Several examples are discussed of asymptotic results illustrating key features of thin shell behavior and possibly extending the benchmark problems for verification of direct numerical procedures. The basic nature of shell solutions for static loading usually can be categorized as membrane, inextensional, and edge effect. With different boundary conditions or geometry, different solution types dominate, with results for the stress and displacement in the shell differing by orders of magnitude. Approximate results for the nonlinear, large displacement of a spherical shell are shown for point, line, and moment loading, obtained by using an "inverted dimple" and edge effect solutions. Results for the maximum bending and maximum direct stress under a nearly concentrated load are included, which demonstrate the transition from slowly varying behavior to the point load with logarithmic singularity for shells of positive, negative, and zero curvature. It is suggested that a minimum condition for a
or be represented.

1. INTRODUCTION

The analysis of shells always presents a challenge because of difficulties in formulating of the equations and computing solutions. The latter arise because a small parameter, the thickness-to-radius ratio, multiplies the high derivative terms, giving a "stiff" system of equations. For such equations, perturbation or asymptotic methods are successful for a number of special cases and are the basis for the standard design formulas used for many years. Because of various restrictions on such formulas, it is important to have general numerical solution methods, such as those offered by finite element analysis. To ascertain the accuracy of the finite element calculations, a few benchmark problems typically have been used. These problems are good in themselves, but do not properly cover the range of behavior of thin shells. Therefore, in this paper problems are selected to expand the range of benchmark problems and clarify the behavior which can be expected from a shell in different situations. Volumes have been written on the latter aspect, which are hardly represented within the limitations of this paper. Nevertheless there may be some value in this summary of some key features and specific numerical values.
Only the static behavior of the isotropic shell is touched upon. In each area such as dynamic response, composites, and fluid-shell interaction, there are many additional interesting features.

2. GENERAL CHARACTERISTICS OF THIN-SHELL SOLUTIONS

As indicated in the lecture by Calladine (Ref. 5), understanding the fundamental behavior of shells did not come easily, and involved substantial differences in opinion among eminent workers in mechanics such as Love, Lamb, and Rayleigh. These differences were resolved, to a considerable extent, in the 1927 publication of the classic work on elasticity by Love (Ref. 17). Most concepts in this paper can be found at least implicitly in that edition. Later works by Flügge (Ref. 10), Gol'denweiser (Ref. 12), Reissner (Ref. 26), Vlasov (Ref. 39), and Timoshenko and Woinowsky-Kreiger (Ref. 38) greatly extended the knowledge, and the process continues with the more recent works of Axelrad (Ref. 2), Bushnell (Ref. 3), Calladine (Ref. 4), Donnell (Ref. 6), Dowell (Ref. 7), Dym (Ref. 8), Gibson (Ref. 11), Gould (Ref. 13), Kraus (Ref. 15), Niordson (Ref. 21), Seide (Ref. 30), and many others. There is a tendency for this information to be overlooked, with the notion that a proper finite element and enough computing power are adequate to tame any structural system. In fact, the thin shell places a severe demand on both element architecture and computing resources. Some reasons for this are evident in the basic behavior of the shell solutions.

A curious feature is that generally correct behavior of shell solutions was understood long before a satisfactory basic theory was obtained. A
comprehensive treatment of the subject is given by Naghdi (Ref. 20)

For minimal notational complexity, we consider the linear, elastic solution for the shell of revolution, and use a Fourier series in the circumferential angle $\theta$. Because for each harmonic the equations form an eighth order system of ordinary differential equations, the normal displacement component $w$ for the one harmonic can be written as:

$$w(s, \theta) = w_n(s) \cos n\theta = \sum_{j=1}^{9} C_j w_{jn}(s) \cos n\theta$$

(1)

in which $s$ is the meridional arc length, and the coefficient $C_j = 1$ multiplies the particular solution for nonzero surface loads. The remaining constants are the constants of integration which must be determined from the boundary conditions. A dominant parameter for the response of shells is:

$$\lambda = \left( \frac{r_2}{c} \right)^{1/2}$$

(2)

where $c$ is the reduced thickness $t$, and the representative size of the shell is taken as the normal radius of circumferential curvature $r_2$:

$$c = t / \left[ 12 \left( 1 - v^2 \right) \right]^{1/2} \quad R = r_2 = \frac{r}{\sin \phi}$$

