Chapter 7
Interest Rate Forwards and Futures
Bond Basics

• U.S. Treasury
  – Bills (<1 year), no coupons, sell at discount
  – Notes (1–10 years), Bonds (10–30 years), coupons, sell at par
  – STRIPS: claim to a single coupon or principal, zero-coupon
Bond Basics (cont’d)

- Notation
  - $r_t(t_1,t_2)$: interest rate from time $t_1$ to $t_2$ prevailing at time $t$
  - $P_{to}(t_1,t_2)$: price of a bond quoted at $t = t_0$ to be purchased at $t = t_1$ maturing at $t = t_2$
  - Yield to maturity: percentage increase in dollars earned from the bond
• Zero-coupon bonds make a single payment at maturity

Table 7.1 Five ways to present equivalent information about default free interest rates. All rates but those in the last column are effective annual rates.

<table>
<thead>
<tr>
<th>Years to Maturity</th>
<th>Zero-Coupon Bond Yield</th>
<th>Zero-Coupon Bond Price</th>
<th>One-Year Implied Forward Rate</th>
<th>Par Coupon</th>
<th>Continuously Compounded Zero Yield</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6.00%</td>
<td>0.943396</td>
<td>6.00000%</td>
<td>6.00000%</td>
<td>5.82689%</td>
</tr>
<tr>
<td>2</td>
<td>6.50</td>
<td>0.881659</td>
<td>7.00236</td>
<td>6.48423</td>
<td>6.29748</td>
</tr>
<tr>
<td>3</td>
<td>7.00</td>
<td>0.816298</td>
<td>8.00705</td>
<td>6.95485</td>
<td>6.76586</td>
</tr>
</tbody>
</table>

– One year zero-coupon bond: \( P(0,1)=0.943396 \)
  • Pay \$0.943396 today to receive \$1 at \( t=1 \)
  • Yield to maturity (YTM) = \( 1/0.943396 - 1 = 0.06 = 6\% = r(0,1) \)

– Two year zero-coupon bond: \( P(0,2)=0.881659 \)
  • YTM=\( 1/0.881659 - 1=0.134225=(1+r(0,2))^2=>r(0,2)=0.065=6.5\% \)
Bond Basics (cont’d)

- Zero-coupon bond price that pays $C_t$ at $t$:
  \[ P(0,t) = \frac{C_t}{[1 + r(0,t)]^t} \]

- Yield curve: graph of annualized bond yields against time

- Implied forward rates
  - Suppose current one-year rate $r(0,1)$ and two-year rate $r(0,2)$
  - Current forward rate from year 1 to year 2, $r_0(1,2)$, must satisfy
    \[ [1 + r_0(0,1)][1 + r_0(1,2)] = [1 + r_0(0,2)]^2 \]
Figure 7.1  An investor investing for 2 years has a choice of buying a 2-year zero-coupon bond paying \([1 + r_0(0, 2)]^2\) or buying a 1-year bond paying \(1 + r_0(0, 1)\) for 1 year, and reinvesting the proceeds at the implied forward rate, \(r_0(1, 2)\) between years 1 and 2. The implied forward rate makes the investor indifferent between these alternatives.
Bond Basics (cont’d)

• In general

\[ [1 + r_0(t_1, t_2)]^{t_2-t_1} = \frac{[1 + r_0(0,t_2)]^{t_2}}{[1 + r_0(0,t_1)]^{t_1}} = \frac{P(0,t_1)}{P(0,t_2)} \]

• Example 7.1

– What are the implied forward rate \( r_0(2,3) \) and forward zero-coupon bond price \( P_0(2,3) \) from year 2 to year 3? (use Table 7.1)

\[ r_0(2,3) = \frac{P(0,2)}{P(0,3)} - 1 = \frac{0.881659}{0.816298} - 1 = 0.0800705 \]

\[ P_0(2,3) = \frac{P(0,3)}{P(0,2)} = \frac{0.816298}{0.881659} = 0.925865 \]
• Coupon bonds
  – The price at time of issue of \( t \) of a bond maturing at time \( T \) that pays \( n \) coupons of size \( c \) and maturity payment of $1 is
  \[
  B_t(t,T,c,n) = \sum_{i=1}^{n} c P_t(t,t_i) + P_t(t,T)
  \]
  where \( t_i = t + i(T - t)/n \)
  – For the bond to sell at par the coupon size must be
  \[
  c = \frac{1 - P_t(t,T)}{\sum_{i=1}^{n} P_t(t,t_i)}
  \]
Forward Rate Agreements

- FRAs are over-the-counter contracts that guarantee a borrowing or lending rate on a given notional principal amount.

