Linear Programming

Formulation:
Find vectors \( x = x_1, \ldots, x_n \) such that

\[
\begin{align*}
\text{max } & \quad c'x \\
\text{s.t. } & \quad Ax \leq b \\
& \quad x \geq 0
\end{align*}
\]

\( (A \text{ is an } m \times n \text{ matrix, } c \text{ an } n \text{ vector, } b \text{ an } m \text{ vector}) \)
Example: Leontief input output model

For products 1,...,n, we wish to set production levels $x_1,...,x_n$
so as to maximize our earnings $c_1x_1 + \cdots + c_nx_n$.
To produce we use resources 1,...,m.
To produce a unit of $j$ we need $a_{ij}$ units of i,
$$a_{1j} x_j$$
$$a_{2j} x_j$$
Hence production of $x_j$ requires various resources.
$$a_{mj} x_j$$
Production of $x = x_1,...,x_n$ requires $a_{i1}x_1 + \cdots + a_{in}x_n$ of resource i.
Resources are limited by levels $b_1,...,b_m$.
The resulting LP is:

$$\max \quad c_1 x_1 + \cdots + c_n x_n$$
$$a_{11} x_1 + \cdots + a_{1n} x_n \leq b_1$$
$$\vdots$$
$$a_{m1} x_1 + \cdots + a_{mn} x_n \leq b_m$$
$$x_1 \geq 0 \quad \cdots \quad x_n \geq 0$$
Geometry of LP

In 2 dimensions:

\[
\begin{align*}
\text{max} & \quad 3x_1 + 2x_2 \\
-x_1 & \quad + 3x_2 \leq 12 \\
x_1 & \quad + x_2 \leq 8 \\
2x_1 & \quad - x_2 \leq 10 \\
x_1, x_2, & \quad \geq 0
\end{align*}
\]
Dantzig’s Simplex, 40s.

\[
\begin{align*}
\text{max} & \quad 5x_1 + 4x_2 + 3x_3 \\
2x_1 + 3x_2 + x_3 & \leq 5 \\
4x_1 + x_2 + 2x_3 & \leq 11 \\
3x_1 + 4x_2 + 2x_3 & \leq 8 \\
x_1, x_2, x_3 & \geq 0
\end{align*}
\]

Add slack variables:

\[
\begin{align*}
\text{max} & \quad 5x_1 + 4x_2 + 3x_3 \\
2x_1 + 3x_2 + x_3 + w_1 & = 5 \\
\text{s.t.} & \quad 4x_1 + x_2 + 2x_3 + w_2 = 11 \\
3x_1 + 4x_2 + 2x_3 + w_3 & = 8 \\
x_1, x_2, x_3, w_1, w_2, w_3 & \geq 0
\end{align*}
\]

Write a dictionary:

\[
\begin{align*}
\zeta &= 5x_1 + 4x_2 + 3x_3 \\
w_1 &= 5 - 2x_1 - 3x_2 - x_3 \\
w_2 &= 11 - 4x_1 - x_2 - 2x_3 \\
w_3 &= 8 - 3x_1 - 4x_2 - 2x_3
\end{align*}
\]

In the current solution: \( w_1 = 5, \ w_2 = 11, \ w_3 = 8, \ \zeta = 0. \) It can be improved:

Increasing \( x_1 \) will increase \( \zeta \) at a rate of 5 per unit.
\( x_1 \) **enters the basis**

This will change all the values:

To keep \( w_1 \) positive: \( \Rightarrow x_1 \leq \frac{5}{2} \)

To keep \( w_2 \) positive: \( \Rightarrow x_1 \leq \frac{11}{4} \)

To keep \( w_3 \) positive: \( \Rightarrow x_1 \leq \frac{8}{3} \)

We set \( x_1 = 2.5 \) and we then get \( w_1 = 0 \)

\( w_1 \) **leaves the basis**

We construct the **Next Dictionary**

\[
\begin{align*}
\zeta &= 12.5 -2.5w_1 -3.5x_2 +0.5x_3 \\
x_1 &= 2.5 -0.5w_1 -1.5x_2 -0.5x_3 \\
w_2 &= 1 +2w_1 +5x_2 \\
w_3 &= 0.5 +1.5w_1 +0.5x_2 -0.5x_3 
\end{align*}
\]

The current solution is \( x_1 = 2.5, \ w_2 = 1, \ w_3 = 0.5, \ \zeta = 12.5 \). It can be importved increasing \( x_3 \) will increase \( \zeta \) at a rate of 0.5 per unit.
$x_3$ enters the basis

\[ x_3 \leq \frac{2.5}{0.5}, \leq \frac{1}{0}, \leq \frac{0.5}{0.5} \]

We set $x_3 = 1$ and we then have $w_3 = 0$

$w_3$ leaves the basis

**Pivot on the $(k,l)$ element:**

\[
\begin{align*}
\bar{a}_{kl} &:= \frac{1}{\tilde{a}_{kl}} \\
\bar{a}_{kj} &:= -\frac{\tilde{a}_{kj}}{\tilde{a}_{kl}} \\
\bar{a}_{il} &:= \frac{\tilde{a}_{il}}{\tilde{a}_{kl}} \\
\bar{a}_{ij} &:= \bar{a}_{ij} - \frac{\bar{a}_{kj}\bar{a}_{il}}{\bar{a}_{kl}}
\end{align*}
\]