(3)

Unlike beam, plane stress, plain strain, or three-dimensional solid problems, the order of magnitude of the constants depends very much on the type of boundary conditions which are applied to the thin shell. The cause is the drastically different
behavior of the different solutions $w_j(s)$. Two of these (say for $j = 1, 2$) are "membrane" solutions, for which the significant stress is "direct", i.e., the average tangential stress through the thickness. The relative magnitude of the displacement and stress quantities is shown in the first column of Table 1, taken from Steele (Ref. 33). This type of solution is sometimes referred to as "momentless"; however, keep in mind that bending and transverse shear stresses are usually present, but at a reduced magnitude. The second type of solution (say for $j = 3, 4$) is "inextensible bending", for which the midsurface strains are nearly zero. This is indicated in the second column in Table 1. The direct stress of the membrane solution and the curvature change of the inextensional solution satisfy the same equation, so these have similar variation over the surface. Specifically these are "slowly varying". As indicated by the first two rows in Table 1, taking the derivative of the displacement, or any other quantity, does not increase the order of magnitude. Thus the inextensional solution is similar to the membrane solution for displacement, bending stress and transverse shear, but has direct stress reduced by four orders of magnitude. The third type of solution is referred to as "edge effect" because it is characterized by exponential decay in magnitude with the distance from a boundary. The distance in which the solution decreases to about 4 percent of the edge magnitude is:

$$\delta = \pi (2 r_c)^{1/2} \approx 2.5 (r_c t)^{1/2} = O(R/\lambda)$$  \hspace{1cm} (4)$$

The relative magnitudes of stress and displacement are indicated by the third column in Table 1. The particular solution for a smooth distribution of
surface load \((j =9)\) is generally similar to the membrane solution.

An anomaly of thin shell behavior is that the load can be carried efficiently over a broad expanse by the membrane solution. However, peak stress occurs in the narrow edge zone, which often is the site of failure initiation. To capture properly the edge effect with finite elements using polynomials of low order, a mesh spacing no greater than about \(d/10\) must be used. For static problems, the elements can be kept to a reasonable number by using a fine mesh near the boundaries and a course mesh elsewhere. For problems of vibration and wave propagation, significant bending waves occur everywhere in the shell which have the wave length of the order of magnitude of \(d\). A proper solution requires a fine mesh of elements everywhere. Nonlinear material and geometric behavior can be significant in distances small in comparison with \(d\), requiring even a finer mesh of elements.

The preceding discussion concerns the relative sizes of the components in a single column of Table 1. The total solution is the sum of the three columns, for which the relative magnitudes between columns is important. This depends strongly on the type of boundary conditions prescribed for the shell.

2.1 Displacement Boundary Conditions

The edge effect makes a small contribution to the meridional and circumferential components of displacement; thus, the four constants of the membrane and inextensional solutions are used to satisfy these conditions with constants \(O(1)\). The edge effect solutions then are used to satisfy the conditions on the normal displacement and rotation, also with
constants $O(1)$. Thus, the contribution of each type of solution to the normal component of displacement is the same order of magnitude, producing the relative magnitudes both within each column and between the first three columns of Table 1. An important conclusion is that the edge effect stress is the same order of magnitude as the interior membrane solution. The bending stress of the inextensional solution is negligible. In this situation, the shell is a very efficient structure for spanning space and carrying load.
Table 1 - Relative order of magnitude of stress and displacement components of basic shell solutions. The parameter \( l \) is the square root of the ratio of radius to reduced thickness, and \( R \) is a representative radius of the surface. For displacement boundary conditions, as indicated by the first three columns, the shell generally carries the load efficiently by membrane stress, with edge effect stresses of the same order of magnitude. In contrast, for pure stress boundary conditions, as indicated in the last three columns, the load is carried by inextensional bending, with high amplitude of stress and displacement.