- Can be settled at maturity (in arrears) or the initiation of the borrowing or lending transaction:
  - FRA settlement in arrears: \((r_{qrtly} - r_{FRA}) \times \text{notional principal}\)
  - At the time of borrowing: \(\text{notional principal} \times (r_{qrtly} - r_{FRA})/(1+r_{qrtly})\)

- FRAs can be synthetically replicated using zero-coupon bonds.
**Forward Rate Agreements (cont’d)**

Table 7.3  Investment strategy undertaken at time 0, resulting in net cash flows of −$1 on day t, and receiving the implied forward rate, 1 + r(t, t + s) at t + s. This synthetically creates the cash flows from entering into a forward rate agreement on day 0 to lend at day t.

<table>
<thead>
<tr>
<th>Transaction</th>
<th>Cash Flows</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>0</strong></td>
<td><strong>t</strong></td>
</tr>
<tr>
<td>Buy 1 + r_0(t, t + s) zeros</td>
<td>−P(0, t + s) × (1 + r_0(t, t + s))</td>
</tr>
<tr>
<td>maturing at t + s</td>
<td></td>
</tr>
<tr>
<td>Short 1 zero maturing at t</td>
<td>+P(0, t)</td>
</tr>
<tr>
<td>Total</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Forward Rate Agreements (cont’d)

**Table 7.4** Example of synthetic FRA. The transactions in this table are exactly those in Table 7.3, except that all bonds are sold at time $t$.

<table>
<thead>
<tr>
<th>Transaction</th>
<th>Cash Flows</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Transaction</strong></td>
<td>0</td>
</tr>
<tr>
<td><strong>Buy $1 + r_0(t, t + s)$ zeros</strong></td>
<td>$-P(0, t + s) \times [1 + r_0(t, t + s)]$</td>
</tr>
<tr>
<td>maturing at $t + s$</td>
<td></td>
</tr>
<tr>
<td>Short 1 zero maturing at $t$</td>
<td>$+P(0, t)$</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>0</td>
</tr>
</tbody>
</table>
Interest Rate Strips and Stacks

- Suppose we will borrow $100 million in 6 months for a period of 2 years by rolling over the total every 3 months.

\[ r_1 = ? \quad r_2 = ? \quad r_3 = ? \quad r_4 = ? \quad r_5 = ? \quad r_6 = ? \quad r_7 = ? \quad r_8 = ? \]

\[
\begin{array}{cccccccc}
0 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & 27 \\
\$100 & \$100 & \$100 & \$100 & \$100 & \$100 & \$100 & \$100 & \$100 \\
\end{array}
\]
Interest Rate Strips and Stacks (cont’d)

• Two alternatives to hedge the interest rate risk
  – Strip: eight separate $100 million FRAs for each 3-month period
  – Stack: enter 6-month FRA for ~$800 million. Each quarter enter into another FRA decreasing the total by ~$100 each time
  – Strip is the best alternative but requires the existence of FRA far into the future. Stack is more feasible but suffers from basis risk
Duration

- **Duration**
  - $\text{Change in price for a unit change in yield}

- **Modified Duration**
  - %. Change in price for a unit change in yield

- **Macaulay Duration**
  - Size-weighted average of time until payments

- \( y \): yield per period; to annualize divide by \# of payments per year

- \( B(y) \): bond price as a function of yield \( y \)

\[
\frac{\text{Change in bond price}}{\text{Unit change in yield}} = -\frac{1}{1 + y} \sum_{i=1}^{n} \frac{C_i}{(1 + y)^i}
\]

\[
\text{Duration} \times \frac{1}{B(y)} = \frac{1}{1 + y} \sum_{i=1}^{n} \left[ \frac{C_i}{(1 + y)^i} \times \frac{1}{B(y)} \right]
\]

\[
\text{Duration} \times \frac{1 + y}{B(y)} = \sum_{i=1}^{n} \left[ \frac{C_i}{(1 + y)^i} \times \frac{1}{B(y)} \right]
\]
Duration (cont’d)

