Conic Linear Programming

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Conic LP

\[(CLP) \quad \text{minimize} \quad c \bullet x\]
\[\text{subject to} \quad a_i \bullet x = b_i, i = 1, 2, \ldots, m, \ x \in C,\]

where \(C\) is a convex cone.

Linear Programming (LP): \(c, a_i, x \in \mathcal{R}^n\) and \(C = \mathcal{R}^n_+\)

Second-Order Cone Programming (SOCP): \(c, a_i, x \in \mathcal{R}^n\) and \(C = SOC\)

Semidefinite Programming (SDP): \(c, a_i, x \in \mathcal{M}^n\) and \(C = \mathcal{M}_+^n\)

Note that cone \(C\) can be a product of many (different) convex cones.
Convex Optimization or Convex Programming

Convex Optimization: minimize a convex function over a convex constraint set/region.

An important fact for CO: any local minimizer is a global minimizer.

\[
(CO) \quad \text{minimize} \quad c_0(x) \\
\text{subject to} \quad c_i(x) \leq b_i, i = 1, 2, ..., m,
\]

where \( c_i(x), i = 0, 1, ..., m, \) are convex functions of \( x. \)

**Proof.** Let \( \hat{x} \) be a local minimizer and \( x^* \) be the global minimizer such that \( c_0(\hat{x}) > c_0(x^*). \) Let \( x(\alpha) = \alpha x^* + (1 - \alpha)\hat{x}. \) Then it is feasible and

\[
c_0(x(\alpha)) \leq \alpha c_0(x^*) + (1 - \alpha)c_0(\hat{x}) < c_0(\hat{x}), \ \forall \alpha > 0.
\]

This contradicts to \( \hat{x} \) being a local minimizer, since \( \alpha \) can be small enough such that \( x(\alpha) \) is in the neighborhood of \( \hat{x}. \)
The convex program can be rewritten as

\[(CO) \quad \text{minimize} \quad \alpha \]

subject to

\[c_0(x) - \alpha \leq 0, \]
\[c_i(x) - b_i \leq 0, \quad i = 1, 2, \ldots, m.\]

Thus, it is sufficient to consider convex optimization in a form

\[(CO) \quad \text{minimize} \quad c^T x \]

subject to

\[c_i(x) \leq 0, \quad i = 1, 2, \ldots, m,\]

where \(c_i(x), \quad i = 1, \ldots, m,\) are convex functions of \(x.\)

Consider set

\[C_i = \{(t; x) : t > 0, \quad tc_i(x/t) \leq 0.\}\]

It is a convex cone!
Then, (CO) can be equivalently written as

\[
\begin{align*}
\text{minimize} & \quad (0; c) \bullet (t; x) \\
\text{subject to} & \quad (1; 0) \bullet (t; x) = 1, \\
& \quad (t; x) \in C_1 \cap \ldots \cap C_m.
\end{align*}
\]

This is a Conic LP!

We now develop theories for (CLP).
The dual problem to

\[(CLP) \quad \text{minimize} \quad c \bullet x \]
\[\text{subject to} \quad a_i \bullet x = b_i, \ i = 1, 2, \ldots, m, \ x \in C.\]

is

\[(CLD) \quad \text{maximize} \quad b^T y \]
\[\text{subject to} \quad \sum_i^m y_i a_i + s = c, \ s \in C^*,\]

where \( y \in \mathbb{R}^m \) are the dual variables, \( s \) is called the dual slack vector/matrix, and \( C^* \) is the dual cone of \( C \).

**Theorem 1** *(Weak duality theorem)*

\[c \bullet x - b^T y = x \bullet s \geq 0\]

for any feasible \( x \) of \((CLP)\) and \((y, s)\) of \((CLD)\).
Frequently, $C^* = C$, that is, they are self-dual.

The dual of the $n$-dimensional non-negative orthant, $\mathcal{R}_+^n = \{x \in \mathbb{R}^n : x \geq 0\}$, is $\mathcal{R}_+^n$; it is self-dual.

The dual of the positive semi-definite matrices cone in $\mathcal{M}_+^n$, $\mathcal{M}_+^n$, is $\mathcal{M}_+^n$; it is self-dual.

