Mathematics Department Stanford University
Math 51H Second Mid-Term, November 10, 2015

75 MINUTES

Unless otherwise indicated, you can use results covered in lecture or homework, provided they are clearly stated.

If necessary, continue solutions on backs of pages
Note: work sheets are provided for your convenience, but will not be graded

Q.1
Q.2
Q.3
Q.4
T/25

Name (Print Clearly): ____________________________

I understand and accept the provisions of the honor code (Signed) ____________________________
1(a) (3 points.) (i) Give the definition of “U is open” and “C is closed” as applied to subsets $U, C \subset \mathbb{R}^n$, and (ii) give the proof that if $C_1, C_2$ are closed then $C_1 \cup C_2$ is closed, and if $U_1, U_2$ are open then $U_1 \cap U_2$ is open.

Note: In (ii), at least one of the two statements should be shown directly from the definition. You may either show the other directly, or by using an appropriate theorem.

1(b) (3 points) (i) For $U \subset \mathbb{R}^n$ open, give the definition of $f : U \to \mathbb{R}^k$ being continuous, and (ii) show that if $f : U \to V \subset \mathbb{R}^k$ is continuous, $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^k$ are open, $g : V \to \mathbb{R}^m$ is continuous then $g \circ f$, defined by $(g \circ f)(x) = g(f(x))$, is continuous.
2(a) (3 points.) Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be defined by \( f(x, y) = \frac{1}{4}(x^7 + y^7) - 64x - y \). Find all the critical points (i.e. points where \( \nabla f = 0 \)) of \( f \), and discuss whether these points are local max/min for \( f \). Justify all claims either with proof or by using a theorem from lecture.

2(b) (3 points.) Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be defined by \( f(x, y) = 1 + 3x^2 + y^6 + 4(x - 1)^4 \). Show that \( f \) is bounded below and it attains its minimum.

Note: you do not need to find where the minimum is attained. Hint: show first that if \(|x| \geq 3\) or \(|y| \geq 2\) then \( f(x, y) \geq 65 \). What is \( f(0, 0) \)?
3(a) (3 points) Consider the power series \( \sum_{n=1}^{\infty} \frac{x^n}{n} \). (i) Find its radius of convergence \( \rho \). (ii) Let \( f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \), \( |x| < \rho \). Show that \( f'(x) = \frac{1}{1-x} \) for \( |x| < \rho \).

3(b) (3 points): (i) A sequence of functions \( f_n : [a, b] \to \mathbb{R} \) converges uniformly to a function \( f : [a, b] \to \mathbb{R} \) if for all \( \varepsilon > 0 \) there is \( N \in \mathbb{N}^+ \) such that \( n \geq N \) implies that sup\{\( |f_n(x) - f(x)| : x \in [a, b] \)\} < \( \varepsilon \). Show that if \( f_n \) are continuous and \( f_n \to f \) uniformly then \( f \) is continuous.

Hint: continuity of \( f \) at \( x \) requires given \( x \in [a, b] \) and \( \varepsilon > 0 \) finding \( \delta > 0 \) with certain properties. Express \( |f(y) - f(x)| \) in terms of \( |f_n(y) - f_n(x)| \) and other expressions, and choose \( n \) well.
4(a) (3 points.) (i) Give the definition of a curve \( \gamma : [a, b] \to \mathbb{R}^n \) having finite length, and for curves of finite length state the definition of the “length of a curve \( \gamma : [a, b] \to \mathbb{R}^n \).” (ii) Show that if \( \gamma : [a, b] \to \mathbb{R}^n \) has the property that there is a constant \( K > 0 \) such that \( \| \gamma(t) - \gamma(t') \| \leq K |t - t'| \) for \( t, t' \in [a, b] \) (one says \( \gamma \) is Lipschitz) then \( \gamma \) has finite length.

4(b) (4 points.) (i) Show directly (without using the corollary of the implicit function theorem that we have not proved) that the set \( M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z^2 + 1\} \) is a 2-dimensional \( C^1 \) submanifold of \( \mathbb{R}^3 \). (ii) Find the tangent space of \( M \) at the point \((1, 1, 1)\), and give a basis for it.

Note: in fact, \( M \) is a \( C^\infty \) submanifold. You may use that \( \sqrt{\cdot} : (0, \infty) \to (0, \infty) \) is \( C^\infty \).