- Example 7.4 & 7.5
  - 3-year zero-coupon bond with maturity value of $100
    - Bond price at YTM of 7.00%: $100/(1.0700^3)=$81.62979
    - Bond price at YTM of 7.01%: $100/(1.0701^3)=$81.60691
  - Duration:
    \[
    \Delta = \frac{-1}{1.07} \times 3 \times \frac{100}{1.07^3} = -228.87
    \]
    - For a basis point (0.01%) change: -$228.87/10,000=-$0.02289
  - Macaulay duration:
    \[
    \frac{-($228.87) \times 1.07}{81.62979} = 3.000
    \]

- Example 7.6
  - 3-year annual coupon (6.95485%) bond
    - Macaulay Duration:
      \[
      (1 \times \frac{0.0695485}{1.0695485}) + (2 \times \frac{0.0695485}{1.0695485^2}) + (3 \times \frac{1.0695485}{1.0695485^3}) = 2.80915
      \]
Duration (cont’d)

• What is the new bond price \( B(y+\varepsilon) \) given a small change \( \varepsilon \) in yield?

  – Rewrite the Macaulay duration

  \[
  D_{Mac} = - \frac{[B(y + \varepsilon) - B(y)]}{\varepsilon} \cdot \frac{1 + y}{B(y)}
  \]

  – And rearrange

  \[
  B(y + \varepsilon) = B(y) - \left[ D_{Mac} \times \frac{B(y)\varepsilon}{1 + y} \right]
  \]
Duration (cont’d)

- Example 7.7
  - Consider the 3-year zero-coupon bond with price $81.63 and yield 7%
  - What will be the price of the bond if the yield were to increase to 7.25%?
  - \( B(7.25\%) = \$81.63 - (3 \times 81.63 \times 0.0025 / 1.07) = \$81.058 \)
  - Using ordinary bond pricing: \( B(7.25\%) = \$100 / (1.0725)^3 = \$81.060 \)

- The formula is only approximate due to the bond’s convexity
Duration (cont’d)

• Duration matching
  – Suppose we own a bond with time to maturity \( t_1 \), price \( B_1 \), and Macaulay duration \( D_1 \)
  – How many \( (N) \) of another bond with time to maturity \( t_2 \), price \( B_2 \), and Macaulay duration \( D_2 \) do we need to short to eliminate sensitivity to interest rate changes? The hedge ratio
  – The value of the resulting portfolio with duration zero is \( B_1 + NB_2 \)

\[
N = -\frac{D_1 B_1 (y_1)}{D_2 B_2 (y_2)} / (1 + y_1)
\]

• Example 7.8
  – We own a 7-year 6% annual coupon bond yielding 7%
  – Want to match its duration by shorting a 10-year, 8% bond yielding 7.5%
  – You can verify that \( B_1 = $94.611 \), \( B_2 = $103.432 \), \( D_1 = 5.882 \), and \( D_2 = 7.297 \)

\[
N = -\frac{5.882 \times 94.611 / 1.07}{7.297 \times 103.432 / 1.075} = -0.7409
\]
Duration (cont’d)

**Figure 7.2** Comparison of the value of three bond positions as a function of the yield to maturity: 2.718 10-year zero-coupon bonds, one 10-year bond paying a 10% annual coupon, and one 25-year bond paying a 10% coupon. The duration (D) and convexity (C) of each bond at a yield of 10% are in the legend.

- 0% coupon 10-year bond ($D = 9.09, C = 90.91$)
- 10% coupon 10-year bond ($D = 6.14, C = 52.79$)
- 10% coupon 25-year bond ($D = 9.08, C = 139.58$)
Treasury Bond/Note Futures

• Contract Specifications

Figure 7.3
Specifications for the Treasury-note futures contract.

<table>
<thead>
<tr>
<th>Specification</th>
<th>Details</th>
</tr>
</thead>
<tbody>
<tr>
<td>Where traded</td>
<td>CBOT</td>
</tr>
<tr>
<td>Underlying</td>
<td>6% 10-year Treasury note</td>
</tr>
<tr>
<td>Size</td>
<td>$100,000 Treasury note</td>
</tr>
<tr>
<td>Months</td>
<td>Mar, Jun, Sep, Dec, out 15 months</td>
</tr>
<tr>
<td>Trading ends</td>
<td>Seventh business day preceding last business day of month. Delivery until last business day of month.</td>
</tr>
<tr>
<td>Delivery</td>
<td>Physical T-note with at least 6.5 years to maturity and not more than 10 years to maturity. Price paid to the short for notes with other than 6% coupon is determined by multiplying futures price by a conversion factor. The conversion factor is the price of the delivered note ($1 par value) to yield 6%. Settlement until last business day of the month.</td>
</tr>
</tbody>
</table>
Treasury Bond/Note Futures (cont’d)