<table>
<thead>
<tr>
<th></th>
<th>Displacement BC</th>
<th>Stress BC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Me</td>
<td>I</td>
<td>E</td>
</tr>
<tr>
<td>m - x</td>
<td>e - f</td>
<td>e - n</td>
</tr>
<tr>
<td>b - t</td>
<td>E</td>
<td>E</td>
</tr>
<tr>
<td>a - e</td>
<td>f</td>
<td>E</td>
</tr>
<tr>
<td>n - e</td>
<td>c</td>
<td>e</td>
</tr>
<tr>
<td>Rotational</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Normal disp w/R</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Merid disp u /R</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

\( l \) represents the load level.
* Denotes items linearly dependent in the leading terms

2.2 Stress Boundary Conditions

For prescribed stresses on the boundaries of beams, plates, and three-dimensional solids, the stress boundary value problem is just the inverse of the displacement problem, with the same order of magnitude of stress and displacement. This is definitely not the case for the thin shell, for which
the matrix for computing the constants of integration is nearly singular. This can be understood if one attempts the same reasoning as used before for prescribed displacement, using the first three columns of Table 1. With all constants the same order of magnitude, the inextensional and edge effect provide small contribution to the meridional and shear stress. However, the membrane solution does not have enough constants to satisfy both stress components at the two edges. Consequently, the constants of integration cannot have the same order of magnitude. Because of the linear dependence of components in the edge effect, the result is that the constants of the inextensional and edge effect solutions must be large \((C_i, C_j = O(l^{1/4}), C_k - C_l = O(l^{3/4})),\) producing the magnitudes shown in the last three columns of Table 1. The total state of stress and displacement is dominated by the inextensional solution. The difference in boundary conditions is summarized by the following: prescribed displacements \(O(1)\) produce stress \(O(1)\); prescribed edge stress \(O(1)\) produces displacements of order \(O(l^{1/4})\) and interior bending stress \(O(l^{3/4})\). This is why the simple inextensional solution of Rayleigh for the first modes of vibration of a hemisphere is quite accurate. Also quite accurate (if the shell is not too long) is the solution in Timoshenko and Woinowsky-Kreiger (Ref. 38) for the cylinder with free ends and loaded by pinch concentrated forces in the center. The asymptotic results for all edge stiffness and the inverse flexibility coefficients for shells of positive, negative and zero curvature are given by Steele (Ref. 34). From these, it is easy to see that the inextensional solutions for a pinch load will be accurate for the displacement because of rapid convergence of the Fourier series solution. The
Fourier series representation is not convergent for the stress, because the stress under a point load has a logarithmic singularity.

Another interpretation of the behavior is that the properly stiffened shell has stress and displacement similar to that in plane stress or axial loading of a straight bar (stiffness = \( Et \)), while the shell with free edges has stress and displacement similar to that of transverse bending of a beam or plate (stiffness = \( E t (l^4) \)).

Note that in the inextensional solution the direct stress resultant tensor is not symmetric; however, the direct stress is virtually negligible compared to the bending stress.

2.3 Mixed Boundary Conditions

For mixed boundary conditions various possibilities exist for combinations of the behavior discussed in the preceding two sections. The simple rule of thumb is that if the tangential displacement conditions permit an inextensional solution, this will, indeed, dominate the total solution. As discussed in many of the references cited, the details depend strongly on the Gaussian curvature of the shell, because the equations for the membrane and inextensional behavior are elliptic, parabolic and hyperbolic for surfaces which are elliptic, parabolic and hyperbolic, respectively.

3. Axisymmetric Loading Examples

For the static, axisymmetric deformation of a shell of revolution, as well as for the first nonaxisymmetric harmonic \( \cos q \), the problem simplifies considerably, because the inextensional solution
consists of rigid body displacements. Thus, only columns 1 and 3 in Table 1 are of concern. The membrane solution is often obtained by simple static equilibrium considerations, while the most relevant information on the edge effect is the knowledge of the decay distance (Eq. 4) and the relation between the edge stress and displacement quantities, i.e., the edge stiffness coefficients.

\[ rM = K \cdot h \approx E t c \left( \frac{2 cr \sin \phi}{\sqrt{2}} \right) \left( \begin{array}{c} \chi \\ h \end{array} \right) \]

The two-term approximation for the coefficients is in Steele and Skogh (Ref. 35), which is:

