CHAPTER SUMMARY

To use effective bond portfolio strategies, it is necessary to understand the price volatility of bonds resulting from changes in interest rates. The purpose of this chapter is to explain the price volatility characteristics of a bond and to present several measures to quantify price volatility.

REVIEW OF THE PRICE-YIELD RELATIONSHIP FOR OPTION-FREE BONDS

An increase (decrease) in the required yield decreases (increases) the present value of its expected cash flows and therefore decreases (increases) the bond’s price. This relationship is not linear. The shape of the price-yield relationship for any option-free bond is referred to as a convex relationship.

PRICE VOLATILITY CHARACTERISTICS OF OPTION-FREE BONDS

There are four properties concerning the price volatility of an option-free bond. (i) Although the prices of all option-free bonds move in the opposite direction from the change in yield required, the percentage price change is not the same for all bonds. (ii) For very small changes in the yield required, the percentage price change for a given bond is roughly the same, whether the yield required increases or decreases. (iii) For large changes in the required yield, the percentage price change is not the same for an increase in the required yield as it is for a decrease in the required yield. (iv) For a given large change in basis points, the percentage price increase is greater than the percentage price decrease.

An explanation for these four properties of bond price volatility lies in the convex shape of the price-yield relationship.

Characteristics of a Bond that Affect Its Price Volatility

There are two characteristics of an option-free bond that determine its price volatility: coupon and term to maturity.

First, for a given term to maturity and initial yield, the price volatility of a bond is greater, the lower the coupon rate. This characteristic can be seen by comparing the 9%, 6%, and zero-coupon bonds with the same maturity. Second, for a given coupon rate and initial yield, the longer the term to maturity, the greater the price volatility.

Effects of Yield to Maturity

A bond trading at a higher yield to maturity will have lower price volatility. An implication of this is that for a given change in yields, price volatility is greater when yield levels in the market are low, and price volatility is lower when yield levels are high.
MEASURES OF BOND PRICE VOLATILITY

Money managers, arbitrageurs, and traders need to have a way to measure a bond’s price volatility to implement hedging and trading strategies. Three measures that are commonly employed are price value of a basis point, yield value of a price change, and duration.

Price Value of a Basis Point

The price value of a basis point, also referred to as the dollar value of an 01, is the change in the price of the bond if the required yield changes by 1 basis point. Note that this measure of price volatility indicates dollar price volatility as opposed to percentage price volatility (price change as a percent of the initial price). Typically, the price value of a basis point is expressed as the absolute value of the change in price. Price volatility is the same for an increase or a decrease of 1 basis point in required yield.

Because this measure of price volatility is in terms of dollar price change, dividing the price value of a basis point by the initial price gives the percentage price change for a 1-basis-point change in yield.

Yield Value of a Price Change

Another measure of the price volatility of a bond used by investors is the change in the yield for a specified price change. This is estimated by first calculating the bond’s yield to maturity if the bond’s price is decreased by, say, $X$ dollars. Then the difference between the initial yield and the new yield is the yield value of an $X$ dollar price change. The smaller this value, the greater the dollar price volatility, because it would take a smaller change in yield to produce a price change of $X$ dollars.

Duration

The Macaulay duration is one measure of the approximate change in price for a small change in yield.

\[
\text{Macaulay duration} = \frac{1C}{(1+y)^1} + \frac{2C}{(1+y)^2} + \ldots + \frac{nC}{(1+y)^n} + \frac{nM}{(1+y)^n}
\]

where \(P = \text{price of the bond, } C = \text{semiannual coupon interest (in dollars), } y = \text{one-half the yield to maturity or required yield, } n = \text{number of semiannual periods (number of years times 2), and } M = \text{maturity value (in dollars).}

Investors refer to the ratio of Macaulay duration to \(1 + y\) as the modified duration. The equation is:

\[
\text{modified duration} = \frac{\text{Macaulay duration}}{1 + y}.
\]
The modified duration is related to the approximate percentage change in price for a given change in yield as given by:

\[
\frac{dP}{dy} \frac{1}{P} = -\text{modified duration}.
\]

Because for all option-free bonds modified duration is positive, the above equation states that there is an inverse relationship between modified duration and the approximate percentage change in price for a given yield change. This is to be expected from the fundamental principle that bond prices move in the opposite direction of the change in interest rates.

In general, if the cash flows occur \( m \) times per year, the durations are adjusted by dividing by \( m \); that is,

\[
duration \text{ in years} = \frac{duration \text{ in } m \text{ periods per year}}{m}.
\]

We can derive an alternative formula that does not have the extensive calculations of the Macaulay duration and the modified duration. This is done by rewriting the price of a bond in terms of its two components: (i) the present value of an annuity, where the annuity is the sum of the coupon payments, and (ii) the present value of the par value. By taking the first derivative and dividing by \( P \), we obtain another formula for modified duration given by:

\[
\text{modified duration} = \frac{\frac{C}{y^2} \left[ 1 - \frac{1}{(1+y)^m} \right] + \frac{n(100 - C/y)}{(1+y)^{n+1}}}{P}
\]

where the price is expressed as a percentage of par value.

**Properties of Duration**

The modified duration and Macaulay duration of a coupon bond are less than the maturity. The Macaulay duration of a zero-coupon bond is equal to its maturity; a zero-coupon bond’s modified duration, however, is less than its maturity. Also, lower coupon rates generally have greater Macaulay and modified bond durations.

There is a consistency between the properties of bond price volatility and the properties of modified duration. For example, a property of modified duration is that, ceteris paribus, bonds with longer the maturity will have greater modified durations. Also, generally the lower the coupon rate, the greater the modified duration. Thus, greater modified durations will have greater the price volatility. As we noted earlier, all other factors constant, the higher the yield level, the lower the price volatility. The same property holds for modified duration.

**Approximating the Percentage Price Change**

58
The following equation can be used to approximate the percentage price change for a given change in required yield:

\[ \frac{dP}{P} = -\text{modified duration} \, (dy). \]

We can use this equation to provide an interpretation of modified duration. Suppose that the yield on any bond changes by 100 basis points. Then, substituting 100 basis points (0.01) for \( dy \) into the above equation, we get:

\[ \frac{dP}{P} = -\text{modified duration} \, (0.01) = -\text{modified duration} \, (\%). \]

Thus, modified duration can be interpreted as the approximate percentage change in price for a 100-basis-point change in yield.

**Approximating the Dollar Price Change**

Modified duration is a proxy for the percentage change in price. Investors also like to know the dollar price volatility of a bond. For small changes in the required yield, the following equation does a good job in estimating the change in price:

\[ dP = -(\text{dollar duration})(dy). \]

When there are large movements in the required yield, dollar duration or modified duration is not adequate to approximate the price reaction. Duration will overestimate the price change when the required yield rises, thereby underestimating the new price. When the required yield falls, duration will underestimate the price change and thereby underestimate the new price.

**Spread Duration**

Market participants compute a measure called *spread duration*. This measure is used in two ways: for fixed bonds and floating-rate bonds.

A spread duration for a fixed-rate security is interpreted as the approximate change in the price of a fixed-rate bond for a 100-basis-point change in the spread.

**Portfolio Duration**

Thus far we have looked at the duration of an individual bond. The duration of a portfolio is simply the weighted average duration of the bonds in the portfolios.

Portfolio managers look at their interest rate exposure to a particular issue in terms of its contribution to portfolio duration. This measure is found by multiplying the weight of the issue in the portfolio by the duration of the individual issue given as:
contribution to portfolio duration = weight of issue in portfolio x duration of issue.

CONVEXITY

Because all the duration measures are only approximations for small changes in yield, they do not capture the effect of the convexity of a bond on its price performance when yields change by more than a small amount. The duration measure can be supplemented with an additional measure to capture the curvature or convexity of a bond.

Measuring Convexity

Duration (modified or dollar) attempts to estimate a convex relationship with a straight line (the tangent line). We can use the first two terms of a Taylor series to approximate the price change. We get the dollar convexity measure of the bond:

\[ \text{dollar convexity measure} = \frac{d^2 P}{dy^2}. \]

The approximate change in price due to convexity is:

\[ dP = (\text{dollar convexity measure})(dy)^2. \]

The percentage change in the price of the bond due to convexity or the convexity measure is:

\[ \text{convexity measure} = \frac{d^2 P}{dy^2} \cdot \frac{1}{P}. \]

The percentage price change due to convexity is:

\[ \frac{dP}{P} = \frac{1}{2} \left( \text{convexity measure} \right) (dy)^2. \]

In general, if the cash flows occur \( m \) times per year, convexity is adjusted to an annual figure as follows:

\[ \text{convexity measure in year} = \frac{\text{convexity measure in m period per year}}{m^2}. \]

Approximating Percentage Price Change Using Duration and Convexity Measures

Using duration and convexity measures together gives a better approximation of the actual price change for a large movement in the required yield.

Some Notes on Convexity
There are three points that should be kept in mind regarding a bond’s convexity and convexity measure. First, it is important to understand the distinction between the use of the term convexity, which refers to the general shape of the price-yield relationship, and the term convexity measure, which is related to the quantification of how the price of the bond will change when interest rates change. The second point has to do with how to interpret the convexity measure. The final point is that in practice different vendors of analytical systems and different writers compute the convexity measure in different ways by scaling the measure in different ways.

Value of Convexity

Generally, the market will take the greater convexity of bonds into account in pricing them. How much should the market want investors to pay up for convexity? If investors expect that market yields will change by very little—that is, they expect low interest rate volatility—investors should not be willing to pay much for convexity. In fact, if the market prices convexity high, investors with expectations of low interest rate volatility will probably want to “sell convexity.”

Properties of Convexity

All option-free bonds have the following convexity properties. First, as the required yield increases (decreases), the convexity of a bond decreases (increases). This property is referred to as positive convexity. Second, for a given yield and maturity, lower coupon rates will have greater convexity. Third, for a given yield and modified duration, lower coupon rates will have smaller convexity.

ADDITIONAL CONCERNS WHEN USING DURATION

Relying on duration as the sole measure of the price volatility of a bond may mislead investors. There are two other concerns about using duration that we should point out. First, in the derivation of the relationship between modified duration and bond price volatility, we assume that all cash flows for the bond are discounted at the same discount rate. Second, there is misapplication of duration to bonds with embedded options.

DON’T THINK OF DURATION AS A MEASURE OF TIME

Unfortunately, market participants often confuse the main purpose of duration by constantly referring to it as some measure of the weighted average life of a bond. This is because of the original use of duration by Macaulay.

