Outline of Lecture

1. Review of Last Lecture
2. Spatio-temporal Point Processes
3. The Spatio-temporal Poisson Process
4. Modeling Interactions
5. Wrapping Up
Where are we?

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Intensity Estimation in Poisson Processes

$\lambda(s)$ can be estimated nonparametrically...
Intensity Estimation in Poisson Processes

or parametrically, such as...

\[ \log \lambda(s) = \mathbf{x}(s)^T \beta \]

To do this, note that the likelihood of \((s_1, \ldots, s_{N(D)}, N(D))\) is:

\[
L(\lambda(\cdot)) = e^{-\int_D \lambda(s) \, ds} \left( \frac{\int_D \lambda(s) \, ds}{N(D)!} \right)^{N(D)} \prod_{i=1}^{N(D)} \frac{\lambda(s_i)}{\int_D \lambda(s) \, ds}
\]

so the log-likelihood is

\[
\ell(\lambda(\cdot)) = -\int_D \lambda(s) \, ds + \sum_{i=1}^{N(D)} \log \lambda(s_i) + \text{constants},
\]

or in terms of \(\beta\)...

\[
\ell(\beta) = -\int_D \exp\{\mathbf{x}(s)^T \beta\} \, ds + \sum_{i=1}^{N(D)} \mathbf{x}(s_i)^T \beta + \text{constants}
\]
Processes may still exhibit **clustering** or **inhibition**.
**Second-Order Intensity**

\[
\lambda(s) = \lim_{|ds| \to 0} \frac{E(N(ds))}{|ds|}
\]

\[
\mu(s) = E(y(s))
\]

\[
\lambda_2(s, s') = \lim_{|ds| \to 0} \frac{E(N(ds)N(ds'))}{|ds||ds'|}
\]

\[
\Sigma(s, s') = E(y(s)y(s')) - \mu(s)\mu(s')
\]

\(\lambda_2\) is called the **second-order intensity**.

A process is stationary if \(\lambda(s) \equiv \lambda\) and \(\lambda_2(s, s') = \lambda_2(s - s')\).
Ripley’s $K$-function

$$K(r) = \frac{1}{\lambda} \mathbb{E}\{\text{events within distance } r \text{ of a randomly chosen event}\}$$

$$= \frac{1}{\lambda} \mathbb{E}\left[ \frac{1}{N(D)} \sum_{i=1}^{N(D)} \sum_{j \neq i} 1\{d(s_i, s_j) \leq r\} \right]$$

This has a natural estimator:

$$\hat{K}(r) = \frac{1}{\hat{\lambda}} \frac{\#\{(i, j) : d(s_i, s_j) \leq r, i \neq j\}}{N(D)}$$

Provides us with a strategy for fitting models:

$$\min_{\theta} \int_0^{r_0} w(r)(\hat{K}(r) - K_\theta(r))^2 \, dr$$
Relationship between $K$ and $\lambda_2$

If the point process is stationary and isotropic,

$$K(r) = \frac{2\pi}{\lambda^2} \int_0^r \lambda_2(r') r' \, dr'$$
Handling Inhomogeneity

- What if $\lambda(s) \not= \lambda$ but is known?
- Then we might hope $\frac{\lambda_2(s, s')}{\lambda(s) \lambda(s')} = \rho(||s - s'||)$ is stationary and isotropic.
- New definitions:

$$K_I(r) = \mathbb{E} \left[ \frac{1}{|D|} \sum_{i=1}^{N(D)} \sum_{j \neq i} \mathbf{1}\{d(s_i, s_j) \leq r\} \right] = 2\pi \int_0^r \rho(r') r' \, dr'$$

$$\hat{K}_I(r) = \frac{1}{|D|} \sum_{i=1}^{N(D)} \sum_{j \neq i} \mathbf{1}\{d(s_i, s_j) \leq r\} \frac{\lambda(s_i) \lambda(s_j)}{\lambda(s) \lambda(s')}$$

- All computations proceed with $K_I$ and $\hat{K}_I$ instead of $K$ and $\hat{K}$. 
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What is a Spatio-temporal Point Process?

