Solutions

Unless otherwise indicated, you can use results covered in lecture or homework, provided they are clearly stated.

If necessary, continue solutions on backs of pages
Note: work sheets are provided for your convenience, but will not be graded

Q.1
Q.2
Q.3
Q.4
T/25

Name (Print Clearly): __________________________

I understand and accept the provisions of the honor code (Signed) ______________________
1(a) (3 points) (i) Give the definition of “$U$ is open” and “$C$ is closed” as applied to subsets $U, C \subset \mathbb{R}^n$, and (ii) give the proof that if $C_1, C_2$ are closed then $C_1 \cup C_2$ is closed, and if $U_1, U_2$ are open then $U_1 \cap U_2$ is open.

Note: In (ii), at least one of the two statements should be shown directly from the definition. You may either show the other directly, or by using an appropriate theorem.

Solution: (i) $U$ open means that for each $y \in U$ there is a $\rho > 0$ such that $B_\rho(y) \subset U$. $C$ closed means that $C$ contains all its limit points. That is if $\{x_k\}$ is a convergent sequence in $\mathbb{R}^n$ and $x_k \in C$ for each $k$, then $\lim x_k \in C$.

(ii) If $U_1, U_2$ are open and $a \in U_1 \cap U_2$ then $a \in U_j$, $j = 1, 2$, so by the openness of $U_j$ there is $\rho_j > 0$ such that $B_{\rho_j}(a) \subset U_j$. Let $\rho = \min(\rho_1, \rho_2) > 0$, so $B_\rho(a) \subset B_{\rho_j}(a) \subset U_j$ for $j = 1, 2$, and thus $B_\rho(a) \subset U_1 \cap U_2$, proving the openness of $U_1 \cap U_2$.

This implies that if $C_1, C_2$ are closed then $C_1 \cup C_2$ is closed, since by the theorem in lecture, a set is closed iff its complement is open. Thus, $(C_1 \cup C_2)^c = C_1^c \cap C_2^c$ shows that $(C_1 \cup C_2)^c$ is open by what we have shown, and thus $C_1 \cup C_2$ closed by the just stated theorem from lecture.

Alternatively, suppose $\{x_k\}$ is a sequence in $C_1 \cup C_2$ converging to some $x \in \mathbb{R}^n$. Then for each $k$, $x_k \in C_1$ or $x_k \in C_2$, so with $K_j$, $j = 1, 2$, the set of $k$ such that $x_k \in C_j$, $K_1 \cup K_2 = \mathbb{N}^+$, and thus one of $K_j$ is infinite. Let $i$ be such that $K_i$ is infinite, and consider the subsequence $\{x_{k_m}\}_{m=1}^\infty$ of $\{x_k\}$ containing exactly the elements of $\{x_k\}$ with $k \in K_i$. Then $\{x_{k_m}\}_{m=1}^\infty$ is a sequence in $C_i$, converges to $x$ (being a subsequence of sequence so converging), so by the closedness of $C_i$, $x \in C_i$, and thus $x \in C_1 \cup C_2$, showing the claimed closedness.

1(b) (3 points) (i) For $U \subset \mathbb{R}^n$ open, give the definition of $f : U \to \mathbb{R}^k$ being continuous, and (ii) show that if $f : U \to V \subset \mathbb{R}^k$ is continuous, $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^k$ are open, $g : V \to \mathbb{R}^m$ is continuous then $g \circ f$ defined by $(g \circ f)(x) = g(f(x))$, is continuous.

Solution: (i) $f$ is continuous if for all $a \in U$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x - a\| < \delta$, $x \in U$ implies $\|f(x) - f(a)\| < \varepsilon$.

(ii) Suppose $f, g$ are as stated, and let $a \in U$, so $f(a) \in V$. Let $\varepsilon > 0$. By the continuity of $g$ there exists $\delta' > 0$ such that $\|y - f(a)\| < \delta'$, $y \in V$ implies $\|g(y) - g(f(a))\| < \varepsilon$. But then by the definition of continuity of $f$, applied with $\delta'$, there exists $\delta > 0$ such that $\|x - a\| < \delta$, $x \in U$ implies $\|f(x) - f(a)\| < \delta'$. Thus, $\|x - a\| < \delta$, $x \in U$ implies $\|f(x) - f(a)\| < \delta'$ which in turn implies $\|g(f(x)) - g(f(a))\| < \varepsilon$, showing the claimed continuity.
2(a) (3 points.) Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be defined by \( f(x, y) = \frac{1}{4}(x^7 + y^7) - 64x - y \). Find all the critical points (i.e. points where \( \nabla_{\mathbb{R}^2} f = 0 \)) of \( f \), and discuss whether these points are local max/min for \( f \). Justify all claims either by proof or by using a theorem from lecture.