APPROXIMATING A BOND’ S DURATION AND CONVEXITY MEASURE

When we understand that duration is related to percentage price change, a simple formula can be used to calculate the approximate duration of a bond or any other more complex derivative securities or options described throughout this book. All we are interested in is the percentage price change of a bond when interest rates change by a small amount. The equation is:
approximate duration = \frac{P_1 - P_0}{2(P_0)(\Delta y)}

where \Delta y is the change in yield used to calculate the new prices (in decimal form). What the formula is measuring is the average percentage price change (relative to the initial price) per 1-basis-point change in yield. It is important to emphasize here that duration is a by-product of a pricing model. If the pricing model is poor, the resulting duration estimate is poor.

The convexity measure of any bond can be approximated using the following formula:

approximate convexity measure = \frac{P_1 + P_2 - 2P_0}{P_0(\Delta y)^2}.

Duration of an Inverse Floater

The duration of an inverse floater is a multiple of the duration of the collateral from which it is created. Assuming that the duration of the floater is close to zero, it can be shown that the duration of an inverse floater is as follows:

duration of an inverse floater = (1 + L)(duration of collateral) X \frac{\text{collateral prices}}{\text{inverse prices}}

where \(L\) is the ratio of the par value of the floater to the par value of the inverse floater.

MEASURING A BOND PORTFOLIO’S RESPONSIVENESS TO NONPARALLEL CHANGES IN INTEREST RATES

There have been several approaches to measuring yield curve risk. The two major ones are yield curve reshaping duration and key rate duration.

Yield Curve Reshaping Duration

The yield curve reshaping duration approach focuses on the sensitivity of a portfolio to a change in the slope of the yield curve.

Key Rate Duration

The most popular measure for estimating the sensitivity of a security or a portfolio to changes in the yield curve is key rate duration. The basic principle of key rate duration is to change the yield for a particular maturity of the yield curve and determine the sensitivity of a security or portfolio to that change holding all other yields constant.
ANSWERS TO QUESTIONS FOR CHAPTER 4

(Questions are in bold print followed by answers.)

1. The price value of a basis point will be the same regardless if the yield is increased or decreased by 1 basis point. However, the price value of 100 basis points (i.e., the change in price for a 100-basis-point change in interest rates) will not be the same if the yield is increased or decreased by 100 basis points. Why?

The convex relationship explains why the price value of a basis point (i.e., the change in price for a 1-basis-point change in interest rates) will be roughly the same regardless if the yield is increased or decreased by 1 basis point, while the price value of 100 basis points will not be the same if the yield is increased or decreased by 100 basis points. More details are below.

When the price-yield relationship for any option-free bond is graphed, it exhibits a convex shape. However, this relationship is not linear. The convex shape of the price-yield relationship generates four properties concerning the price volatility of an option-free bond. First, although the prices of all option-free bonds move in the opposite direction from the change in yield required, the percentage price change is not the same for all bonds. Second, for very small changes in the yield required (like 1 basis point), the percentage price change for a given bond is roughly the same, whether the yield required increases or decreases. Third, for large changes in the required yield (like 100 basis points), the percentage price change is not the same for an increase in the required yield as it is for a decrease in the required yield. Fourth, for a given large change in basis points, the percentage price increase is greater than the percentage price decrease.

2. Calculate the requested measures in parts (a) through (f) for bonds A and B (assume that each bond pays interest semiannually):

<table>
<thead>
<tr>
<th></th>
<th>Bond A</th>
<th>Bond B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coupon</td>
<td>8%</td>
<td>9%</td>
</tr>
<tr>
<td>Yield to maturity</td>
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<td>8%</td>
</tr>
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<td>Maturity (years)</td>
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<td>5</td>
</tr>
<tr>
<td>Par</td>
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<td>$100.00</td>
</tr>
<tr>
<td>Price</td>
<td>$100.00</td>
<td>$104.055</td>
</tr>
</tbody>
</table>

(a) What is the price value of a basis point for bonds A and B?

For bond A, we get a bond quote of $100.00 for our initial price if we have a 2-year maturity, an 8% coupon rate, and an 8% yield. If we change the yield one basis point so the yield is 8.01%, then we have the following variables and values:

\[ C = 40, \ y = \frac{0.0801}{2} = 0.04005 \text{ and } n = 2(2) = 4. \]

Inserting these values into the present value of the coupon payments formula, we get:
\[
P = C \left[ \frac{1 - \frac{1}{(1 + r)^n}}{r} \right] = \$40 \left[ \frac{1 - \frac{1}{(1.04005)^4}}{0.04005} \right] = \$145.179.
\]

Computing the present value of the par or maturity value of $1,000 gives:

\[
\frac{M}{(1 + r)^n} = \frac{\$1,000}{(1.04005)^4} = \$854.640.
\]

If we add a basis point to the yield, we get the value of bond A as: \(P = \$145.179 + \$854.640 = \$999.819\) with a bond quote of $99.9819. For bond A the price value of a basis point is about \(\$100 - \$99.9819 = \$0.0181\) per $100.

Using the bond valuation formulas as just completed above, the value of bond B with a yield of 8%, a coupon rate of 9%, and a maturity of 5 years is: \(P = \$364.990 + \$675.564 = \$1,040.554\) with a bond quote of $104.0554. If we add a basis point to the yield, we get the value of bond B as: \(P = \$364.899 + \$675.239 = \$1,040.139\) with a bond quote of $104.0139. For bond B, the price value of a basis point is \(\$104.0554 - \$104.0139 = \$0.0416\) per $100.

(b) Compute the Macaulay durations for the two bonds.

For bond A with \(C = \$40, n = 4, y = 0.04, P = \$1,000\) and \(M = \$1,000\), we have:

\[
\text{Macaulay duration (half years)} = \frac{1C}{(1+y)^1} + \frac{2C}{(1+y)^2} + \ldots + \frac{nC}{(1+y)^n} + \frac{nM}{(1+y)^n} = \frac{1(\$40)}{(1.04)^1} + \frac{2(\$40)}{(1.04)^2} + \ldots + \frac{4(\$40)}{(1.04)^4} + \frac{4(\$1,000)}{(1.04)^4} = \frac{\$3,775.09}{\$1,000} = 3.77509.
\]

Macaulay duration (years) = Macaulay duration (half years) / 2 = 3.77509 / 2 = 1.8875.

For bond B with \(C = \$45, n = 10, y = 0.04, P = \$1,040.55\) and \(M = \$1,000\), we have:

\[
\text{Macaulay duration (half years)} = \frac{1C}{(1+y)^1} + \frac{2C}{(1+y)^2} + \ldots + \frac{nC}{(1+y)^n} + \frac{nM}{(1+y)^n} = \frac{1(\$45)}{(1.04)^1} + \frac{2(\$45)}{(1.04)^2} + \ldots + \frac{10(\$45)}{(1.04)^10} + \frac{10(\$1,000)}{(1.04)^10} = \frac{\$8,645.2929}{\$1,040.55} = 8.3084.
\]

64
Macaulay duration (years) = Macaulay duration (half years) / 2 = 8.3084 / 2 = 4.1542.

(c) Compute the modified duration for the two bonds.

Taking our answer for the Macaulay duration in years in part (b), we can compute the modified duration for bond A by dividing by 1.04. We have:

\[
\text{modified duration} = \frac{1.8875}{1.04} = 1.814948.
\]

Taking our answer for the Macaulay duration in years in part (b), we can compute the modified duration for bond B by dividing by 1.04. We have:

\[
\text{modified duration} = \frac{4.1542}{1.04} = 3.994417.
\]

[NOTE. We could get the same answers for both bonds A and B by computing the modified duration using an alternative formula that does not require the extensive calculations required by the procedure in parts (a) and (b). This shortcut formula is:

\[
\text{modified duration} = \frac{C}{y^2} \left[ 1 - \frac{I}{(1+y)^n} \right] + \frac{n(100 - C/y)}{(1+y)^{n+1}}.
\]

where C is the semiannual coupon payment, y is the semiannual yield, n is the number of semiannual periods, and P is the bond quote in 100’s.

For bond A (expressing numbers in terms of a $100 bond quote), we have: C = $4, y = 0.04, n = 4, and P = $100. Inserting these values in our modified duration formula, we can solve as follows:

\[
\frac{4}{0.04^2} \left[ 1 - \frac{1}{(1.04)^4} \right] + \frac{4\times(100 - 4/0.04)}{(1.04)^5} = \frac{\$362.98952 + \$0}{\$100} = 3.6299.\]

Converting to annual number by dividing by two gives a modified duration for bond A of 1.8149 which is the same answer shown above.

For bond B, we have C = $4.5, n = 20, y = 0.04, and P = $104.055. Inserting these values in our modified duration formula, we can solve as follows:

\[
\frac{4.5}{0.04^2} \left[ 1 - \frac{1}{(1.04)^{20}} \right] + \frac{4\times(100 - 4.5/0.04)}{(1.04)^{11}} = \frac{\$912.47578 - \$81.19762}{\$104.055} = 7.988834 \text{ or about } 7.99.\]

Converting to annual number by dividing by two gives a modified duration for bond B of 3.994417 which is the same answer.
(d) Compute the approximate duration for bonds A and B using the shortcut formula by changing yields by 20 basis points and compare your answers with those calculated in part (c).

To compute the approximate measure for bond A, which is a 2-year 8% coupon bond trading at 8% with an initial price \( (P_0) \) of $1,000 (thus, it trades at its par value of $1,000), we proceed as follows.

First, we increase the yield on the bond by 20 basis points from 8% to 8.20%. Thus, \( \Delta y \) is 0.002. The new price \( (P^+) \) can be computed using our bond valuation formula. Doing this we get $996.379 with a bond price quote of $99.6379.

Second, we decrease the yield on the bond by 20 basis points from 8% to 7.8%. The new price \( (P^-) \) can be computed using our bond valuation formula. Doing this we get $1,003.638 with a bond price quote of $100.3638.

Third, with the initial price, \( P_0 \), equal to $100 (when expressed as a bond quote), the duration can be approximated as follows:

\[
\text{approximate duration} = \frac{P - P_0}{2(P_0)(\Delta y)}
\]

where \( \Delta y \) is the change in yield used to calculate the new prices (in decimal form). What the formula is measuring is the average percentage price change (relative to the initial price) per 20-basis-point change in yield. Inserting in our values, we have:

\[
\text{approximate duration} = \frac{\$100.3638 - \$99.6379}{2(\$100)(0.02)} = 1.814948.
\]

This compares with 1.814948 computed in part (c). Thus, the approximate duration measure is virtually the same as the modified duration computed in part (c).

