Math 51, Homework-2

Section numbers are from the course textbook.

1. Write the parametric equation of the plane that contains the following point and line:

\[
\begin{bmatrix}
1 \\
3 \\
2 \\
5
\end{bmatrix}, \quad \left\{ \begin{bmatrix}
1 \\
4 \\
-2 \\
1
\end{bmatrix} + t \begin{bmatrix}
-1 \\
3 \\
0 \\
4
\end{bmatrix} \middle| t \in \mathbb{R} \right\}.
\]

2. Consider the equation \(ax + by + cz = d\), where \(a, b, c, d \in \mathbb{R}\) are known constants.
   (a) Write \(x\) in terms of \(y\) and \(z\).
   (b) Fill in the blank boxes in the following equation:

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \left\{ \begin{bmatrix}
\square \\
\square \\
\square
\end{bmatrix} + \begin{bmatrix}
\square \\
\square \\
1
\end{bmatrix} y + \begin{bmatrix}
\square \\
\square
\end{bmatrix} \middle| \square, z \in \mathbb{R} \right\}.
\]
   (c) Using the result from the previous part, explain the geometry described by the equation \(ax + by + cz = d\).

3. Recall that for nonzero vectors \(u, v \in \mathbb{R}^n\):

\[
u \cdot v = \|u\|\|v\| \cos(\theta),
\]

where \(\theta\) is the angle between the vectors. Use this to find the angle between the diagonal of the unit square in:
   (a) \(\mathbb{R}^2\) and one of the axes,
   (b) \(\mathbb{R}^3\) and one of the axes,
   (c) \(\mathbb{R}^n\) and one of the axes.
   (d) What happens to this angle if \(n \to \infty\)?

4. Consider the matrix:

\[
A = \begin{bmatrix}
1 & -3 & 2 & 4 \\
6 & 9 & 8 & 5 \\
3 & 4 & 1 & 2
\end{bmatrix}.
\]
(a) What is the size of $A$?
(b) What is element in the 2nd row, 3rd column of $A$?
(c) The transpose of a matrix is another matrix obtained by reversing the rows and columns of the original matrix. Write down the transpose of $A$ which is represented as $A^T$.

5. The solution to the system of equations:

\[
\begin{align*}
2x_1 + 6x_2 + x_3 + 2x_4 &= 5, \\
3x_2 + x_3 + 4x_4 &= 1, \\
3x_2 + x_3 + 2x_4 &= 5
\end{align*}
\]

is given as follows:

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} = \begin{cases} 
\begin{bmatrix} 1/2 \\ -1/3 \\ 1 \\ 0 
\end{bmatrix} + x_3 \begin{bmatrix} -9/2 \\ 3 \\ 0 \\ -2 
\end{bmatrix} | x_3 \in \mathbb{R}
\end{cases}
\]

Describe the geometry of the solution set.

6. Write down all possible $3 \times 2$ reduced row echelon form matrices.

7. Consider two planes in $\mathbb{R}^3$ described parametrically as follows:

\[
\begin{align*}
\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} s \mid s, t \in \mathbb{R} \right\}, \\
\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} k + \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} m \mid k, m \in \mathbb{R} \right\}
\end{align*}
\]

(a) Which plane does not pass through the origin?
(b) Do these planes intersect at all? If yes, where do they intersect?

8. For each system of equations below:

- Express the system as an augmented matrix.
- Use Gaussian elimination to put the system in the reduced row echelon form.
- Identify the pivot and free variables.
- Determine whether the system has one of the following:
  - No solutions.
  - One solution (find it).
  - Infinitely many solutions (express the solution form).
- Describe the geometry of the solution:

(a)

\[
\begin{align*}
x_1 + 2x_2 + 3x_3 &= 1, \\
2x_1 + x_2 - 2x_3 &= 1.
\end{align*}
\]
(b) \[
\begin{align*}
x_1 + 2x_2 - x_3 - x_4 &= 1, \\
2x_1 + 4x_2 - 2x_3 + 3x_4 &= 3, \\
-x_1 + x_2 - 2x_3 + 4x_4 &= 2.
\end{align*}
\]

9. Consider the equation:
\[x_1 + x_2 + x_3 + \cdots + x_n = b,\]
where \(b \in \mathbb{R}\) is a constant.

(a) Write the augmented matrix corresponding to the system.

(b) Identify the pivot and free variables.

