear dependence lemma, \( k \leq n \). Moreover, as \( V \) is non-trivial, there exists a non-zero vector \( v \) in it; then \( \{v\} \) is an orthogonal collection (a set with one non-zero element). Now let

\[ S = \{ k : \exists \{v_1, \ldots, v_k\} \text{ mutually orthogonal non-zero in } V \}. \]

Then \( S \) is a non-empty set of positive integers, bounded above by \( n \), thus it has a maximal element; let \( k = \max S \). Let \( \{v_1, \ldots, v_k\} \) be mutually orthogonal in \( V \); these exist by the very definition of \( k \). Now let \( W = \text{span}\{v_1, \ldots, v_k\} + V^\perp \), with \( V^\perp \) the orthocomplement of \( V \) in \( \mathbb{R}^n \). Then \( W \oplus W^\perp = \mathbb{R}^n \). If \( W^\perp \neq \{0\} \) then there exists \( \bar{x} \neq 0 \), \( \bar{x} \in W^\perp \), so \( \bar{x} \) is orthogonal to all \( v_j \), and \( \bar{x} \) is orthogonal to all elements of \( V^\perp \). The latter means \( \bar{x} \in (V^\perp)^\perp = V \). Then \( \{v_1, \ldots, v_k, \bar{x}\} \) are \( k + 1 \) mutually orthogonal non-zero vectors in \( V \), so \( k + 1 \in S \), which contradicts our choice of \( k \). Thus, \( W^\perp = \{0\} \), so \( W = \mathbb{R}^n \), i.e. \( \text{span}\{v_1, \ldots, v_k\} + V^\perp = \mathbb{R}^n \). Notice that the first summand is a subspace of \( V \), so any \( y \in V \) can be written as \( y + \bar{x} \), \( y \in \text{span}\{v_1, \ldots, v_k\} \subset V \), \( \bar{x} \in V^\perp \). Since there is a unique way of writing any element of \( \mathbb{R}^n \), in particular any element \( v \) of \( V \) as a vector in \( V \) plus one in \( V^\perp \), and as \( y = y + 0 \) is such a decomposition, we conclude that \( y = \bar{x} \in \text{span}\{v_1, \ldots, v_k\} \), so \( V = \text{span}\{v_1, \ldots, v_k\} \). Since \( \{v_1, \ldots, v_k\} \) are linearly independent, this means that they form a basis of \( V \), as claimed.

Alternative (simpler) argument using orthocomplements within \( V \): Find \( v_1, \ldots, v_k \) as above, but let \( W = \text{span}\{v_1, \ldots, v_k\} \). Now, \( V = W \oplus W^\perp \) with \( W^\perp = \{v \in V : v \cdot w = 0 \forall w \in W\} \) the orthocomplement of \( W \) in \( V \), so \( W = V \) if and only if \( W^\perp = \{0\} \). But if \( W^\perp \neq \{0\} \) then there is \( x \in W \) such that \( x \neq 0 \), and then \( \{v_1, \ldots, v_k, x\} \) are \( k + 1 \) mutually orthogonal non-zero vectors in \( V \), so \( k + 1 \in S \), which contradicts our choice of \( k \). So \( W^\perp = 0 \), and thus \( V = \text{span}\{v_1, \ldots, v_k\} \). Since \( \{v_1, \ldots, v_k\} \) are linearly independent, this means that they form a basis of \( V \), as claimed.
3(a) (3 points): Suppose $A$ is an $n \times n$ matrix and $b \in \mathbb{R}^n$. Show that if $C(A) = \mathbb{R}^n$ then that $Ax = b$ has a unique solution for each $b \in \mathbb{R}^n$. (You need to show both existence and uniqueness.)

Hint: Use the rank/nullity theorem.

Solution: For any $x \in \mathbb{R}^n$, $Ax = \sum_{j=1}^{n} x_j \alpha_j$, where $\alpha_j$ is the $j$'th column of $A$. Thus $\{Ax : x \in \mathbb{R}^n\} = \text{span}\{\alpha_1, \ldots, \alpha_n\} = C(A)$, so $b \in \{Ax : x \in \mathbb{R}^n\} \iff b \in C(A)$. Correspondingly, if $C(A) = \mathbb{R}^n$, then $Ax = b$ has a solution for all $b \in \mathbb{R}^n$.