To compute the approximate measure for bond B, which is a 5-year 9% coupon bond trading at 8% with an initial price \( (P_0) \) of $104.0554, we proceed as follows.

First, we increase the yield on the bond by 20 basis points from 8% to 8.20%. Thus, \( \Delta y \) is 0.002. The new price \( (P^+) \) can be computed using our bond valuation formula. Doing this we get $1,032.283 with a bond price quote of $103.2283.

Second, we decrease the yield on the bond by 20 basis points from 8% to 7.8%. The new price \( (P^-) \) can be computed using our bond valuation formula. Doing this we get $1,048.909 with a bond price quote of $104.8909.

Third, with the initial price, \( P_0 \), equal to $104.0554 (when expressed as a bond quote), the duration
can be approximated as follows:

\[
\text{approximate duration} = \frac{P - P_o}{2(P_o)(\Delta y)}
\]

where \(\Delta y\) is the change in yield used to calculate the new prices (in decimal form). What the formula is measuring is the average percentage price change (relative to the initial price) per 20-basis-point change in yield. Inserting in our values, we have:

\[
\text{approximate duration} = \frac{104.8909 - 103.2283}{2(104.0554)(0.02)} = 3.994400.
\]

This compares with 3.994417 computed in part (c). Thus, the approximate duration measure is virtually the same as the modified duration computed in part (c).

Besides the above approximate duration approach, there is another approach that is shorter than the Macaulay duration and modified duration approach. With this approach, we proceed as follows. For bond A, we add 20 basis points and get a yield of 8.20%. We now have \(C = 40\), \(y = 4.10\%\), \(n = 4\), and \(M = 1,000\). Before we use this shortcut approach, we first compute \(P\). As given above, we can use our bond valuation formula to get $996.379.

Now we can compute the modified duration for bond A using the below formula:

\[
\text{modified duration} = \frac{C}{y^2} \left[ 1 - \frac{1}{(1 + y)^n} \right] + \frac{n(100 - C / y)}{(1 + y)^n+1}.
\]

Putting in all applicable variables in terms of $100, we have: \(C = 4\), \(n = 4\), \(y = 0.041\) and \(P = 99.6379\). Inserting these values in our modified duration formula, we can solve as follows:

\[
\frac{4}{0.041^2} \left[ 1 - \frac{1}{(1.041)^4} \right] + \frac{4(100 - 4/0.041)}{(1.041)^{4+1}} = \frac{(353.30310 + 79.8036)}{99.6379} = 3.6260 \text{ or about 3.63.}
\]

Converting to annual number by dividing by two gives a modified duration for bond A of 1.8130. This compares with 1.8149 computed in part (c). Since both round off to 1.81, the 20 point change in basis does not exercise any noticeable effect on our computation as the difference is only 0.0019.

For bond B, we add 20 basis points and get a yield of 8.20%. We now have \(C = 45\), \(y = 4.10\%\), \(n = 10\), and \(M = 1,000\). Before we use the modified duration formula, we first compute \(P\). Using our bond valuation formula to get $1,032.283.

Now we can compute the modified duration for bond B as above for bond A. Given \(C = 4.5\), \(n = \)
10, y = 0.041, and P = $103.2237, and inserting these values into the above formula for modified duration gives:

\[
\text{modified duration} = \frac{($885.80511 - $627.0730)}{$103.2283} = 7.97357 \text{ or about } 7.97.
\]

Converting to annual number by dividing by two gives a modified duration for bond B of 3.9868. This compares with 3.9944 computed in part (c). Since both round off to 3.99, the 20-point change in basis does not exercise any noticeable effect on our computation. However, for the two-year bond, we only had a difference of 0.0019, while for the five-year bond, we have a difference of 0.0076. Thus, this shortcut approach gives a wider disagreement for the longer term bond (bond B).

(e) Compute the convexity measure for both bonds A and B.

In half years, the convexity measure is \(\frac{d^2P}{dy^2} \cdot \frac{1}{P}\). Noting that

\[
\frac{d^2P}{dy^2} = \left[ \frac{2C}{y^2} \left(1 - \frac{1}{(1+y)^n}\right) - \frac{2Cn}{y^2(1+y)^{n+1}} + \frac{n(n+1)(100 - C/y)}{(1+y)^{n+2}} \right],
\]

we can insert this quantity into our convexity measure (half year) formula to get:

\[
\text{convexity measure} = \left[ \frac{2C}{y^2} \left(1 - \frac{1}{(1+y)^n}\right) - \frac{2Cn}{y^2(1+y)^{n+1}} + \frac{n(n+1)(100 - C/y)}{(1+y)^{n+2}} \right] \cdot \frac{1}{P}.
\]

For bond A, we have a 2-year, 8% coupon bond trading at 8% with an initial price \((P_0)\) of $1,000 with a bond quote of $100. Expressing numbers in terms of a $100 bond quote, we have: \(C = 4\), \(y = 0.04\), \(n = 4\) and \(P = 100\). Inserting these numbers into our convexity measure formula gives:

\[
\text{convexity measure (half years)} = \left[ \frac{2(4)}{(0.04)^2} \left(1 - \frac{1}{(1.04)^4}\right) - \frac{2(4)}{(0.04)^2(1.04)^3} + \frac{4(5)(100 - 4/0.04)}{(1.04)^6} \right] \cdot \frac{1}{100} = [125,000(0.14519581) - 16,438.54214 + 0] \cdot \frac{1}{100} = 1,710.93[0.01] = 17.109354.
\]

Convexity measure (years) = convexity measure (half years) / 4 = 17.1093 / 4 = 4.2773350. Dollar convexity measure = convexity measure (years) times \(P = 4.2773350(100)\) equals about $427.73.

[NOTE. We can get the same convexity by proceeding as follows. First, we increase the yield on
the bond by 10 basis points from 8% to 8.1%. Thus, \( \Delta y \) is 0.001. The new price \( (P^+) \) can be computed using our bond valuation formula. Doing this we get $998.187 with a bond quote of $99.8187. Second, we decrease the yield on the bond by 10 basis points from 8% to 7.9%. The new price \( (P^-) \) can be computed using our bond valuation formula. Doing this we get $100.1817. Third, with the initial price, \( P_0 \), equal to $100, the convexity measure of any bond can be approximated using the following formula:

\[
\text{approximate convexity measure} = \frac{P_1 + P_2 - 2P_0}{P_0 (\Delta y)^2}.
\]

Inserting in our values, the approximate convexity measure for bond A is

\[
\text{approximate convexity measure} = \frac{\$100.1817 + \$99.8187 - 2(\$100)}{\$100(0.001)^2} = 4.2773384.
\]

The approximate convexity measure of 4.2773384 is almost identical to the convexity measure of 4.2773350 computed above.]

For bond B, we have a 5-year 9% coupon bond trading at 8% with an initial price \( (P_0) \) of $104.055. Expressing numbers in terms of a $100 bond quote, we have: \( C = $4.5, y = 0.04, n = 10, \) and \( P = $104.0554 \). Inserting these numbers into our convexity measure formula gives:

\[
\text{convexity measure (half years)} = \\
\frac{2(\$4.5)}{(0.04)^3} \left[ \frac{1}{(1.04)^{10}} \right] - \frac{2(\$4.5)10}{(0.04)^2(1.04)^{11}} + \frac{10(11)(100 - \$4.5/0.045)}{(1.04)^{12}} \left[ \frac{1}{\$104.0554} \right] = \\
\frac{1}{[\$140,625[0.32443583] - \$36,538.92740 + -\$858.8209]} = \\
8,226.04[0.0096103] = 79.0544.
\]

Convexity measure (years) = convexity measure (half years) / 4 = 79.0544 / 4 = 19.7636077. Dollar convexity measure = convexity measure (years) times \( P = 19.7636077(\$1,04.0554) \) equals about $2,056.51.

[NOTE. We can get the same convexity measure by proceeding as follows. First, we increase the yield on the bond by 10 basis points from 8% to 8.1%. Thus, \( \Delta y \) is 0.001. The new price \( (P^+) \) can be computed using our bond valuation formula. Doing this we get $1,036.408 with a bond quote of $103.6408. Second, we decrease the yield on the bond by 10 basis points from 8% to 7.9%. The new price \( (P^-) \) can be computed using our bond valuation formula. Doing this we get $104.4721. Third, with the initial price, \( P_0 \), equal to $104.0554, the convexity measure of any bond can be approximated using the following formula:
approximate convexity measure = \( \frac{P_+ + P_- - 2P_0}{P_0(\Delta y)^2} \).

Inserting in our values, the approximate convexity measure for bond B is

\[
\text{approximate convexity measure} = \frac{104.4721 + 103.6408 - 2(104.0554)}{104.0554(0.001)^2} = 19.7636548.
\]

19.7636548. The approximate convexity measure of 19.7636548 is almost identical to the convexity measure of 19.763608 computed above.

(f) Compute the approximate convexity measure for bonds A and B using the shortcut formula by changing yields by 20 basis points and compare your answers to the convexity measure calculated in part (e).

To compute the approximate convexity measure for bond A, which is a 2-year 8% coupon bond trading at 8% with an initial price \( P_0 \) of $100, we proceed as follows.

First, we increase the yield on the bond by 20 basis points from 8% to 8.2%. Thus, \( \Delta y \) is 0.002. The new price \( P_+ \) can be computed using our bond valuation formula. Doing this we get $99.6379.

Second, we decrease the yield on the bond by 20 basis points from 8% to 7.8%. The new price \( P_- \) can be computed using our bond valuation formula. Doing this we get $1,003.638 with a bond price quote of $100.3638.

Third, with the initial price, \( P_0 \), equal to $100, the convexity measure of any bond can be approximated using the following formula:

\[
\text{approximate convexity measure} = \frac{P_+ + P_- - 2P_0}{P_0(\Delta y)^2}.
\]

Inserting in our values, the approximate convexity measure is

\[
\text{approximate convexity measure} = \frac{100.3638 + 99.6379 - 2(100)}{100(0.002)^2} = 4.2773486.
\]

This answer of 4.2773486 for the approximate convexity measure is very similar to that computed in part (e) using the convexity measure where we got 4.2773350. [NOTE. The 4.2773486 for a change of 20 basis points is almost identical to the 4.2773384 that we can compute for a change of 10 basis points.]

To compute the approximate convexity measure for bond B which is a 5-year 9% coupon bond trading at 8% with an initial price \( P_0 \) of $104.0554 (worth $1,040.554 and with a par value = M = $1,000), we proceed as follows.
First, we increase the yield on the bond by 20 basis points from 8% to 8.2%. Thus, \( \Delta y \) is 0.001. The new price \((P^+)\) can be computed using our bond valuation formula. Doing this we get $103.2283.