**Solution:** \( Df(x, y) = (x^6 - 64, y^6 - 1) \), so there are 4 critical points \((2, 1), (-2, -1), (2, -1), (-2, 1)\). The Hessian matrix at \((x, y)\) is \[ \begin{pmatrix} 6x^5 & 0 \\ 0 & 6y^5 \end{pmatrix} \] which gives positive definite quadratic form \( 6 \cdot 32 \lambda^2 + 6 \mu^2 \) at \((2, 1)\) and negative definite quadratic form \(-6 \cdot 32 \lambda^2 - 6 \mu^2 \) at \((-2, -1)\). Hence by the Second Derivative test from lecture (applicable because \( f \) is \( C^2 \), in fact \( C^\infty \)), we see that \( f \) has a local minimum at \((2, 1)\) and a local maximum at \((-2, -1)\). At the point \((-2, 1)\) the Hessian quadratic form is \(-6 \cdot 32 \lambda^2 + 6 \mu^2 \) which changes sign (has positive max on \( S^1 \) and a negative min on \( S^1 \)), and hence, as we proved in lecture/section, it is neither a local max nor a local min for \( f \). (Concretely, \( f(x, 1) \) has a local max at \(-2\), \( f(-2, y) \) has a local min at \( y = 1 \).) Similarly the point \((2, -1)\) is neither a local max nor a local min for \( f \).

2(b) (3 points.) Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be defined by \( f(x, y) = 1 + 3x^2 + y^6 + 4(x - 1)^4 \). Show that \( f \) is bounded below and it attains its minimum.

Note: you do not need to find where the minimum is attained. Hint: show first that if \(|x| \geq 3\) or \(|y| \geq 2\) then \( f(x, y) \geq 65 \). What is \( f(0, 0) \)?

**Solution:** Since all terms in the expression for \( f \) are squares of real numbers, we have \( f(x, y) \geq 1 \), so \( f \) is bounded below. Moreover, if \(|x| \geq 3\) then \(|x - 1| \geq |x| - 1 \geq 2\) (since \(|x| \leq |x - 1| + 1\) by the triangle inequality) so \( f(x, y) \geq 1 + 4 \cdot 16 = 65 \) (using that all other terms are \( \geq 0 \)). If \(|y| \geq 2\) then \( f(x, y) \geq 1 + 64 = 65 \) (again using that all other terms are \( \geq 0 \)). Thus, if \(|x| \geq 3\) or \(|y| \geq 2\) then \( f(x, y) \geq 65 \). On the other hand \( R = \{(x, y) : |x| \leq 3, |y| \leq 2\} \) is a closed and bounded subset of \( \mathbb{R}^2 \); it is bounded directly from the definition and closed because it is the intersection of the inverse images of the closed intervals \([-3, 3]\) resp. \([-2, 2]\) under the continuous maps \( g(x, y) = x \) and \( h(x, y) = y \), i.e. it is the intersection of two closed sets, thus closed. Correspondingly, by the theorem in lecture, \( R = \{(x, y) : |x| \leq 3, |y| \leq 2\} \) is compact, and as \( f \) is continuous, \( f|_R \) attains its minimum there, say at the point \((x_0, y_0)\). Note that as \( f(0, 0) = 1 + 4 = 5 \) and \((0, 0) \in R \), the minimum value \( f(x_0, y_0) \leq 5 < 65 \). Since \( f(x, y) \geq 65 \) when \((x, y) \notin R \), we conclude that the minimum of \( f \) over \( \mathbb{R}^2 \) (and not just \( R! \)) is indeed attained at \((x_0, y_0)\).
3(a) (3 points) Consider the power series $\sum_{n=1}^\infty \frac{x^n}{n}$. (i) Find its radius of convergence $\rho$. (ii) Let $f(x) = \sum_{n=1}^\infty \frac{x^n}{n}$, $|x| < \rho$. Show that $f'(x) = \frac{1}{1-x}$ for $|x| < \rho$.

Solution: (i) First, recall that the series $\sum_{n=1}^\infty \frac{1}{n}$ diverges, and this is just the power series evaluated at 1, so as a power series converges absolutely in $(-\rho, \rho)$, if $\rho$ is its radius of convergence, we must have $\rho \leq 1$. On the other hand, $|x^n/n| \leq |x^n|$, and $\sum_{n=1}^\infty |x^n|$ converges for $x$ with $|x| < 1$ (this being a geometric series with common ratio $|x|$), by the comparison theorem for series with non-negative terms (i.e. the convergence theorem for increasing sequences which are bounded above), $\sum_{n=1}^\infty |x^n/n|$ converges for $|x| < 1$, thus (absolute convergence implies convergence) $\sum_{n=1}^\infty \frac{x^n}{n}$ converges for $|x| < 1$. Hence the radius of convergence is $\geq 1$, so in summary $\rho = 1$.