We now observe locations and times \((s_i, t_i)\).

```r
library(stpp)
data(fmd, northcumbria)
fmd <- as.3dpoints(fmd)
plot(fmd, s.region=northcumbria)
```

xy-locations

cumulative number
Summarizing spatial and temporal information jointly

We now observe locations and times \((s_i, t_i)\).

plot(fmd, s.region=northcumbria, pch=19, mark=T)
Animations tell the best story

We now observe locations and times \((s_i, t_i)\).

\[
\text{animation}(\text{fmd}, \text{s.region=northcumbria})
\]
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Spatio-temporal Poisson Process

- Model events as occurring in space time with intensity $\lambda(s, t)$.
- If $A \subset D \times [0, T]$, then $N(A) \sim \text{Pois} \left( \int_A \lambda(s, t) \, ds \, dt \right)$.
- How do we estimate $\lambda(\cdot, \cdot)$?
Estimating the Intensity Function

Exactly the same as before!

How can differences between space and time be captured when estimating $\lambda(\cdot, \cdot)$?

What if $\lambda(\cdot, \cdot)$ is separable?

$$\lambda(s, t) = \lambda_s(s)\lambda_t(t)$$

The need for modeling interactions between observations becomes even more acute in the space-time setting.
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Approach 1: $K$-functions

The second-order spatio-temporal intensity function is:

$$\lambda_2((s, t), (s', t')) \overset{\text{def}}{=} \lim_{|ds|, |ds'|, dt, dt' \to 0} \frac{E(N(ds \times dt)N(ds' \times dt'))}{|ds| |ds'| dt dt'}$$

If the point process is stationary and isotropic in space and in time:

$$K(r, h) = \frac{1}{\lambda} \mathbb{E} \left( \text{# events within radius } r \text{ and time } h \text{ of randomly chosen event} \right)$$

$$= \frac{1}{\lambda} \mathbb{E} \left[ \frac{1}{N} \sum_{i=1}^{N} \sum_{j>i} 1\{d(s_i, s_j) \leq r\} 1\{t_j - t_i \leq h\} \right]$$

which can be estimated in the usual way. We also have the relation:

$$K(r, h) = \frac{2\pi}{\lambda^2} \int_0^h \int_0^r \lambda_2(r', h') r' \, dr' \, dh'$$
Approach 1: $K$-functions

- Suppose we have a process that depends on some parameters $\theta$.
- e.g., Parents generate $S \sim \text{Pois}(\mu)$ offspring. Each of the $S$ offspring appear at times $T_i$ according to a Poisson process with rate $r$.
- If we can calculate the theoretical $K$-function $K_\theta$, then we can estimate $\theta$ by solving

$$\minimize_{\theta} \int_0^{r_0} \int_0^{h_0} w(r, h)(\hat{K}(r, h) - K_\theta(r, h))^2$$
Approach 2: Conditional Intensity Function

- In space-time, it’s natural to model the conditional intensity given the past $\mathcal{H}_t = \{(s_i, t_i) : t_i < t\}$.

$$\lambda_c(s, t|\mathcal{H}_t) = \lim_{|ds| \to 0, dt \to 0} \frac{E(N(ds \times dt)|\mathcal{H}_t)}{|ds|dt}$$

- For a Poisson process, $\lambda_c(s, t|\mathcal{H}_t) = \lambda(s, t)$.

- e.g., pairwise interaction model:

$$\lambda_c(s, t|\mathcal{H}_t) = \alpha(t) \prod_{i=1}^{\mathcal{H}_t} h_\theta(s, s_i)$$

1. $h_\theta(s, s_i) = 1 - e^{-\theta||s-s_i||}$
2. $h_\theta(s, s_i) = e^{-\theta||s-s_i||}$
Approach 2: Conditional Intensity Function

We typically estimate the intensity function using maximum likelihood. The log-likelihood of \((s_i, t_i)\) is

\[
\ell(\lambda_c(\cdot, \cdot | \mathcal{H}.) ) = - \int_0^T \int_D \lambda_c(s, t | \mathcal{H}_t) \, ds \, dt + \sum_{i=1}^n \log \lambda_c(s_i, t_i | \mathcal{H}_{t_i}) + \text{constants}.
\]

**NB.** The integral will probably have to be approximated.
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Summary

- Spatio-temporal point processes are a messy and emerging field.
- Poisson processes are not viable models for spatio-temporal processes; must take into account interactions across space-time.
- There are two main approaches for modeling and fitting interaction models: $K$-functions and conditional intensity.
- The stpp package contains many useful routines for visualizing, simulating, and (less so) fitting spatio-temporal point process models.
Homeworks

- Homework 3 due Friday. Don’t forget to upload your files to the website (should be in .csv format).
- Homework 4a will be posted tonight.