Second, we decrease the yield on the bond by 20 basis points from 8% to 7.8%. The new price \((P^-)\) can be computed using our bond valuation formula. Doing this we get $1,048.909 with a bond price quote of $104.8909.

Third, with the initial price, \(P_0\), equal to $104.0554, the convexity measure of any bond can be approximated using the following formula:

\[
\text{approximate convexity measure} = \frac{P_+ + P_- - 2P_0}{P_0(\Delta y)^2}.
\]

Inserting in our values, the approximate convexity measure is:

\[
\text{approximate convexity measure} = \frac{\$104.8909 + \$103.2283 - 2(\$104.0554)}{\$104.0554(0.002)^2} = 19.763824.
\]

This answer of 4.2773486 for the approximate convexity measure is very similar to that computed in part (e) using the convexity measure where we got 19.7636077 [NOTE. The 19.7636077 for a change of 20 basis points is almost identical to the 19.7636548 computed for a change of 10 basis points.]

3. Can you tell from the following information which of the following three bonds will have the greatest price volatility, assuming that each is trading to offer the same yield to maturity?

<table>
<thead>
<tr>
<th>Bond</th>
<th>Coupon Rate (%)</th>
<th>Maturity (years)</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>Y</td>
<td>10</td>
<td>11</td>
</tr>
<tr>
<td>Z</td>
<td>11</td>
<td>12</td>
</tr>
</tbody>
</table>

The price of a bond will change over time as a result of a change in the perceived credit risk of the issuer. Thus, if one of the three bonds undergoes greater change in credit risk then that bond might be expected to experience more volatility unless other factors dominate. Below we describe these factors: term to maturity, coupon rate, and yield to maturity.

For a given term to maturity and initial yield, the price volatility of a bond will increase as the coupon rate becomes smaller. Thus, ceteris paribus, we would expect bond X to have greater price volatility that bond Y, and bond Y to have greater price volatility than bond Z. However, the differences in coupon rates for bonds Y and Z are not that great. Thus, if just looking at these two bonds, the differences in price volatilities may not be that recognizable.

For a given coupon rate and initial yield, longer terms to maturity will produce greater price
volatility. Thus, ceteris paribus, bond Z will have more price volatility than bond Y and bond Y will have more price volatility than bond X. However, the differences in maturities for bonds Y and Z are not that great. Thus, if just comparing these two bonds, the differences in price volatilities may not be that identifiable.

We should note that the expectations of price volatility based upon coupon rates and maturities are the reverse for bonds X and Z. For example, we expect bond X to have the greatest price volatility based upon coupon rate but the lowest based upon maturity. For bond Z, we expect it to have lowest price volatility based upon coupon rate but the greatest based upon maturity. If the coupon rate and the maturity factors or characteristics balance out then it is possible all three bonds will experience price volatilities that are very similar.

[NOTE. An implication of the maturity factor is that investors who want to increase a portfolio’s price volatility because they expect interest rates to fall, all other factors being constant, should hold bonds with long maturities in the portfolio. To reduce a portfolio’s price volatility in anticipation of a rise in interest rates, bonds with shorter term maturities should be held in the portfolio.]

Although the yield to maturity is held constant for bonds X, Y, and Z, the yield to maturity can also play a role as a factor impacting a bond’s price volatility. Ceteris paribus, the higher the yield to maturity at which a bond trades, then the lower the price volatility should be.

In addition to the above factors, we have to keep in mind four important properties concerning the price volatility of an option-free bond that result from the convex shape of the price-yield relationship.

First, although the prices of all option-free bonds move in the opposite direction from the change in yield required, the percentage price change (e.g., price volatility) depends on the convexity relationship between price and yield for each of the three bonds. Thus, for bonds X, Y, and Z, we will not expect the same price volatility due to likely differences in convexity. Whether the market yield rises or falls, the bond with the greatest convexity will achieve a higher price. That is, if the required yield rises, the capital loss for this bond will be less while a fall in the required yield will generate greater price appreciation.

Second, for very small changes in the yield required, the percentage price change for a given bond is roughly the same, whether the yield required increases or decreases. Thus, for bonds X, Y, and Z if the percentage price change is very small, we will not likely detect which bond has the greatest price volatility.

Third, for large changes in the required yield, the percentage price change is not the same for an increase in the required yield as it is for a decrease in the required yield. Thus, whichever bond or bonds change, the price volatility will depend on the direction of the change.

Fourth, for a given large change in basis points, the percentage price increase is greater than the percentage price decrease. Thus, whichever bond or bonds change, the price volatility will be relatively greater if there is a percentage price increase as opposed to a decrease. The implication
of this fourth property is that if an investor owns a bond, the price appreciation that will be realized (if the required yield decreases) is greater than the capital loss that will be realized if the required yield rises by the same number of basis points. For an investor who is “short” a bond, the reverse is true: the potential capital loss is greater than the potential capital gain if the required yield changes by a given number of basis points.

4. Answer the following questions for bonds A and B.

<table>
<thead>
<tr>
<th></th>
<th>Bond A</th>
<th>Bond B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coupon</td>
<td>8%</td>
<td>9%</td>
</tr>
<tr>
<td>Yield to maturity</td>
<td>8%</td>
<td>8%</td>
</tr>
<tr>
<td>Maturity (years)</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>Par</td>
<td>$100.00</td>
<td>$100.00</td>
</tr>
<tr>
<td>Price</td>
<td>$100.00</td>
<td>$104.055</td>
</tr>
</tbody>
</table>

(a) Calculate the actual price of the bonds for a 100-basis-point increase in interest rates.

For bond A, we get a bond quote of $100 for our initial price if we have an 8% coupon rate and an 8% yield. If we change the yield 100 basis point so the yield is 9%, then the value of the bond (P) is the present value of the coupon payments plus the present value of the par value. We have C = $40, y = 4.5%, n = 4, and M = $1,000. Inserting these numbers into our present value of coupon bond formula, we get:

\[ P = C \left[ \frac{1 - \frac{1}{(1 + r)^n}}{r} \right] = 40 \left[ \frac{1 - \frac{1}{(1.045)^4}}{0.045} \right] = 143.501. \]

The present value of the par or maturity value of $1,000 is:

\[ \frac{M}{(1 + r)^n} = \frac{1,000}{(1.045)^4} = 838.561 \]

Thus, the value of bond A with a yield of 9%, a coupon rate of 8%, and a maturity of 2 years is: P = $143.501 + $838.561 = $982.062. Thus, we get a bond quote of $98.2062.

We already know that bond B will give a bond value of $1,000 and a bond quote of $100 since a change of 100 basis points will make the yield and coupon rate the same, For example, inserting the values of C = $45, y = 4.5%, n = 10, and M = $1,000 into our bond valuation formula gives: P = $356.072 + $643.928 = $1,000.00 with a bond quote of $100.

(b) Using duration, estimate the price of the bonds for a 100-basis-point increase in interest rates.

To estimate the price of bond A, we begin by first computing the modified duration. We can use an alternative formula that does not require the extensive calculations required by the Macaulay
The formula is:

\[
\text{modified duration} = \frac{C}{y^2} \left[ 1 - \frac{1}{(1+y)^n} \right] + \frac{n(100 - C / y)}{(1+y)^{n+1}}.
\]

Putting all applicable variables in terms of $100, we have $C = 4$, $n = 4$, $y = 0.045$, and $P = 98.2062$. Inserting these values, in the modified duration formula gives:

\[
\frac{4}{0.045^2} \left[ 1 - \frac{1}{(1+0.045)^4} \right] + \frac{4(100 - 4/0.04)}{(1.045)^{4}} = \frac{\$318.89117 + \$35.66449}{\$98.2062} = 3.6103185 \text{ or about } 3.61.
\]

Converting to annual number by dividing by two gives a modified duration of 1.805159 (before the increase in 100 basis points it was 1.814948).

We next solve for the change in price using the modified duration of 1.805159 and $dy = 100$ basis points $= 0.01$. We have:

\[
\frac{dP}{P} = -\text{modified duration} (dy) = -1.805159(0.01) = -0.0180515.
\]

We can now solve for the new price of bond A as shown below:

\[
\left(1 + \frac{dP}{P}\right)P = (1 + (-0.0180515))\$1,000 = (0.9819484)\$1,000 = \$981.948.
\]

This is slightly less than the actual price of $982.062. The difference is $982.062 – $981.948 = $0.114.

To estimate the price of bond B, we follow the same procedure just shown for bond A. Using the alternative formula for modified duration that does not require the extensive calculations required by the Macaulay procedure and noting that $C = 45$, $n = 10$, $y = 0.045$, and $P = 100$, we get:

\[
\frac{4.5}{0.045^2} \left[ 1 - \frac{1}{(1+0.045)^{10}} \right] + \frac{10(100 - 45/0.045)}{(1.045)^{11}} = \frac{\$791.27182 + \$0}{\$100} = 7.912718 \text{ or about } 7.91 \text{ (before the increase in 100 basis points it was } 7.988834 \text{ or about 7.99).}
\]

Converting to an annual number by dividing by two gives a modified duration of 3.956359 (before the increase in 100 basis points it was 3.994417). We will now estimate the price of bond B using the modified duration measure. With 100 basis points giving $dy$
\[ \frac{dP}{P} = -\text{modified duration} \ dy = -3.956359(0.01) = -0.0395635. \]

Thus, the new price is 
\[ \left(1 - \frac{dP}{P}\right)P = (1 - 0.0395635)\$1,040.55 = (0.9604364)\$1,040.55 = \text{\$999.382}. \] This is slightly less than the actual price of \$1,000. The difference is \$1,000 – \$999.382 = \$0.618.

(c) Using both duration and convexity measure, estimate the price of the bonds for a 100-basis-point increase in interest rates.

For bond A, we use the duration and convexity measures as given below.

First, we use the duration measure. We add 100 basis points and get a yield of 9%. We now have \( C = \$40, \ y = 4.5\%, \ n = 4, \) and \( M = \$1,000. \) NOTE. In part (a) we computed the actual bond price and got \( P = \$982.062. \) Prior to that, the price sold at par \( (P = \$1,000) \) since the coupon rate and yield were then equal. The actual change in price is: 
\( (\$982.062 – \$1,000) = -$17.938 \) and the actual percentage change in price is: 
\( -$17.938 / \$1,000 = -0.017938\%. \)

We will now estimate the price by first approximating the dollar price change. With 100 basis points giving \( dy = 0.01 \) and a modified duration computed in part (b) of 1.805159, we have:
\[ dP \ P = -\text{modified duration} \ dy = -1.805159(0.01) = -0.01805159 \text{ or about } -1.805159\%. \]

This is slightly more negative than the actual percentage decrease in price of \(-1.7938\%). The difference is \(-1.7938\% – (-1.805159\%) = -1.7938\% + 1.805159\% = 0.011359\%.