10. Exercise 6.11.

11. For the matrix \(A\) and vector \(x\) given below, compute the matrix-vector product \(Ax\) in two ways:

- Writing it as a linear combination of the columns of \(A\),
- Writing it as dot product of rows of \(A\) (expressed as vectors) and \(x\).

\[
A = \begin{bmatrix}
5 & 5 \\
-7 & -1 \\
2 & 3 \\
0 & 1 
\end{bmatrix}, \quad x = \begin{bmatrix}
-2 \\
5 
\end{bmatrix}.
\]

12. Graph the result of multiplying the vector \(x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\) by the following matrices:

\[
A = \begin{bmatrix}
\cos(\theta) & -\sin(\theta) \\
\sin(\theta) & \cos(\theta)
\end{bmatrix}, \quad \theta = \pi/4, \quad \text{and} \quad B = \begin{bmatrix}
4 & 0 \\
0 & 4
\end{bmatrix}.
\]

Explain in words the effect of these matrix multiplications with the vector \(x\).

13. Describe in words the effect of multiplying a vector \(x \in \mathbb{R}^5\) by the matrix \(A\):

\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}.
\]

What happens when you multiply \(x\) by the transpose of \(A\)?

15. Exercise 8.25.

16. Exercise 9.3

[Optional Challenge Problems.]

17. Am. Math. Mon., Feb. 1933: A ship is sailing with velocity vector $v_1$; the wind blows apparently (judging by the vane on the mast) in the direction of a vector $a$; on changing the direction and speed of the ship from $v_1$ to $v_2$, the apparent wind is in the direction of a vector $b$. Find the velocity vector of the wind.

18. Consider three vectors $x_1, x_2,$ and $x_3$. Suppose $x_1$ perpendicular to $x_2$, and $x_2$ is not parallel to $x_3$, then can the vectors $x_1, x_2,$ and $x_3$ form a linearly independent set? Give an example or a counterexample.

19. Consider vector $x \in \mathbb{R}^n$ whose components $x_k$ represent the value of a signal at time stamp $k = 1, 2, \ldots, n$. Assuming $n$ is even, we want to construct a $2 \times$ downsampling signal $y$ of size $n/2$ by multiplying $x$ by an appropriate matrix $A$ such that:

$$y = Ax,$$

where $y_k = x_{2k}$, $k = 1, 2, \ldots, n/2$.

Using $n = 6$, write the elements of matrix $A$.

20. Consider an $m \times n$ matrix $A$ and an $n \times p$ matrix $B$. The product of the two matrices is an $m \times p$ matrix $C$ that is defined as follows:

$$C = AB = [Ab_1, Ab_2, \ldots, Ab_p].$$

(a) Write the elements of the matrix $C$ in terms of dot products rows of matrix $A$ (written in a vertical fashion) and columns of matrix $B$.

(b) Two vectors $x$ and $y$ are said to be orthogonal if $x \cdot y = 0$; that is the angle between them is 90°. Suppose both matrices $A$ and $B$ are of size $n \times n$. Comment on the nature of orthogonality between the rows and columns of the matrices $A$ and $B$ under each one of the following conditions:

$$AB = 0, \ A^T B = 0, \ AB^T = 0, \ A^T B^T = 0.$$

We are looking for a statement like: the columns of $A$ are orthogonal to the columns of $B$ etc.

21. The system shown in the figure below consists of $n$ linear springs that support $n$ masses:

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Sudarsan N.S. Acharya

acharyan@stanford.edu
For \( i = 1, 2, \ldots, n \), the spring stiffnesses are denoted by \( k_i \), the weights of the masses are \( W_i \), and \( x_i \) are the unknown displacements of the masses (measured from the positions where the springs are undeformed). The so-called displacement formulation is obtained by writing the equilibrium equation of each mass and substituting \( F_i = k_i(x_{i+1} - x_i) \) for the spring forces. The result is the following system of equations:

\[
\begin{align*}
(k_1 + k_2)x_1 - k_2x_2 &= W_1 \\
-k_i x_{i-1} + (k_i + k_{i+1})x_i - k_{i+1}x_{i+1} &= W_i, \quad i = 2, 3, \ldots, n-1 \\
-k_n x_{n-1} + k_n x_n &= W_n.
\end{align*}
\]

(a) Write the above set of equations in the matrix–vector form \( \mathbf{Ax} = \mathbf{b} \).

(b) Is there something unique about the structure of the matrix \( \mathbf{A} \) if the spring constants are all equal.