(ii) By the theorem from class, a power series is infinitely differentiable within its radius of convergence with derivatives given by term-by-term differentiation. Hence, for $|x| < 1$, $f'(x)$ exists and is $f'(x) = \sum_{n=1}^\infty \frac{x^{n-1}}{n} = \sum_{n=0}^\infty x^n = \frac{1}{1-x}$, where the last equality comes from the sum of a convergent geometric series.

3(b) (3 points): (i) A sequence of functions $f_n : [a, b] \to \mathbb{R}$ converges uniformly to a function $f : [a, b] \to \mathbb{R}$ if for all $\varepsilon > 0$ there is $N \in \mathbb{N}^+$ such that $n \geq N$ implies that $\sup \{|f_n(x) - f(x)| : x \in [a, b]\} < \varepsilon$. Show that if $f_n$ are continuous and $f_n \to f$ uniformly then $f$ is continuous.

Hint: continuity of $f$ at $x$ requires given $x \in [a, b]$ and $\varepsilon > 0$ finding $\delta > 0$ with certain properties. Express $|f(y) - f(x)|$ in terms of $|f_n(y) - f_n(x)|$ and other expressions, and choose $n$ well.

Solution: Suppose $f_n$ continuous for all $n$, $f_n$ converges to $f$ uniformly. We need to show that $f$ is continuous. So let $x \in [a, b]$ and $\varepsilon > 0$. For any $y \in [a, b]$ and any $n$ we have

$$|f(y) - f(x)| \leq |f(y) - f_n(y)| + |f_n(y) - f_n(x)| + |f_n(x) - f(x)|$$

by the triangle inequality. So first choose $n$ such that the first and the last terms are guaranteed to be small, namely choose $n$ such that $\sup \{|f_n(x) - f(x)| : x \in [a, b]\} < \varepsilon/3$, we can do this due to the uniform convergence of $f_n$ to $f$. Then the first and last terms are $< \varepsilon/3$. Now for this $n$, using the continuity of $f_n$ at $x$, we get $\delta > 0$ such that $|y - x| < \delta$, $y \in [a, b]$ implies $|f_n(y) - f_n(x)| < \varepsilon/3$. Thus, $|y - x| < \delta$, $y \in [a, b]$ implies $|f(y) - f(x)| < \varepsilon$, which proves that $f$ is continuous, completing the proof.
4(a) (3 points.) (i) Give the definition of a curve \( \gamma : [a, b] \to \mathbb{R}^n \) having finite length, and for curves of finite length state the definition of the “length of a curve \( \gamma : [a, b] \to \mathbb{R}^n \),” (ii) Show that if \( \gamma : [a, b] \to \mathbb{R}^n \) has the property that there is a constant \( K > 0 \) such that \( \|\gamma(t) - \gamma(t')\| \leq K|t - t'| \) for \( t, t' \in [a, b] \) (one says \( \gamma \) is Lipschitz) then \( \gamma \) has finite length.

Solution: (i) A curve (a continuous map) \( \gamma : [a, b] \to \mathbb{R}^n \) has finite length if the set \( \{\ell(\gamma, \mathcal{P}) : \mathcal{P} \text{ partition of } [a, b]\} \) is bounded above, in which case \( \ell(\gamma) \) is the supremum of this set. Here \( \ell(\gamma, \mathcal{P}) = \sum_{j=1}^{N} \|\gamma(j) - \gamma(j-1)\| \), where \( \mathcal{P} \) is the partition \( a = t_0 < t_1 < \ldots < t_N = b \).

(ii) Suppose \( \gamma \) is as above. For any partition \( \mathcal{P} \) of \([a, b]\), say \( a = t_0 < t_1 < \ldots < t_N = b \), we have

\[
\ell(\gamma, \mathcal{P}) = \sum_{j=1}^{N} \|\gamma(j) - \gamma(j-1)\| \leq \sum_{j=1}^{N} K|t_j - t_{j-1}| = \sum_{j=1}^{N} K(t_j - t_{j-1}) = K(t_N - t_0) = K(b - a).
\]

Thus \( \{\ell(\gamma, \mathcal{P}) : \mathcal{P} \text{ partition of } [a, b]\} \) is bounded above, with \( K(b - a) \) being an upper bound, and correspondingly \( \gamma \) has finite length; in fact \( \ell(\gamma) \leq K(b - a) \).