Using the \(-1.805159\% \) just given by the duration measure, the new price for bond A is:
\[ \left(1 + \frac{dP}{P}\right)P = (1 +(-0.018051598)\$1,000 = (0.9819484)\$1,000 = \$981.948. \]

This is slightly less than the actual price of \$982.062. The difference is \$982.062 – \$981.948 = \$0.114.

Next, we use the convexity measure to see if we can account for the difference of 0.011359%. We have: convexity measure (half years) = \[ \frac{d^2 P}{dy^2} \ P = \]
\[ \left[ \frac{2C}{y} \left[1 - \frac{1}{(1+y)^n}\right] - \frac{2Cn}{y^2(1+y)^{n+1}} + \frac{n(n+1)(100-C/y)}{(1+y)^{n+2}} \right] \left[ \frac{1}{P}\right]. \]
For bond A, we add 100 basis points and get a yield of 9%. We now have $C = 40$, $y = 4.5\%$, $n = 4$, and $M = 1,000$. NOTE. In part (a) we computed the actual bond price and got $P = 982.062$. Prior to that, the price sold at par ($P = 1,000$) since the coupon rate and yield were then equal. Expressing numbers in terms of a $100$ bond quote, we have: $C = 4$, $y = 0.045$, $n = 4$, and $P = 98.2062$. Inserting these numbers into our convexity measure formula gives:

$$
\text{convexity measure (half years)} = \\
\left[ \frac{2(4)}{0.045^2} \left( 1 - \frac{1}{(1.045)^4} \right) - \frac{2(4)4}{(0.045)^2(1.045)^5} + \frac{4(5)(100 - \frac{4}{0.045})}{(1.045)^6} \right] \left( \frac{1}{98.2062} \right) = \\
1,662.88 \times 0.0101827 = 16.9325. 
$$

The convexity measure (years) = convexity measure (half years) / 4 = 16.9325 / 4 = 4.233125.

[NOTE. Dollar convexity measure = convexity measure (years) times $P = 4.233125(98.2062) = 415.7192$.]

The percentage price change due to convexity is: $\frac{dP}{P} = \frac{1}{2} \text{convexity measure} \cdot (dy)^2$. Inserting in our values, we get: $\frac{dP}{P} = \frac{1}{2} \times (16.9325)(0.01)^2 = 0.00021166$. Thus, we have 0.021166% increase in price when we adjust for the convexity measure.

Adding the duration measure and the convexity measure, we get −1.805159% + 0.021166% = −1.783994%. Recall the actual change in price is: ($982.062 − 1,000$) = −$17.938$ and the actual percentage change in price is: −$17.938$ / $1,000 = −0.017938$ or approximately −1.7938%. Using the −1.783994% resulting from both the duration and convexity measures, we can estimate the new price for bond A. We have:

$$
\text{new price is} \left( 1 + \frac{dP}{P} \right)P = (1 + (−0.01783994))1,000 = (0.9819484)1,000 = 982.160.
$$

This estimated price for bond A of $982.160 is slightly more than the actual price of $982.062. The difference is $982.160 − $982.062 = $0.098. Thus, using the convexity measure along with the duration measure has narrowed the estimated price from a difference of −$0.114 to $0.098.

For bond B, we can also estimate its price using both the duration and convexity measures as just shown for bond A.

First, we use the duration measure. We add 100 basis points and get a yield of 9%. We now have $C = 45$, $y = 4.5\%$, $n = 10$, and $M = 1,000$. NOTE. In part (a) we computed the actual bond price and got $P = 1,000$ since the coupon rate and yield were then equal. Prior to that, the price sold at par ($P = 1,040.55$). The actual change in price is: ($1,000 − 1,040.55$) = −$40.55$ and the actual percentage change in price is: −$40.55$ / $1,040.55$ = −0.0389697 or about −3.896978%.  

76
We will now estimate the price by first approximating the dollar price change. With 100 basis points giving \( dy = 0.01 \) and a modified duration computed in part (b) of 3.956359, we have:

\[
\frac{dP}{P} = -\text{modified duration} (dy) = -3.956359(0.01) = -0.0395635 \text{ or about } -3.95635\%.
\]

This is slightly more negative than the actual percentage decrease in price of -3.896978%. The difference is -3.896978 - -3.95635% = -3.896978 + 3.95635% = 0.059382%.

Using the -3.95635% just given by the duration measure, the new price for bond B is:

\[
\left(1 - \frac{dP}{P}\right) P = (1 - 0.0395635)\$1,040.55 = (0.96043641)\$1,040.55 = \$999.382.
\]

This is slightly less than the actual price of $1,000. The difference is $1,000 - $999.382 = $0.618.

Next, we use the convexity measure to see if we can account for the difference of 0.0594%. We have: convexity measure (half years) = \( \frac{d^2P}{dy^2} \frac{1}{P} = \)

\[
\left[ \frac{2C}{y^4} \left( 1 - \frac{1}{(1+y)^6} \right) - \frac{2Cn}{y^2(1+y)^{2n+1}} + \frac{n(n+1)(100 - C/y)}{(1+y)^{n+2}} \right] \left[ \frac{1}{P} \right].
\]

For bond B, we add 100 basis points and get a yield of 9%. We now have \( C = $45, y = 4.5\%, n = 10, \text{ and } M = $1,000. \) NOTE. In part (a) we computed the actual bond price and got \( P = $1,000 \) since the coupon rate and yield were then equal. Prior to that, the price sold at \( P = $1,040.55. \) Expressing numbers in terms of a $100 bond quote, we have: \( C = $4.5, y = 0.045, n = 10, \text{ and } P = $100. \) Inserting these numbers into our convexity measure formula gives:

\[
\text{convexity measure (half years)} = \left[ \frac{2(4.5)}{(0.045)^4} \left( 1 - \frac{1}{(1.045)^{10}} \right) - \frac{2(4.5)4}{(0.045)^2(1.045)^{11}} + \frac{10(11)(100 - 4.5/0.045)}{(1.045)^{12}} \right] \left[ \frac{1}{100} \right] = 7,781.03[0.01000] = 77.8103.
\]

The convexity measure (years) = convexity measure (half years) / 4 = 77.8103 / 4 = 19.452564.

[NOTE. Dollar convexity measure = convexity measure (years) times \( P = 19.452564($100) = $1,945.2564. \)]
The percentage price change due to convexity is: \( \frac{dP}{P} = \frac{1}{2} \text{convexity measure} \cdot (dy)^2 \). Inserting in our values, we get: \( \frac{dP}{P} = \frac{1}{2}(77.8103)(0.01)^2 = 0.00097463 \). Thus, we have 0.097463\% increase in price when we adjust for the convexity measure.

Adding the duration measure and the convexity measure, we get –3.956359\% + 0.097263\% equals –3.859096\%. Recall the actual change in price is: \( ($1,000 – $1,040.55) = –$40.55 \) and the actual percentage change in price is: \( –$40.55 / $1,040.55 = –0.0389697 \) or about –3.896978\%.

Thus, the new price is \( (1 – 0.03859096)$1,040.55 = (0.9614091)$1,040.55 = $1,000.394 \) for bond A. This is about the same as the actual price of $1,000. The difference is $1,000.394 – $1,000 = $0.394. Thus, using the convexity measure along with the duration measure has narrowed the estimated price from a difference of –$0.618 to $0.394.

(d) Comment on the accuracy of your results in parts b and c, and state why one approximation is closer to the actual price than the other.

For bond A, the actual price is $982.062. When we use the duration measure, we get a bond price of $981.948 that is $0.114 less than the actual price. When we use duration and convex measures together, we get a bond price of $982.160. This is slightly more than the actual price of $982.062. The difference is $982.160 – $982.062 = $0.098. Thus, using the convexity measure along with the duration measure has narrowed the estimated price from a difference of –$0.114 to $0.098.

For bond B, the actual price is $1,000. When we use the duration measure, we get a bond price of $999.382 that is $0.618 less than the actual price. When we use duration and convex measures together, we get a bond price of $1,000.394. This is slightly more than the actual price of $1,000. The difference is $1,000.394 – $1,000 = $0.394. Thus, using the convexity measure along with the duration measure has narrowed the estimated price from a difference of –$0.618 to $0.394.

As we see, using the duration and convexity measures together is more accurate. The reason is that adding the convexity measure to our estimate enables us to include the second derivative that corrects for the convexity of the price-yield relationship. More details are offered below.

Duration (modified or dollar) attempts to estimate a convex relationship with a straight line (the tangent line). We can specify a mathematical relationship that provides a better approximation to the price change of the bond if the required yield changes. We do this by using the first two terms of a Taylor series to approximate the price change as follows:

\[
\frac{dP}{dy} = \frac{1}{2} \frac{d^2P}{dy^2} + (dy)^2 + \text{error (1)}.
\]

Dividing both sides of this equation by \( P \) to get the percentage price change gives us:
\[
\frac{dP}{P} = \frac{dP}{dy} \cdot \frac{1}{P} \cdot dy + \frac{1}{2} \frac{d^2P}{dy^2} \cdot (dy)^2 + \frac{\text{error}}{P}. \quad (2)
\]

The first term on the right-hand side of equation (1) is the equation for the dollar price change based on dollar duration and is our approximation of the price change based on duration. In equation (2), the first term on the right-hand side is the approximate percentage change in price based on modified duration. The second term in equations (1) and (2) includes the second derivative of the price function for computing the value of a bond. It is the second derivative that is used as a proxy measure to correct for the convexity of the price-yield relationship. Market participants refer to the second derivative of bond price function as the dollar convexity measure of the bond. The second derivative divided by price is a measure of the percentage change in the price of the bond due to convexity and is referred to simply as the convexity measure.

(e) Without working through calculations, indicate whether the duration of the two bonds would be higher or lower if the yield to maturity is 10% rather than 8%.

Like term to maturity and coupon rate, the yield to maturity is a factor that influences price volatility. Ceteris paribus, the higher the yield level, the lower the price volatility. The same property holds for modified duration. Thus, a 10% yield to maturity will have both less volatility than an 8% yield to maturity and also a smaller duration.