3.1 Edge Stiffness Coefficients

Figure 1 indicates the shell of revolution with the edge quantities: radial force \( H \), radial displacement \( h \), moment \( M_s \) and rotation \( c \), which can be related by the edge stiffness matrix. The first approximation, i.e., with the neglect of terms \( O(l^2) \), is:
\[ k_{11} = Et c \sqrt{2 \pi r_1 \sin \phi} \left[ 1 + \nu \cot \phi \left( \frac{2}{r_2} \right)^{1/2} + O\left( \lambda^{-2} \right) \right] \]

\[ k_{12} = k_{21} = Et c \left[ 1 + \left( \frac{1}{4} + \frac{r_2}{4 r_1} + \nu \right) \cot \phi \left( \frac{2}{r_2} \right)^{1/2} + O\left( \lambda^{-2} \right) \right] \tag{6} \]

\[ k_{22} = k_{12} \frac{2}{\sqrt{2 \pi r_1 \sin \phi}} \left[ 1 + O\left( \lambda^{-2} \right) \right] \]

in which \( r_1 \) is the meridional radius of curvature and \( r_2 = r / \sin f \). These approximations are accurate when the correction term in brackets is less than about 0.2, which covers a wide range of shell geometry, but excludes the vicinity of the apex of toroid and sphere, where \( \sin f = 0 \). Of interest is that the first term depends only on the meridional slope at the edge, while the correction terms bring in the effect of the meridional curvature.

The effect of transverse shear deformation and membrane prestress can be added:

\[ k_{11} = (k_{11})_6 \frac{f \sqrt{f + \rho + \beta / 2}}{f + \beta} \]

\[ k_{12} = (k_{12})_6 \frac{1}{f + \beta} \]

\[ k_{22} = (k_{22})_6 \frac{\sqrt{f + \rho + \beta / 2}}{f + \beta} \tag{7} \]

\[ f = \sqrt{1 + 2 \rho \beta} \]

The subscript 6 denotes the expressions in Eq. 6, the prestress factor is:
\[ \rho = \frac{N_s r_2}{2Et c} \]  

(8)

and the transverse shear factor is:

\[ \beta = \frac{Ec}{G_t r_2} \]  

(9)

in which \( G_t \) is the equivalent transverse shear modulus. For a thick shell or a composite with a relatively soft matrix, \( \beta \) is significant.

When the meridional membrane stress increases in tension, \( r \) becomes more positive and the stiffness coefficients increase in magnitude. For compression, the stiffness approaches zero as the critical value is approached:

\[ \rho_{cr} = -\left[ 1 - \beta / 2 \right] \]  

(10)

Eq. 10 provides the "classical" buckling load. The inverse matrix of flexibility coefficients has a singularity when \( r \) approaches the value which is half of Eq. 10. Thus the free edge has an instability at one-half of classical buckling load, as pointed out by Hoff and Soong (Ref. 14) for the cylindrical shell.

The first geometric nonlinearities can be obtained from the Reissner equations for moderate rotation theory. Results for the edge coefficients are in Ranjan and Steele (Ref. 25). As an example, the term providing the edge moment for a prescribed rotation with the radial displacement fixed, has the approximation:

\[ k_{11} = (k_{11})_6 \left[ 1 - \frac{3 \chi}{10} \cot \phi \right] \]  

(11)

This nonlinearity is of the softening type.
3.2 Pressure Vessel With Clamped Edge

Certainly, an important shell structure is the pressure vessel. Despite the fact that the shell generally carries the load by membrane action, the maximum stress concentration occurs at the attachments with stiffeners. Because these points are often the site for failure initiation, it would seem that a first requirement is the proper computation of such a region. For example, the rigidly clamped edges in Fig. 2 are considered. The membrane solution is well known. To this the edge effect must be added, which must have radial displacement and rotation which cancel those of the membrane solution. Thus Eq. 5 readily gives the simplest, one-term approximation for the edge meridional bending stress:

\[ \sigma_B = \frac{6M_s}{r^2} \approx \sigma_D \left( \frac{3}{1 - \nu^2} \right)^{1/2} \left( 2 - \frac{r_2}{r_1} - \nu \right) \]  

in which the reference meridional membrane (direct) stress is the familiar value:

\[ \sigma_D = \frac{p r_2}{2t} \]  

and the error is given by the correction terms in Eq. 6. Thus, for \( n = 0.3 \), Eq. 12 gives the bending stress factors:

\[ \sigma_B = \sigma_D \times \begin{cases} \ 1.27 & \text{spherical cap} \\ \ 3.10 & \text{cylinder} \end{cases} \]  

which is a specific demonstration of the \( O(1) \) edge effect in Table 1, column 3.
Fig. 2 – Cylindrical and spherical pressure vessels with rigid plates clamped to edges. These are examples of $O(1)$ edge effects for prescribed edge displacements.