**Next dictionary**

\[
\begin{align*}
\zeta &= 13  & -w_1 & -3x_2 & -w_3 \\
x_1 &= 2  & -2w_1 & -2x_2 & +w_3 \\
w_2 &= 1  & +2w_1 & +5x_2 \\
x_3 &= 1  & +3w_1 & +x_2 & -2w_3
\end{align*}
\]

**Current solution** $x_1 = 2, w_2 = 1, x_3 = 1, \zeta = 13$.

Solution is optimal.
Simplex Algorithm for LP.

\[ \text{max } \zeta = c'x \]
\[ \text{s.t. } Ax + Iw = b \]
\[ x, w \geq 0 \]

Start from a feasible basic solution. Write a dictionary expressing the basic variables in terms of the non-basics.

\[ \text{max } \zeta = \bar{\zeta} + \bar{c}'x_N \]
\[ \text{s.t. } x_B = \bar{b} - \bar{A}x_N \]
\[ x_B, x_N \geq 0 \]

**Optimality test** If all \( \bar{c} \leq 0 \) the solution is optimal.
If some \( \bar{c}_l > 0 \) (often there will be more than one choice) \( x_l \) enters the basis

calculate \( x_l = \left( \max_i \frac{\bar{a}_{il}}{\bar{b}_i} \right)^{-1} \).
this is the value of the new basic variable.

determine the variable \( x_k : k = \left( \arg \max_i \frac{\bar{a}_{il}}{\bar{b}_i} \right)^{-1} \)

\( x_k \) leaves the basis

Pivot on the element \( (k, l) \).
Geometry of LP - various cases:

**Infeasible problem:**

\[ Ax \leq b \]

No solution exists for
\[ x \geq 0 \]

The feasible region is empty.

**Unbounded problem:**

\[ Ax \leq b \]

Feasible region of
\[ x \geq 0 \]

is unbounded and contains a ray along which the objective increases to infinity.

**Feasible and bounded problem**
we have \( n + m \) variables, \( x_1, \ldots, x_n, w_1, \ldots, w_m \)

The feasible region is a convex polyhedron.
One facet of the polyhedron corresponds to one variable which is nil.

A vertex, corner, extreme point of the polyhedron corresponds to \( n \) variables which are nil.

If more than \( n \) variables are nil, i.e. more than \( n \) facets intersect at one corner, this is a degenerate corner.

The solution at an extreme point (vertex) is determined by the \( m \) that are not \( 0 \).

put differently, by a choice of \( m \) independent columns.
This is a **basic** solution.
Duality

For the problem:

\[
\begin{align*}
\max & \quad c'x \\
\text{s.t.} & \quad Ax \leq b \\
& \quad x \geq 0
\end{align*}
\]

The dual problem is

\[
\begin{align*}
\min & \quad b'y \\
\text{s.t.} & \quad A'y \geq c \\
& \quad y \geq 0
\end{align*}
\]

**Theorem: Weak duality** for any pair of feasible solutions for the primal problem and for the dual problem, \(x, y\) one has:

\[
b'y \geq c'x.
\]

**Proof:** \(b'y \geq x'A'y \geq x'c\)

for the first inequality we use: \(b' \geq x'A', \; y \geq 0\).

for the second inequality we use: \(x' \geq 0, \; A'y \geq c\).

Equality holds if and only if **Complementary Slackness**: 

(a) for each \(i\): \(y_i \times [b_i - (Ax)_i] = 0\)

(b) for each \(j\): \(x_j \times [(A\otimes)_j - c_j] = 0\)

**Theorem: Strong duality** for an LP which is feasible and bounded, \(\max c'x = \min b'y\) and a pair of solutions \(x^*, y^*\) is optimal if and only if it satisfied (a) and (b).

We can read \(x^*, y^*\) from the final dictionary: The optimal \(x^*\) are the \(\widehat{b}\). The optimal \(y^*\) are \(\widehat{c}\).

At all steps: \(\widehat{b}, \widehat{c}\) are complementary slack solutions.
Efficiency of simplex algorithm

Klee & Minty cube:
one can construct a cube in \( n \) space, and distort it, so that the simplex algorithm in its search for the optimal vertex of the cube will step through all the \( 2^n \) corners of the cube.

One can construct such a Klee&Minty cube for every known deterministic version of the simplex.

Gil Kalai has achieved a worst case bound of \( 2^{\sqrt{n}} \) with a randomized simplex - best so far but far from polynomial.

In the late 70’s the Russians Schor, Nemirovsky and Khachyan discovered an external ellipsoid method to solve LP which had polynomial worst case performance. It was very inefficient in practice.

In the early 80’s Karmarkar has devised a polynomial interior ellipsoid method. This motivated further interior point methods which are currently competitive with simplex.

Brogwardt, Smale, Haimowitz, Adler and Megido have shown in the 80’s that for various probabilistic models simplex will require \( n \) to \( n^2 \) pivot iterations on the average.

In practice simplex requires some \( 2n \) pivot iterations. The simplex should be good for prism shaped regions. interior point methods should be good for disco-balls.

New: Spielman & Teng:
Simplex has polynomial smoothed complexity.

Hirsch conjecture: The diameter of a polyhedron with \( n \) facets in \( d \) space (\( n > d \geq 2 \)), is bounded by \( n - d \).