There is consistency between the properties of bond price volatility and the properties of modified duration. When all other factors are constant, a bond with a longer maturity will have greater price volatility. A property of modified duration is that when all other factors are constant, a bond with a longer maturity will have a greater modified duration. Also, all other factors being constant, a bond with a lower coupon rate will have greater bond price volatility. Also, generally, a bond with a lower coupon rate will have a greater modified duration. Thus, bonds with greater durations will have greater price volatilities.

5. State why you would agree or disagree with the following statement: As the duration of a zero-coupon bond is equal to its maturity, the price responsiveness of a zero-coupon bond to yield changes is the same regardless of the level of interest rates.

As seen in Exhibit 4-3, the price responsiveness of a zero-coupon bond is different as yields change. Like other bonds, zero-coupon bonds have greater price responsiveness for changes at higher levels of maturity as interest rates change. Like other bonds, zero-coupon bonds have greater price responsiveness for changes at lower levels of interest rates compared to higher levels of interest rates.

Except for long-maturity deep-discount bonds, bonds with lower coupon rates will have greater modified and Macaulay durations. Also, for a given yield and maturity, zero-coupon bonds have higher convexity and thus greater price responsiveness to changes in yields.

6. State why you would agree or disagree with the following statement: When interest rates are low, there will be little difference between the Macaulay duration and modified duration measures.
The Macaulay duration is equal to the modified duration times one plus the yield. Rearranging this expression gives:

\[
\text{modified duration} = \frac{\text{Macaulay duration}}{1 + y}
\]

It follows that the modified duration will approach equality with the Macaulay duration as yields approach zero. Thus, if by low interest rates one means rates approaching zero, then one would agree with the statement.

[NOTE. Like term to maturity and coupon rate, the yield to maturity is a factor that will influence price volatility. All other factors constant, the higher the yield level, the lower the price volatility. The same property holds for duration. There is also consistency between the properties of bond price volatility and the properties of modified duration. When all other factors are constant, a bond with a longer maturity will have greater price volatility. A property of modified duration is that when all other factors are constant, a bond with a longer maturity will have a greater modified duration. Also, all other factors being constant, the lower the coupon rate, the greater the bond price volatility. Also, generally, lower coupon rates will render greater modified durations. Thus, bonds with greater modified durations will have greater price volatilities.]

7. State why you would agree or disagree with the following statement: If two bonds have the same dollar duration, yield, and price, their dollar price sensitivity will be the same for a given change in interest rates.

If the two bonds have the same dollar durations, then their percentage change in price is the same. This implies they will have the same dollar price sensitivity. This possibility is seen from the following equation:

\[
\frac{dP}{dy} = -(\text{modified duration})P
\]

where the expression on the right-hand side is the estimated dollar duration. By having the same dollar duration, price \((P)\), and yield, we see they can have the same price change \((dP)\) for a given change in yield \((dy)\). Thus, their dollar price change or dollar price sensitivity can be the same.

There are possible caveats to the above argument that make it possible that the dollar price sensitivity can be different for a given change in interest rates. For example, for an increase in the required yield, the estimated dollar price change is more than the actual price change. The reverse is true for a decrease in the required yield. Thus, to the extent the estimated changes can differ, their percentage changes in price can differ. This makes it possible that their dollar price sensitivity will be different for a given change in interest rates. Also, coupon rates are a factor in determining price behavior. For an option-free bond with a given term to maturity and initial yield, the price volatility of a bond will increase as the coupon rate decreases. Thus, unless coupon rates are actually the same when the dollar duration, yield, and price are the same, then their dollar price sensitivity will not necessarily be the same for a given change in interest rates.
8. State why you would agree or disagree with the following statement: For a 1-basis point change in yield, the price value of a basis point is equal to the dollar duration.

The validity of the above statement is discussed below.

The price value of a basis point, also referred to as the dollar value of a 01, is the change in the price of the bond if the required yield changes by 1 basis point. For small changes in the required yield, the below equation does a good job in estimating the change in price:

\[ dP = -\text{(dollar duration)}(dy). \]

Consider a 6% 25-year bond selling at $70.3570 to yield 9%. The dollar duration is 747.2009. For a 1-basis-point (0.0001) increase in the required yield, the estimated price change per $100 of face value is

\[ dP = -\text{(dollar duration)}(dy) = -747.2009(0.0001) = -$0.07472. \]

If we change the yield one basis point so the yield is 9.01%, then the value of the bond is: \( P = 702.824 \) with a bond quote of $70.2824. The price value of a basis point is about \( 70.2824 - 70.3570 = -$0.07464 \). The dollar duration for a 1-basis point change gives about the same value as the price value of a basis point as both round off to \(-$0.0747\).

If we add a basis point to the yield, we get the value of bond A as: \( P = 999.819 \) with a bond quote of $99.9819. For bond A the price value of a basis point is about \$0.0181 per $100.

9. The November 26, 1990, issue of BondWeek includes an article, “Van Kampen Merritt Shortsens.” The article begins as follows:

“Peter Hegel, first v.p. at Van Kampen Merritt Investment Advisory, is shortening his $3 billion portfolio from 110% of his normal duration of 6½ years to 103–105% because he thinks that in the short run the bond rally is near an end.”

Explain Hegel’s strategy and the use of the duration measure in this context.

If Hegel thinks the bond rally is over, it implies that he thinks bond prices will not go up. This implies the belief that Hegel thinks interest rates will stop falling.

If interest rates begin going up, then one does not want to lock in longer term bonds at lower rates. This implies you want your portfolio of bonds to focus more on shorter term bonds. Thus, you want a portfolio with a shorter duration. A shorter duration will mean not only less sensitivity to interest rates, but if interest rates go up then Hegel will later capitalize on this because as bonds in his portfolio mature quicker (than would be achieved with a portfolio with a higher duration) he will be able to buy new bonds and lock in higher rates.

In brief, Hegel uses the duration measure to optimize the value of his portfolio based upon his
belief about how interest rates change.

10. Consider the following two Treasury securities:

<table>
<thead>
<tr>
<th>Bond</th>
<th>Price</th>
<th>Modified duration (years)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$100</td>
<td>6</td>
</tr>
<tr>
<td>B</td>
<td>$80</td>
<td>7</td>
</tr>
</tbody>
</table>

Which bond will have the greater dollar price volatility for a 25-basis-point change in interest rates?

The estimated dollar price change can be obtained by using the below equation:

\[ dP = -(\text{modified duration})P(dy). \]

Inserting in our values for bond A, we have:

\[ dP = -(6)(100)(0.0025) = -$1.50 \] which is the estimated dollar price change or volatility for a 25-basis-point change. The percentage change in price is:

\[ \frac{dP}{P} = \frac{$1.50}{$100} = -0.0150 \text{ or } -1.50\%. \]

Inserting in our values for bond B, we have:

\[ dP = -(7)(80)(0.0025) = -$1.40 \] which is the estimated dollar price change or volatility for a 25-basis-point change. The percentage change in price is:

\[ \frac{dP}{P} = \frac{$1.40}{$80} = -0.0175 \text{ or } -1.750\%. \]

Thus, we see that while bond A has a greater estimated dollar price volatility compared to bond B, it has a lower percentage change in price. From an investor’s point of view, every dollar invested in bond B has greater volatility.

11. What are the limitations of using duration as a measure of a bond’s price sensitivity to interest-rate changes?

Below we discuss three limitations of using duration.

First, duration measures are only approximations for small changes in yield. They do not capture the effect of the convexity of a bond on its price performance when yields change by more than a small amount. To get improved accuracy, the duration measure should be supplemented with an additional measure to capture the curvature or convexity of a bond. It is important to note that investors can be misled if they rely on duration as the sole measure of the price volatility of a bond.
Second, in the derivation of the relationship between modified duration and bond price volatility, we started with the bond price equation that assumes that all cash flows for the bond are discounted at the same discount rate. In essence we are assuming that the yield curve is flat and all shifts are parallel. This assumption does not always hold. This is very important when we try to use a portfolio’s duration to quantify the responsiveness of a portfolio’s value to a change in interest rates. If a portfolio has bonds with different maturities, the duration measure may not provide a good estimate for unequal changes in interest rates of different maturities.

Third, we must be careful when applying our duration equations to bonds that are not option-free bonds. When changes in yields result in a change in the expected cash flow for a bond, which is the case for bonds with embedded options, the duration and convexity measures are appropriate only in certain circumstances.

12. The following excerpt is taken from an article titled “Denver Investment to Make $800 Million Treasury Move” that appeared in the December 9, 1991, issue of BondWeek, p. 1:

“Denver Investment Advisors will swap $800 million of long zero-coupon Treasuries for intermediate Treasuries. . . . The move would shorten the duration of its $2.5 billion fixed-income portfolio. . . .”

Why would the swap described here shorten the duration of the portfolio?

Duration captures the price sensitivity of a fixed-income investment to changes in yields. Thus, lowering the duration should lower the sensitivity. This is desired if one feels interest rates are going to increase, in which case the value of your fixed-income investment would decline.

Denver Investment Advisors are swapping $800 million long zero-coupon Treasuries for intermediate Treasuries. As a percentage of its portfolio, the proposed swap involves $800 million \( \div $2.5 \text{ billion} = 0.32 \) or 32%. Because the portfolio duration is the weighted average of its individual investments, the swap of $800 million long zero-coupon Treasuries for intermediate Treasuries will lower its duration if the $800 million being swapped actually has a lower duration or price sensitivity.

As seen in Exhibit 4-3, the price responsiveness of a zero-coupon bond is different as yields change. Like other bonds, zero-coupon bonds have greater price responsiveness for changes at higher levels of maturity as interest rates change. Furthermore, like other bonds, zero-coupon bonds have greater price responsiveness for changes at lower levels of interest rates. However, the exhibit also shows that zero-coupon bonds have greater percentage price changes especially for longer term securities. This indicates that swapping its long zero-coupon Treasuries for intermediate Treasuries could have an important impact on lowering Denver Investment’s duration.

13. You are a portfolio manager who has presented a report to a client. The report indicates the duration of each security in the portfolio. One of the securities has a maturity of 15 years but a duration of 25. The client believes that there is an error in the report because he believes that the duration cannot be greater than the security’s maturity. What would be
your response to this client?