The dual of the second-order cone, $\{(t; x) \in \mathcal{R}^{n+1} : t \geq \|x\|\}$, is also the second-order cone; it is self-dual.
SOCP Examples

minimize \[
\begin{pmatrix}
2 \\
1 \\
1 \\
1
\end{pmatrix} \bullet x
\]

subject to \[
\begin{pmatrix}
1 \\
1 \\
1 \\
1
\end{pmatrix} \bullet x = 1, \ x \in SOC.
\]

Dual:

maximize \[
y
\]

subject to \[
\begin{pmatrix}
2 \\
1 \\
1
\end{pmatrix} - y \cdot \begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix} = s \in SOC.
\]
**SDP Examples**

minimize \[
\begin{pmatrix}
2 & .5 \\
.5 & 1
\end{pmatrix}
\bullet X
\]

subject to \[
\begin{pmatrix}
1 & .5 \\
.5 & 1
\end{pmatrix}
\bullet X = 1, \; X \succeq 0.
\]

Dual:

maximize \[ y \]

subject to \[
\begin{pmatrix}
2 & .5 \\
.5 & 1
\end{pmatrix}
- y \cdot \begin{pmatrix}
1 & .5 \\
.5 & 1
\end{pmatrix} = S \succeq 0,
\]
Farkas’ Lemma for General Cones?

Given \( \mathbf{a}_i, i = 1, \ldots, m, \) and \( \mathbf{b} \in \mathbb{R}^m. \)

Then, the system \( \{ \mathbf{x} : \mathbf{a}_i \cdot \mathbf{x} = b_i, \ i = 1, \ldots, m, \ \mathbf{x} \in C \} \) has a feasible solution \( \mathbf{x} \) if and only if that
\[- \sum_i^m y_i \mathbf{a}_i \in C^* \text{ and } \mathbf{b}^T \mathbf{y} > 0 \] has no feasible solution \( \mathbf{y} \)?

It is necessary but not sufficient!

Let’s write equations in a compact form:

\[
\mathbf{A} \mathbf{x} = (\mathbf{a}_1 \cdot \mathbf{x}; \ldots; \mathbf{a}_m \cdot \mathbf{x}) \in \mathbb{R}^m
\]

and

\[
\mathbf{A}^T \mathbf{y} = \sum_i^m y_i \mathbf{a}_i.
\]
Alternative Systems for General Cones?

Alternative System Pair I?:

\[ Ax = b, \ x \in C, \]

and

\[ -A^T y \in C^*, \ b^T y = 1 \]

Alternative System Pair II?:

\[ Ax = 0, \ x \in C, \ c \cdot x = -1 (< 0) \]

and

\[ c - A^T y \in C^* \]
An SDP Cone Example when “Alternative System” failed

\[ C = \mathcal{M}_+^2. \]

\[ a_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

and

\[ b = \begin{pmatrix} 0 \\ 2 \end{pmatrix}. \]
Let $C$ be a closed convex cone in the rest of the course.

If there is $y$ such that $-A^T y \in \text{int} \ C^*$, then Alternative System Pair I is true:

$$Ax = b, \quad x \in C,$$

and

$$-A^T y \in C^*, \quad b^T y = 1$$

And if there is $x$ such that $A^T x = 0$, $x \in \text{int} \ C$, then Alternative System Pair II is true:

$$Ax = 0, \quad x \in C, \quad c \cdot x = -1 (< 0)$$

and

$$c - A^T y \in C^*$$
Conic Linear Programming in Compact Form

\[(CLP) \quad \text{minimize} \quad c \cdot x \]
\[\text{subject to} \quad Ax = b, \quad x \in C.\]

\[(CLD) \quad \text{maximize} \quad b^T y \]
\[\text{subject to} \quad A^T y + s = c, \quad s \in C^*.\]

Denote by \(F_p\) and \(F_d\) the primal and dual feasible sets, respectively.
The weak duality theorem shows that a feasible solution to either problem yields a bound on the value of the other problem. We call $c \bullet x - b^T y$ the duality gap.

**Corollary 1** Let $x^* \in \mathcal{F}_p$ and $(y^*, s^*) \in \mathcal{F}_d$. Then, $c \bullet x^* = b^T y^*$ implies that $x^*$ is optimal for (CLP) and $(y^*, s^*)$ is optimal for (CLD).