If $C(A) = \mathbb{R}^n$ then the rank/nullity theorem tells us that the dimension of $N(A)$ is 0, i.e. $N(A) = \{0\}$. But, if $Ax = b$ and $Ay = b$ then $A(x - y) = Ax - Ay = b - b = 0$, so $x - y \in N(A)$. So $N(A) = \{0\}$ gives $x = y$, which is the desired uniqueness.

3(b) (3 points): Suppose that $V, W$ are subspaces of $\mathbb{R}^n$ and $V \subset W$. Show that if $\dim V = \dim W$ then $V = W$.

Solution: If $V$ is the trivial subspace of $\mathbb{R}^n$, then $\dim W = 0$ and thus $W$ is also the trivial subspace, completing the proof in this case.

If $V$ is not the trivial subspace, then $V$ has a basis $\{v_1, \ldots, v_k\}$. These are linearly independent and lie in $W$, thus by the basis theorem, applied to $W$, there exists a basis $\{w_1, \ldots, w_k, \ldots, w_m\}$ of $W$ with $m \geq k$ and with $w_j = v_j$ for $j \leq k$. In particular, $\dim W = m \geq k = \dim V$. Since we know $\dim V = \dim W$, we have $m = k$, i.e. $\{v_1, \ldots, v_k\}$ is a basis of $W$ as well. Thus, $W$ is the span of these vectors, i.e. $W = V$. 


4 (6 points): Find (i) $\text{rref } A$ (showing all row operations), (ii) a basis for the null space $N(A)$ and (iii) a basis for the column space of $A$, if

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 3 \\ 0 & 2 & 4 & 1 & -1 \\ -1 & 0 & 0 & 1 & 2 \end{pmatrix}$$

Solution: (i)

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 3 \\ 0 & 2 & 4 & 1 & -1 \\ -1 & 0 & 0 & 1 & 2 \end{pmatrix} \rightarrow r_3 \leftrightarrow r_3 + r_1 \begin{pmatrix} 1 & 1 & 0 & 0 & 3 \\ 0 & 2 & 4 & 1 & -1 \\ 0 & 1 & 0 & 1 & 5 \end{pmatrix} \rightarrow r_2 \leftrightarrow r_2/2 \begin{pmatrix} 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & 2 & 1/2 & -1/2 \\ 0 & 0 & 1 & 1/4 & -1/4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & 2 & 1/2 & -1/2 \\ 0 & 0 & -2 & 1/2 & 11/2 \end{pmatrix} \rightarrow r_3 \rightarrow r_3 - r_2 \begin{pmatrix} 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & 2 & 1/2 & -1/2 \\ 0 & 0 & -2 & 1/2 & 11/2 \end{pmatrix} \rightarrow r_1 \leftrightarrow r_1 + 2r_3 \begin{pmatrix} 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & 2 & 1/2 & -1/2 \\ 0 & 0 & 1 & 1/4 & -1/4 \end{pmatrix} \rightarrow r_2 \leftrightarrow r_2 - 2r_3 \begin{pmatrix} 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & 2 & 1/2 & -1/2 \\ 0 & 0 & 1 & 1/4 & -1/4 \end{pmatrix}$$

(ii)

$$\text{rref } A \mathbf{x} = 0 \iff \begin{pmatrix} x_1 = x_4 + 2x_5, x_2 = -x_4 - 5x_5, x_3 = \frac{1}{4}x_4 + \frac{11}{4}x_5 \end{pmatrix}$$

$$\iff \mathbf{x} = x_4(1, -1, \frac{1}{4}, 1, 0)^T + x_5(2, -5, \frac{11}{4}, 0, 1)^T$$

with $x_4, x_5$ arbitrary, so $N(A) = N(\text{rref } A) = \text{span}\{(1, -1, \frac{1}{4}, 1, 0)^T, (2, -5, \frac{11}{4}, 0, 1)^T\}$, and these two vectors are indeed linearly independent (by inspection of the last two components, or the general result from lecture), so they give a basis for $N(A)$.

(iii) The pivot columns of $\text{rref } A$ are the first, second and third columns, so from lecture a basis for $C(A)$ is obtained by taking the first, second and third columns of $A$; that is, a basis for $C(A)$ is $(1, 0, -1)^T, (1, 2, 0)^T, (0, 4, 0)^T$.