Unfortunately, market participants often confuse the main purpose of duration by constantly referring to it as some measure of the weighted average life of a bond. This is because of the original use of duration by Macaulay where the cash flow for each period divided by the market value formed a weight with the weights adding up to one. If you rely on this interpretation of duration, it will be difficult for you to understand why a security with a maturity of 15 years can have a duration greater than 25 years. For example, consider collateralized mortgage obligation (CMO) bond classes. Certain CMO bond classes have a greater duration than the underlying mortgage loans (because CMO bond classes are leveraged instruments whose price sensitivity or duration are a multiple of the underlying mortgage loans from which they were created). That is, a CMO bond class can have a duration of 25 although the underlying mortgage loans from which the CMO is created can have a maturity of 15 years.

The answer to the puzzle (about duration being greater than maturity) is that duration is the approximate percentage change in price for a small change in interest rates. Thus, a CMO bond class with a duration of 25 does not mean that it has some type of weighted average life of 15 years. Instead, it means that for a 100-basis-point change in yield, that bond’s price will change by roughly 40%. Similarly, we interpret the duration of an option in the same way. A call option can have a duration of 25 when the time to expiration of the option is much less than 25 years. This is confusing to someone who interprets duration as some measure of the life of an option.

14. Answer the following questions.

(a) Suppose that the spread duration for a fixed-rate bond is 2.5. What is the approximate change in the bond’s price if the spread changes by 50 basis points?

A measure of how a non-Treasury bond’s price will change if the spread sought by the market changes is referred to as spread duration. A spread duration for a fixed-rate security is interpreted as the approximate change in the price of a fixed-rate bond for a 100-basis-point change in the spread. If the change is 2.5% (as given by a duration of 2.5) for 100 basis points then it would be about 1.25% for 50 basis points as shown below in more detail.

Let us begin by noting that

$$\frac{dP}{P} = -\text{modified duration} \times (dy).$$

Substituting spread duration for modified duration to approximate the percentage price change for a given change in the yield we get:

$$\frac{dP}{P} = -\text{spread duration} \times (dy).$$

Putting in 2.5 for the spread duration and 0.005 for $dy$ (since the spread changes by 50 basis points), we get:
\[
\frac{dP}{P} = -2.5(0.005) = -0.0125.
\]

Thus, the change in the bond’s price if the spread changes by 50 basis points in percentage terms is \(-1.25\%\), which can be interpreted as the approximate percentage change in price for a 50-basis-point change in yield.

**(b) What is the spread duration of a Treasury security?**

The spread represents compensation for credit risk. The price of a non-Treasury bond is exposed to a change in the spread that is called credit spread risk. For a Treasury security, there is no credit risk and thus the spread duration for a Treasury security is zero.

**15. What is meant by the spread duration for a floating-rate bond?**

A floating-rate bond’s price sensitivity will depend on whether the spread that the market wants changes. The spread is reflected in the quoted margin in the coupon reset formula. The quoted margin is fixed over the life of a typical floating-rate bond (or floater). Spread duration is a measure used to estimate the sensitivity of a floater’s price sensitivity to a change in the spread. A spread duration of 1.4 for a floater would mean that if the spread the market requires changes by 100 basis points, the floater’s price will change by about 1.4%.

**16. Explain why the duration of an inverse floater is a multiple of the duration of the collateral from which the inverse floater is created.**

In general, an inverse floater is created from a fixed-rate security. The security from which the inverse floater is created is called the collateral. From the collateral two bonds are created: a floater and an inverse floater. The two bonds are created such that (i) the total coupon interest paid to the two bonds in each period is less than or equal to the collateral’s coupon interest in each period, and (ii) the total par value of the two bonds is less than or equal to the collateral’s total par value. Equivalently, the floater and inverse floater are structured so that the cash flow from the collateral will be sufficient to satisfy the obligation of the two bonds.

The duration of the inverse floater is related to the duration of the collateral and the duration of the floater. Assuming that the duration of the floater is close to zero, it can be shown that the duration of an inverse floater is as follows:

\[
\text{duration of an inverse floater} = (1 + L)(\text{duration of collateral}) \times \frac{\text{collateral prices}}{\text{inverse prices}}
\]

where \(L\) is the ratio of the par value of the floater to the par value of the inverse floater. For example, if collateral with a par value of $100 million is used to create a floater with a par value of $80 million and an inverse floater with a par value of $20 million, then \(L = (\$80 \text{ million} / \$20 \text{ million}) = 4\).

We can illustrate why an inverse’s duration is a multiple of the collateral. Suppose that the par
value of the collateral of $50 million is split as follows: $40 million for the floater and $10 million for the inverse floater. Suppose also that the collateral and inverse are trading at par so that the ratio of the prices is 1 and that the duration for the collateral is 8. For a 100-basis-point change in interest rates, the collateral’s price will decline by 8% or 0.08 ($500 million) = $4 million. Assuming that the floater’s price does not change when interest rates increase, the $4 million decline must come from the inverse floater. For the inverse floater to realize a decline in value of $4 million when its value is $10 million, the duration must be 40. That is, a duration of 40 will produce a 40% change in value or 0.04($10 million) = $4 million. Thus, the duration is five times the collateral’s duration of 8. Or equivalently, because \( L \) is 4, it is \((1 + 4)\) times the collateral’s duration.

17. Consider the following portfolio:

<table>
<thead>
<tr>
<th>Bond</th>
<th>Market Value</th>
<th>Duration (years)</th>
</tr>
</thead>
<tbody>
<tr>
<td>W</td>
<td>$13 million</td>
<td>2</td>
</tr>
<tr>
<td>X</td>
<td>$27 million</td>
<td>7</td>
</tr>
<tr>
<td>Y</td>
<td>$60 million</td>
<td>8</td>
</tr>
<tr>
<td>Z</td>
<td>$40 million</td>
<td>14</td>
</tr>
</tbody>
</table>

(a) What is the portfolio’s duration?

The portfolio duration is equivalent to the weighted average of the duration for bond W \((D_w)\), bond X \((D_x)\), bond \((D_y)\), and bond Z \((D_z)\). We proceed as follows to calculate the portfolio duration.

First, we calculate the total market value (where M equal million). We have: total market value = \( \sum \text{market value of all four bonds} = MV_w + MV_x + MV_y + MV_z = 13M + 27M + 60M + 40M = 140M. \)

Second, we compute the portfolio weights as given by the following formula: weight \((W)\) = market value of bond \((MV)\) / total market value \((TMV)\) or \(W_i = MV_i / TMV\) for \(i = W, X, Y\) and \(Z\). For the four weights we have:

- weight bond W = \(W_w = MV_w / TMV = 13M / 140M = 13/140\);
- weight bond X = \(W_x = MV_x / TMV = 27M / 140M = 27/140\);
- weight bond Y = \(W_y = MV_y / TMV = 60M / 140M = 3/7\);
- weight bond Z = \(W_z = MV_z / TMV = 40M / 140M = 2/7\).

The portfolio duration equals the weighted average of the duration for bond W \((D_w)\), bond X \((D_x)\), bond \((D_y)\), and bond Z \((D_z)\). We have:

\[(W_w)D_w + (W_x)D_x + (W_y)D_y + (W_z)D_z = (13/140)2 + (27/140)7 + (3/7)8 + (2/7)14 = 0.1857142 + 1.350000 + 3.4285714 + 4.0000 = 8.9643\ or about 9 years.\]
The portfolio’s duration is 8.9643 and interpreted as follows: If all the yields affecting the four bonds in the portfolio change by 100 basis points, the portfolio’s value will change by approximately 9%.

(b) If interest rates for all maturities change by 50 basis points, what is the approximate percentage change in the value of the portfolio?

The total change in value of a portfolio if all rates (for each point on the yield curve) change by the same number of basis points is simply the duration of a single security. Thus, we can proceed as follows when computing the approximate percentage change in the value of the portfolio by using the below formula where \(dy\) is 0.005 because all interest rates for all maturities change by 50 basis point. We have:

\[
\frac{dP}{P} = -(\text{modified duration})(dy) = -(8.9643)(0.005) = -0.0448215 \text{ or about } -4.4821\%.
\]

[NOTE. If all the yields affecting the four bonds in the portfolio change by 100 basis points, the portfolio’s value will change by \(-(8.9643)(0.01) = -0.089643\) or about –8.9643%.

(c) What is the contribution to portfolio duration for each bond?

Portfolio managers look at their interest rate exposure to a particular issue in terms of its contribution to portfolio duration. This measure is found by multiplying the weight of the issue in the portfolio by the duration of the individual issue. We get:

\[
\text{contribution to portfolio duration} = (\text{weight of issue in portfolio})(\text{duration of issue}).
\]

In part (a) we computed the contribution to portfolio duration for each bond as \((\text{weight of issue in portfolio})(\text{duration of issue}) = (W_i)D_i\), for \(i = W, X, Y\) and \(Z\). We have: \((W_w)D_w = (13/140)2 = 0.1857\); \((W_x)D_x = (27/140)7 = 1.3500\); \((W_y)D_y = (+3/7)8 = 3.4286\); \((W_z)D_z = (2/7)14 = 4.0000\).

The contribution to portfolio duration for each issue is in the last column of the following table:

<table>
<thead>
<tr>
<th>Bond</th>
<th>Market Value</th>
<th>Portfolio Weight</th>
<th>Duration (years)</th>
<th>Contribution to Duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>W</td>
<td>$13 million</td>
<td>13 / 140</td>
<td>2</td>
<td>0.1857</td>
</tr>
<tr>
<td>X</td>
<td>$27 million</td>
<td>27 / 140</td>
<td>7</td>
<td>1.3500</td>
</tr>
<tr>
<td>Y</td>
<td>$60 million</td>
<td>2 / 7</td>
<td>8</td>
<td>3.4286</td>
</tr>
<tr>
<td>Z</td>
<td>$40 million</td>
<td>3 / 7</td>
<td>14</td>
<td>4.0000</td>
</tr>
<tr>
<td>Total</td>
<td>$140 million</td>
<td>1.00</td>
<td></td>
<td>8.9643</td>
</tr>
</tbody>
</table>

18. “If two portfolios have the same duration, the change in their value when interest rates change will be the same.” Explain why you agree or disagree with this statement.

While not the sole or best measure, duration attempts to measure an asset’s price sensitivity to yield changes. Duration does a good job of estimating an asset’s percentage price change for a small change in yield. However, it does not do as good a job for a large change in yield. The
percentage price change due to convexity can be used to supplement the approximate price change using duration. For a portfolio, its duration is the weighted average of the duration for each asset. If the two portfolios with the same duration have the same weighted average it does not imply that it has the same assets and/or the same proportion of assets and/or assets with the same maturities. Thus, if there is a change in interest rate it can affect the duration of each portfolio’s assets differently. This is particularly true if the change in interest rates is different for different maturities. Thus, there is certainly no guarantee that a change in interest rates (when all is said and done) will produce the same duration for each portfolio.