Is the reverse also true? That is, given $x^*$ optimal for (CLP), then there is $(y^*, s^*)$ feasible for (CLD) and $c \bullet x^* = b^T y^*$?

This is called the **Strong Duality Theorem** and it is “true” for LP, but it is “False” in general cases.
SDP Example with a Duality Gap

\[ \mathbf{c} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{a}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \]

and

\[ \mathbf{b} = \begin{pmatrix} 0 \\ 10 \end{pmatrix}. \]
When Strong Duality Theorems Holds for CLP

**Theorem 2**  (Strong duality theorem) Let $\mathcal{F}_p$ and $\mathcal{F}_d$ be non-empty and at least one of them has an interior. Then, $x^*$ is optimal for (CLP) and $(y^*, s^*)$ is optimal for (CLD) if and only if

$$c \cdot x^* = b^T y^*. $$

There are cases that the duality gap tends to zero but the optimal solution is not attainable.
Theorem 3  (CLP duality theorem) If one of (CLP) or (CLD) is unbounded then the other has no feasible solution.

If (CLP) and (CLD) are both feasible, then both have bounded optimal objective values and the optimal objective values may have a duality gap.

If one of (CLP) or (CLD) has a strictly or interior feasible solution and it has an optimal solution, then the other is feasible and has an optimal solution with the same optimal value.
Optimality Conditions for SDP

\[ c \cdot X - b^T y = 0 \]
\[ AX = b \]
\[ -A^T y - S = -c \]
\[ X, S \succeq 0 \] (1)

\[ XS = 0 \]
\[ AX = b \]
\[ -A^T y - S = -c \]
\[ X, S \succeq 0 \] (2)
Rank of SDP Solutions

At any optimal solution pair \((X^*, S^*)\)

\[ \text{rank}(X^*) + \text{rank}(S^*) \leq n. \]

If the equality holds, they are a strictly complementary solution pair.

There are optimal solutions of \(X^*\) and \(S^*\) such that the rank of \(X^*\) and the rank of \(S^*\) are minimal, respectively.

There are optimal solutions of \(X^*\) and \(S^*\) such that the rank of \(X^*\) and the rank of \(S^*\) are maximal, respectively.

If the SDP problem has an optimizer, then it has an optimizer whose rank \(r\) satisfies \(r(r + 1) \leq 2m\).

Given an optimizer, a such low-rank optimizer can be found in polynomial time.
Potential and Duality Gap in SDP

For any $X \in \text{int } \mathcal{F}_p$ and $(y, S) \in \text{int } \mathcal{F}_d$, let parameter $\rho > 0$ and

$$\psi_{n+\rho}(X, S) := (n + \rho) \log(X \cdot S) - \log(\det(X) \cdot \det(S)),$$

then

$$\psi_{n+\rho}(X, S) = \rho \log(X \cdot S) + \psi_n(X, S) \geq \rho \log(X \cdot S) + n \log n.$$

Then, $\psi_{n+\rho}(X, S) \to -\infty$ implies that $X \cdot S \to 0$. More precisely, we have

$$X \cdot S \leq \exp\left(\frac{\psi_{n+\rho}(X, S) - n \log n}{\rho}\right).$$
Primal-Dual SDP Alternative Systems

A pair of SDP has two alternatives under mild conditions

(Solvable) \[ AX - b = 0 \]

\[ -A^T y + C \succeq 0, \]

\[ b^T y - C \bullet X = 0, \]

\[ y \text{ free, } X \succeq 0 \]

(Infeasible) \[ AX = 0 \]

\[ -A^T y \succeq 0, \]

\[ b^T y - C \bullet X > 0, \]

\[ y \text{ free, } X \succeq 0 \]
An Integrated Homogeneous System

The two alternative systems can be homogenized as one:

\[ (HSDP) \quad \mathcal{A}X - b\tau = 0 \]
\[ -\mathcal{A}^T y + C\tau = s \geq 0, \]
\[ b^T y - C \cdot X = \kappa \geq 0, \]
\[ y \text{ free}, \; X \succeq 0, \; \tau \geq 0, \]

where the three alternatives are

(Solvable) : \quad (\tau > 0, \kappa = 0)

(Infeasible) : \quad (\tau = 0, \kappa > 0)

(All others) : \quad (\tau = \kappa = 0).