4(b) (4 points.) (i) Show directly (without using the corollary of the implicit function theorem that we have not proved) that the set \( M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z^2 + 1 \} \) is a 2-dimensional \( C^1 \) manifold of \( \mathbb{R}^3 \), (ii) Find the tangent space of \( M \) at the point \((1, 1, 1)\), and give a basis for it.

Note: in fact, \( M \) is a \( C^\infty \) submanifold. You may use that \( \sqrt{t} : (0, \infty) \to (0, \infty) \) is \( C^\infty \).

Solution: (i) It is often convenient to use the notation \((x_1, x_2, x_3)\) below. By the equivalent statement to the definition discussed in section, for each point \( a \in M \), we need to find an open set \( V \subset \mathbb{R}^3 \) containing it, a permutation map \( P \), an open subset \( U \) of \( \mathbb{R}^2 \) and a \( C^1 \) map \( g \) such that \( V \cap M = P(G(U)) \), where \( G(x_1, x_2) = (x_1, x_2, g(x_1, x_2)) \). This is equivalent to saying that one of the coordinates \( x, y, z \) has to be expressed as a graph over an open subset \( U \) of the remaining coordinates’ plane. We can write \( M = M_{1,+} \cup M_{1,-} \cup M_{2,+} \cup M_{2,-} = \cup_{j=1,2} \cup \pm M_{j,\pm} \), where \( M_{j,\pm} = \{(x_1, x_2, x_3) \in M : \pm x_j > 0\} \). Indeed, certainly \( M_{j,\pm} \subset M \) for all \( j \) and \( \pm \), and conversely if \((x_1, x_2, x_3) \in M \) then \( x_1^2 + x_2^2 \geq 1 \), so at least one of \( x_1 \) and \( x_2 \) is nonzero, thus either positive or negative, so the point is in one of \( M_{j,\pm} \). Let \( V_{j,\pm} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \pm x_j > 0\} \); this is open being the inverse image of the open set \((0, \infty)\) under the map \( h_{j,\pm}(x_1, x_2, x_3) = \pm x_j \); then \( M \cap V_{j,\pm} = M_{j,\pm} \). Thus, it suffices to show that \( M_{j,\pm} \) is the image of a permuted graph map. For the sake of definiteness, consider \( M_{1,+} \); all others are similar. Points in \( M_{1,+} \) satisfy \( x_1 > 0 \) and \( x_1^2 + x_2^2 = x_3^2 + 1 \), thus \( x_2^2 < x_3^2 + 1 \), i.e. \( x_2 < \sqrt{x_3^2 + 1} \), and \( x_1 = \sqrt{x_3^2 + 1 - x_2^2} \), with all square roots being the non-negative square roots of non-negative reals. Now the set \( U_{1,+} = \{(x_2, x_3) : x_2^2 < x_3^2 + 1 \} \subset \mathbb{R}^2 \) is open, being the inverse image of \((0, \infty)\) under the continuous map \( h_{1,+}(x_2, x_3) = x_2^2 + 1 - x_2^2 \), and \( M_{1,+} \) is the permuted graph of the \( C^\infty \) function \( g_{1,+}(x_2, x_3) = \sqrt{x_3^2 + 1 - x_2^2} \) over \( U_{1,+} \), with the \( C^\infty \) statement due to being the composition of \( C^\infty \) functions, \( \sqrt{\cdot} \) defined over \((0, \infty)\), and a polynomial. This, together with completely analogous considerations for the other \( M_{j,\pm} \) proves that \( M \) is a 2-dimensional \( C^\infty \) submanifold of \( \mathbb{R}^3 \).

(ii) Notice that \((1, 1, 1) \in M_{1,+} \), so by the theorem in lecture the tangent space to \( M \) at \((1, 1, 1)\) is the span of the partial derivatives of the graph map, with the latter being linearly independent and thus forming a basis. Concretely, the permuted graph map is \( G(x_2, x_3) = (\sqrt{x_3^2 + 1 - x_2^2}, x_2, x_3) \), \((x_2, x_3) \in U_{1,+} \), so a basis of the tangent space at \( G(x_2, x_3) \) is given by

\[
(-x_2/\sqrt{x_3^2 + 1 - x_2^2}, 1, 0)^T, (x_3/\sqrt{x_3^2 + 1 - x_2^2}, 0, 1)^T,
\]

i.e. at \((1, 1, 1)\) (corresponding to \( G(1, 1) \)) by \((-1, 1, 0)^T, (1, 0, 1)^T\). Note that these vectors are indeed orthogonal to the gradient of \( f(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2 - 1 \), which is \( \nabla f = (2x_1, 2x_2, -2x_3)^T \), i.e. is \((1, 1, -1)^T\) at \((1, 1, 1)\), thus their span (being 2-dimensional) is exactly the orthocomplement of the span of \( \nabla f \).