To further understand why two portfolios with the same duration can be differently influenced by change in interest rates consider the derivation of duration. In the derivation of the relationship between modified duration (which is the approximate percentage change in price for a 100-basis-point change in yield) and bond price volatility, we started with the bond price equation. This price equation assumes that all cash flows for the bond are discounted at the same discount rate. Essentially, the derivation assumes that the yield curve is flat and all shifts are parallel. There are limitations of applying duration when this assumption does not hold, and the yield curve does not shift in a parallel fashion. This is extremely important when we try to use a portfolio’s duration to quantify the responsiveness of a portfolio’s value to a change in interest rates. If a portfolio has bonds with different maturities, the duration measure may not provide a good estimate for unequal changes in interest rates of different maturities. Thus, if two portfolios have the same duration, the change in their value when interest rates change will not necessarily be the same.

19. In the fifth edition of The Handbook of Fixed Income Securities (Irwin Professional Publishing, 1997), page 104 gives the following formula for the approximate convexity measure:

\[
\frac{P_r + P_s - 2P_0}{P_0(\Delta y)^2}
\]

where the variables are defined as in equation (4.24) of this chapter. Compare this formula with the approximate convexity measure given by equation (4.24). Which formula is correct?

Below is equation 4.24:

approximate convexity measure = \[\frac{P_r + P_s - 2P_0}{P_0(\Delta y)^2}\].

We see that the two equations are identical except that the denominator in the handbook’s equation is twice as large. The two different expressions can be explained in terms of how one decides to scale the answer. If one chooses to write equation (4.24) with the extra multiple of 2 in the denominator, all it really means is that when the percentage change in price due to convexity is computed using equation (4.20), the 1/2 should be eliminated. Below more details are provided.

In practice, different vendors of analytical systems and different writers compute the convexity measure in different ways. To see why, look back at equation (4.16) and focus on the second term on the right-hand side of the equation. In equation (4.19), we used part of that equation to define
the convexity measure. Specifically, the convexity measure is the product of the second derivative and the reciprocal of the price. Suppose instead that we defined the convexity measure from the second term of equation (4.16) to be

\[
\text{convexity measure} = \frac{1}{2} \frac{d^2 P}{dy^2} \frac{1}{D}.
\]

That is, the convexity measure shown is just one-half the convexity measure given by equation (4.19). Does it make a difference? Not at all. We must make sure that we adjust the relationship between the approximate percentage price change due to convexity and the convexity measure accordingly. Specifically, in equation (4.20), the relationship would be changed as follows:

\[
\frac{dP}{P} = \text{(convexity measure)}(dy)^2.
\]

The bottom line is that the approximate percentage price change due to convexity is the same regardless of whether the preceding equation or equation (4.20) is used. The interpretation of the convexity measure on a stand-alone basis is not meaningful because different vendors and writers may scale the measure in different ways. What is important is relating the convexity measure and the change in yield (squared).

20. Answer the following questions.

(a) **How is the short-end duration of a portfolio computed?**

The shortcoming of duration is that this measure may be inadequate in measuring how a security’s price or a portfolio’s value will change when interest rates do not change in a parallel manner. This is particularly the case for a bond portfolio. As a result, it is necessary to be able to measure the exposure of a bond or bond portfolio to shifts in the yield curve. There have been several approaches that have been suggested for measuring this exposure. There have been several approaches to measuring yield curve risk. The two major ones are yield curve reshaping duration and key rate duration.

Yield curve reshaping duration concentrates on the sensitivity of a portfolio to a change in the slope of the yield curve. The initial step in this approach is to define what is meant by the slope of the yield curve. Market participants have used different definitions. Some define yield curve slope as the difference in the Treasury yield curve at two maturity levels. For instance, the yield curve slope can be defined as the difference between the yield on a proxy for the long-term Treasury bond (30-year Treasury) and the 2-year on-the-run Treasury.

One of the first measures of this approach was introduced by Klaffky, Ma, and Nozari at Salomon Smith Barney. They called their measure by the name of yield curve reshaping duration. They focus on three maturity points on the yield curve: 2-year, 10-year, and 30-year. Using these three points they then calculate the spread between the 10-year and 2-year yield and refer to this as the spread for the short end of the yield curve; the spread between the 30-year and the 10-year is computed and referred to as the spread for the long end of the yield curve. Klaffky, Ma, and Nozari
refer to the sensitivity of a portfolio to changes in the short end of the yield curve as short-end duration (SEDUR) and to changes in the long-end of the yield curve as long-end duration (LEDUR).

To calculate the SEDUR of the portfolio, the change in each security’s price is calculated both for a steepening of the yield curve at the short end by x basis points and also for a flattening of the yield curve at the short end by x basis points.

The portfolio value for a steepening of the yield curve is then computed by adding up the value of every security in the portfolio after the steepening. We denote this value as \( V_{SE,F} \) where \( V \) stands for portfolio value, \( SE \) for short end of the yield curve, and \( S \) for steepening. Similarly, the portfolio value after the flattening is obtained by summing up the value of each security in the portfolio and the resulting value will be denoted by \( V_{SE,F} \) where \( F \) denotes flattening. The SEDUR is then computed as follows:

\[
\text{SEDUR} = \frac{V_{SE,F} - V_{SE,S}}{2(V_0)(\Delta y)}
\]

where \( V_0 \) is the initial value of the portfolio (the value before any steepening or flattening) and \( \Delta y \) is the number of basis points used to compute the steepening and flattening of the yield curve (x).

SEDUR is interpreted as the approximate percentage change in the value of a portfolio for a 100-basis-point change in the slope of the short end of the yield curve.

(b) How is the long-end duration of a portfolio computed?

To compute the LEDUR of the portfolio, the change in each security’s price is calculated both for a flattening of the yield curve at the long end by x basis points and also for a steepening of the yield curve at the long end by x basis points.

The value for the portfolio after each shift is computed and denoted by \( V_{LE,F} \) and \( V_{LE,S} \) where \( LE \) denotes the long-end of the yield curve. Then LEDUR is calculated from the following formula:

\[
\text{LEDUR} = \frac{V_{LE,F} - V_{LE,S}}{2(V_0)(\Delta y)}
\]

LEDUR is interpreted as the approximate percentage change in the value of a portfolio for a 100-basis-point change in the slope of the long end of the yield curve.

(c) How is the short end and long end of a portfolio defined?

Klaffky, Ma, and Nozari of Salomon Brothers (now Salomon Smith Barney) focus on three maturity points on the yield curve: 2-year, 10-year, and 30-year. Using these three points they then calculate the spread between the 10-year and 2-year yield and refer to this as the spread for the short end of the yield curve; the spread between the 30-year and the 10-year is computed and
referred to as the spread for the long end of the yield curve. Klaffky, Ma, and Nozari refer to the sensitivity of a portfolio to changes in the short end of the yield curve as short end duration (SEDUR) and to changes in the long end of the yield curve as long-end duration (LEDUR). These concepts, however, are applicable to other points on the yield curve.

(d) Suppose that the SEDUR of a portfolio is 3. What is the approximate change in the portfolio’s value if the slope of the short end of the yield curve changed by 25 basis points?

The portfolio value for a steepening of the yield curve is computed by adding up the value of every security in the portfolio after the steepening. We denote this value as \( V_{SE,F} \) where \( V \) stands for portfolio value, \( SE \) for short end of the yield curve, and \( S \) for steepening. Similarly, the portfolio value after the flattening is obtained by summing up the value of each security in the portfolio and the resulting value will be denoted by \( V_{SE,F} \) where \( F \) denotes flattening. The SEDUR is then computed as follows:

\[
\text{SEDUR} = \frac{V_{SE,S} - V_{SE,F}}{2(V_0)(\Delta y)}
\]

where \( V_0 \) is the initial value of the portfolio (the value before any steepening or flattening) and \( \Delta y \) is the number of basis points used to compute the steepening and flattening of the yield curve.

Noting that the change in price is approximated by \( V_{SE,S} - V_{SE,F} \) and SEDUR represents the duration, then we can substitute in the formula

\[
\frac{dP}{P} = -\text{modified duration} \times (\text{dy})
\]

to get:

\[
V_{SE,S} - V_{SE,F} = -\text{SEDUR} \times (\text{dy}).
\]

Inserting 3 for SEDUR and 0.0025 for \( \text{dy} \) (since the spread changes by 25 basis points), we get:

\[
\frac{dP}{P} \text{ or } V_{SE,S} - V_{SE,F} = -3(0.0025) = -0.0075.
\]

Thus, the approximate percentage change in the portfolio’s value if the slope of the short end of the yield curve changed by 25 basis points would be \(-0.75\%\).

21. Answer the following two questions.

(a) Explain what a 10-year key rate duration of 0.35 means.

The key rate duration is an approach to measure the exposure of a bond or bond portfolio to shifts in the yield curve. The basic principle of key rate duration is to change the yield for a particular maturity of the yield curve and determine the sensitivity of a security or portfolio to that change.
holding all other yields constant. The sensitivity of the change in value to a particular change in yield is called rate duration. There is a rate duration for every point on the yield curve. Consequently, there is not any single rate duration, but a vector of durations representing each maturity on the yield curve. The total change in value of a bond or a portfolio if all rates change by the same number of basis points is simply the duration of a security or portfolio.

A key rate duration for a particular portfolio maturity should be interpreted as follows: Holding the yield for all other maturities constant, the key rate duration is the approximate percentage change in the value of a portfolio (or bond) for a 100-basis-point change in the yield for the maturity whose rate has been changed. Thus, a 10-year key rate duration of 0.35 means that if the 10-year spot rate changes by 100 basis points and the spot rate for all other maturities does not change, the portfolio’s value will change by approximately 0.35%.

(b) How is a key rate duration computed?

Holding the yield for all other maturities constant, the key rate duration is the approximate percentage change in the value of a portfolio (or bond) for a 100-basis-point change in the yield for the maturity whose rate has been changed. Thus, a key rate duration is quantified by changing the yield of the maturity of interest and determining how the value or price changes. Thus, letting $P_0$ be the initial price and $\Delta y$ be the change in yield used to calculate the new prices (in decimal form), key rate duration can be approximated using the following formula:

$$\text{approximate key rate duration} = \frac{P_+ - P_-}{2(P_0)(\Delta y)}.$$  

The prices denoted by $P_-$ and $P_+$ in the equation are the prices in the case of a bond and the portfolio values in the case of a bond portfolio found by holding all other interest rates constant and changing the yield for the maturity whose key rate duration is sought.