- Long T-note futures position is an obligation to buy a 6% bond with maturity between 6.5 and 10 years to maturity
- The short party is able to choose from various maturities and coupons: the “cheapest-to-deliver” bond
- In exchange for the delivery the long pays the short the “invoice price.”
  - Invoice price = (Futures price x conversion factor) + accrued interest

Table 7.5 Prices, yields, and the conversion factor for two bonds. The futures price is 97.583. The short would break even delivering the 8-year 7% bond and lose money delivering the 7-year 5% bond. Both bonds make semiannual coupon payments.
Repurchase Agreements

• A repurchase agreement or a repo entails selling a security with an agreement to buy it back at a fixed price

• The underlying security is held as collateral by the counterparty => A repo is collateralized borrowing

• Frequently used by securities dealers to finance inventory

• Speculators and hedge funds also use repos to finance their speculative positions

• A “haircut” is charged by the counterparty to account for credit risk
Chapter 7

Additional Art
**Table 7.2** Yields and prices on bills, notes, and bonds issued by the U.S. government, October 12, 2007. Bid and asked yields are reported for bills. Prices for notes, bonds, and strips are ask prices. Note and bond prices are quoted in 32nds (e.g., 100:14 is 100 and 14/32, or 100.4375.) Source: *Wall Street Journal* Online.

<table>
<thead>
<tr>
<th>Maturity Date</th>
<th>Bills</th>
<th>Notes and Bonds</th>
<th>STRIPS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Asked</td>
<td>Bid</td>
<td>Price</td>
</tr>
<tr>
<td>Oct 18, 2007</td>
<td>3.54</td>
<td>3.59</td>
<td></td>
</tr>
<tr>
<td>Oct 25, 2007</td>
<td>3.71</td>
<td>3.76</td>
<td></td>
</tr>
<tr>
<td>Nov 01, 2007</td>
<td>3.88</td>
<td>3.88</td>
<td></td>
</tr>
<tr>
<td>Nov 08, 2007</td>
<td>3.96</td>
<td>3.96</td>
<td></td>
</tr>
<tr>
<td>Nov 15, 2008</td>
<td>100:14</td>
<td>4.750</td>
<td>4.28</td>
</tr>
<tr>
<td>Nov 15, 2010</td>
<td>100:20</td>
<td>4.500</td>
<td>4.27</td>
</tr>
<tr>
<td>May 15, 2017</td>
<td>130:23</td>
<td>8.875</td>
<td>4.72</td>
</tr>
<tr>
<td>Feb 15, 2036</td>
<td>93:30</td>
<td>4.500</td>
<td>4.88</td>
</tr>
<tr>
<td>Feb 15, 2037</td>
<td>97:16</td>
<td>4.750</td>
<td>4.91</td>
</tr>
</tbody>
</table>
Equation 7.1

\[ P(0, n) = \frac{1}{[1 + r(0, n)]^n} \]
1 + r_0(1, 2) = \frac{[1 + r_0(0, 2)]^2}{1 + r_0(0, 1)}
Equation 7.3

\[
[1 + r_0(t_1, t_2)]^{t_2-t_1} = \frac{[1 + r_0(0, t_2)]^{t_2}}{[1 + r_0(0, t_1)]^{t_1}} = \frac{P(0, t_1)}{P(0, t_2)}
\]
Equation 7.4

\[ P_0(t_1, t_2) = \frac{1}{[1 + r_0(t_1, t_2)]^{t_2-t_1}} = \frac{[1 + r_0(0, t_1)]^{t_1}}{[1 + r_0(0, t_2)]^{t_2}} = \frac{P(0, t_2)}{P(0, t_1)} \]
Equation 7.5

\[ B_t(t, T, c, n) = \sum_{i=1}^{n} c P_t(t, t_i) + P_t(t, T) \]
Equation 7.6

\[ c = \frac{1 - P_t(t, T)}{\sum_{i=1}^{n} P_t(t, t_i)} \]
Equation 7.7

\[ B_t(t, T, c, n) = \sum_{i=1}^{n} \frac{c}{(1 + y_m)^i} + \frac{1}{(1 + y_m)^n} \]
Equation 7.8

