Unless otherwise indicated, you can use results covered in lecture or homework, provided they are clearly stated.

If necessary, continue solutions on backs of pages
Note: work sheets are provided for your convenience, but will not be graded

Q.1
Q.2
Q.3
Q.4
T/25

Name (Print Clearly): ________________________

I understand and accept the provisions of the honor code (Signed) ________________________
1(a) (3 points) State the chain rule for the composite function \( g \circ f \), where \( f : U \to V \) and \( g : V \to \mathbb{R}^p \), where \( U \subset \mathbb{R}^n \) and \( V \subset \mathbb{R}^m \) are open. Using the chain rule, or otherwise, prove that if \( g : \mathbb{R}^n \to \mathbb{R} \) is differentiable on \( \mathbb{R}^n \), if \( a, b \in \mathbb{R}^n \), and if \( h(t) = g(a + tb) \) for \( t \in \mathbb{R} \), then \( h'(0) \) exists, and find its value in terms of the components \( b_1, \ldots, b_n \) of \( b \) and the partial derivatives of \( g \) at \( a \).

1(b) (3 points.) (i) Give the definition of “\( U \) is open” and “\( C \) is closed” as applied to subsets \( U, C \subset \mathbb{R}^n \), and (ii) give the proof that \( \mathbb{R}^n \setminus U \) closed implies \( U \) open.

Note: In lecture we proved \( \mathbb{R}^n \setminus U \) closed \( \iff \) \( U \) open; in (ii) you are only being asked to prove “\( \Rightarrow \)”.
2(a) (3 points): Suppose $\delta > 0$ and $\sum_{n=0}^{\infty} a_n x^n, \sum_{n=0}^{\infty} b_n x^n$ are convergent power series in $(-\delta, \delta)$. Prove $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$ for each $x \in (-\delta, \delta)$ implies that $a_n = b_n$ for each $n = 0, 1, 2, \ldots$. 

Hint: Since we can take $c_n = a_n - b_n$, it suffices to prove $\sum_{n=0}^{\infty} c_n x^n = 0 \forall x \in (-\delta, \delta) \Rightarrow c_n = 0 \forall n = 0, 1, 2, \ldots$.

2(b) (3 points.) (i) Prove that the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is AC on all of $\mathbb{R}$.

(ii) If we define $\exp x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, prove that $\exp(x + t) = (\exp x)(\exp t)$.

Hint for (ii): For fixed $t$ let $f(x) = \exp(x + t)$ and $g(x) = (\exp t)(\exp x)$. Start by checking that $f^{(n)}(0) = g^{(n)}(0)$ for each $n = 0, 1, 2, \ldots$. 
3(a) (4 points.) Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $f(x, y) = \frac{1}{3}y^3 + xy + x^2$. Find the critical points (i.e. points where $\nabla_{\mathbb{R}^2} f = 0$) of $f$, and state whether each critical point is a local max, local min or neither. Make sure you justify all claims you make in your argument, either with a proof or by quoting the appropriate theorem from lecture.

3(b) (3 points.) Give the definition of “length of a curve $\gamma : [a, b] \rightarrow \mathbb{R}^n$.” Using any theorem from lecture that you need, find the length of $\gamma$ in case $n = 2$ and $\gamma(t) = (\sin t^2, \cos t^2), t \in [0, 2]$. 
4(a) (3 points.) Prove that $M = \{ \mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3^2 = 1 + x_1^2 + x_2^2 \}$ is a $C^1$ manifold, and find the tangent space $T_aM$ at the point $a = (2, 2, -3)$.

Note: You should give a basis for the tangent space.

4(b) (3 points.) (i) If $M$ is a $k$-dimensional $C^1$ submanifold of $\mathbb{R}^n$ ($n \geq 2$ and $1 \leq k \leq n - 1$ given), and $f : W \to \mathbb{R}$ is $C^1$ with $W \subset \mathbb{R}^n$ open, $W \supset M$, give the definition of “the tangential gradient $\nabla_M f$” and “a critical point of $f|_M$.” (ii) In the special case when $M = S^{n-1}$ (so $k = n - 1$ and $f$ is $C^1$ on an open set $W \supset S^{n-1}$) prove that $f|_{S^{n-1}}$ has at least two distinct critical points $a, b \in S^{n-1}$. 