\[
\text{Notional principal} \times \frac{(r_{\text{quarterly}} - r_{\text{FRA}})}{1 + r_{\text{quarterly}}}
\]
Equation 7.9

\[
\text{Change in bond price} \over \text{Unit change in yield} = - \sum_{i=1}^{n} \frac{i}{m} \frac{C/m}{(1 + y/m)^{i+1}} - \frac{n}{m} \frac{M}{(1 + y/m)^{n+1}} \\
= - \frac{1}{1 + y/m} \left[ \sum_{i=1}^{n} \frac{i}{m} \frac{C/m}{(1 + y/m)^i} + \frac{n}{m} \frac{M}{(1 + y/m)^n} \right]
\]
Equation 7.10

Modified duration = \[-\frac{\text{Change in bond price}}{\text{Unit change in yield}} \times \frac{1}{B(y)}\]

= \[\frac{1}{B(y)} \times \frac{1}{1 + y/m} \left[ \sum_{i=1}^{n} \frac{i}{m} \frac{C/m}{(1 + y/m)^i} + \frac{n}{m} \frac{M}{(1 + y/m)^n} \right]\]
Equation 7.11

Macaulay duration = \(-\frac{\text{Change in bond price}}{\text{Unit change in yield}} \times \frac{1 + y/m}{B(y)}\)

\[= \frac{1}{B(y)} \left[ \sum_{i=1}^{n} \frac{i}{m} \frac{C/m}{(1 + y/m)^i} + \frac{n}{m} \frac{M}{(1 + y/m)^n} \right] \]
Equation 7.12

\[ B(y + \epsilon) = B(y) - \left[ D \times B(y)\epsilon \right] = B(y) - \left[ D_{Mac}/(1 + y) \times B(y)\epsilon \right] \]
Equation 7.13

\[ N = - \frac{D_1 B_1(y_1)/(1 + y_1)}{D_2 B_2(y_2)/(1 + y_2)} \]
Equation 7.14

\[
\text{Convexity} = \frac{1}{B(y)} \left[ \sum_{i=1}^{n} \frac{i(i + 1)}{m^2} \frac{C/m}{(1 + y/m)^{i+2}} + \frac{n(n + 1)}{m^2} \frac{M}{(1 + y/m)^{n+2}} \right]
\]
Equation 7.15

\[ B(y + \epsilon) = B(y) - [D \times B(y) \times \epsilon] + 0.5 \times \text{Convexity} \times B(y) \times \epsilon^2 \]
Equation 7.16

\[
\text{Futures price} = \frac{\text{Price of cheapest to deliver}}{\text{Conversion factor for cheapest to deliver}}
\]
Equation 7.17

\[ B(y) = \sum_{i=1}^{n} \frac{C/2}{(1 + y/2)^{i-1} + d/d'} + \frac{M}{(1 + y/2)^{n-1} + d/d'} \]
Equation 7.18

\[
B(y) = \left(\frac{1}{1 + \frac{y}{2}}\right)^{d/d'} \left(\frac{C/2}{y/2} \left[1 - \frac{1}{(1 + \frac{y}{2})^{n-1}}\right] + \frac{M}{(1 + \frac{y}{2})^{n-1}}\right)
\]
Equation 7.19

\[ B(y) = \sum_{i=1}^{n} \frac{C/2}{(1 + y/2)^i} + \frac{M}{(1 + y/2)^n} \]
Table 7.6  Treasury-bill quotations.

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Days to Maturity</th>
<th>Ask Discount</th>
<th>Ask Yield</th>
</tr>
</thead>
<tbody>
<tr>
<td>February 13</td>
<td>43</td>
<td>3.65</td>
<td>3.72</td>
</tr>
<tr>
<td>December 18</td>
<td>351</td>
<td>3.87</td>
<td>4.04</td>
</tr>
</tbody>
</table>
Equation 7.20

\[ r_{be} = -\frac{2 \times \text{days}}{365} + 2\sqrt{\left(\frac{\text{days}}{365}\right)^2 - \left(\frac{2 \times \text{days}}{365} - 1\right)\left(1 - \frac{100}{P}\right)} \]