3.3 Pressure Vessel With Free Edge

The result for prescribed edge loading of the shell is generally not as benign as for constrained edges. From column 3 in Table 1, it is seen that a prescribed transverse shear stress will cause edge effect stresses of the magnitude $O(l)$ larger. A demonstration is the spherical cap on "ice" with external pressure in Fig. 3. At the edge, the moment and radial force are zero. Thus, the edge effect solution must cancel the radial component of the membrane solution. The two-term approximation (Eq. 6) gives the radial edge displacement $h$ and vertical displacement $v$ at the center of the cap:

$$h = h_{ref} \left[ \frac{(\lambda / \sqrt{2}) \sin 2\phi - 1 + \nu \sin 2\phi}{(1 + \beta)} \right]$$

$$v = h_{ref} \left[ \frac{(\lambda / \sqrt{2}) \tan \phi \sin 2\phi + (1 + \nu) \tan \phi + \left( \frac{1 - \nu}{\sin \phi} \right) \left( \frac{1 - \cos \phi}{\sin \phi} \right)}{(1 + \beta)} \right]$$

(15)
\[ h_{\text{ref}} = \frac{p R^2 \sin \phi}{2 E t} \]

in which \( \phi \) is the angle at the edge. Specific results are in Table 2 for a fixed edge angle. Fig. 3 demonstrates the distribution of stress and displacement for \( R / t = 100 \). The one-term asymptotic results in Table 2 are computed from the leading term \( O(1) \) in Eq. 15, while the two-term results are computed from all terms in Eq. 15, including the transverse shear deformation term for an isotropic material. A comparison of the stress factor for the constrained edge (Eq. 14) with Table 2 indicates the penalty paid for inadequate stiffening of a shell.

Results from FAST1 are also in Table 2. This is a prototype computer program combining asymptotic and numerical methods. In the solution algorithm, if an error condition is satisfied, the asymptotic solution is used directly, with some direct numerical integration. Otherwise, the shell is divided into sections in which a power series solution is used. For the \( R / t = 10 \) results in Table 2, the FAST1 program is using primarily the power series.

3.4 Pressure Vessel With Slope Discontinuity

The same behavior as in Fig. 3 occurs in a shell with a discontinuity in the meridional slope. Such a discontinuity causes a discontinuity in the radial force resultant of the membrane solution. Thus, the edge effect solutions must have a radial force discontinuity of the opposite sign and same magnitude. Again from Table 1, column 3, it is seen that this produces the displacement and stress
Table 2 – Radial and vertical displacements for spherical cap with external pressure and without radial or moment constraint at the edge $f = p / 4$. The one- and two-term asymptotic results and the calculation with FAST1 are shown. The reference displacement is the membrane displacement (with $n = 0$). The stress concentration factors are similar in magnitude to these displacement factors.

<table>
<thead>
<tr>
<th>$R / t$</th>
<th>Dis $p$</th>
<th>1-ter $m$</th>
<th>2-ter $m$</th>
<th>FAS T1</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 $h / h_{ref}$</td>
<td>4.0 5</td>
<td>4.0 3</td>
<td>3.4 0</td>
<td></td>
</tr>
<tr>
<td>10 $v / h_{ref}$</td>
<td>4.0 5</td>
<td>7.0 2</td>
<td>7.2 5</td>
<td></td>
</tr>
<tr>
<td>100 $v / h_{ref}$</td>
<td>12. 8</td>
<td>12. 3</td>
<td>11. 9</td>
<td></td>
</tr>
<tr>
<td>100 $h / h_{ref}$</td>
<td>12. 8</td>
<td>14. 7</td>
<td>15. 3</td>
<td></td>
</tr>
<tr>
<td>100 $v / h_{ref}$</td>
<td>40. 5</td>
<td>39. 7</td>
<td>39. 5</td>
<td></td>
</tr>
<tr>
<td>100 $h / h_{ref}$</td>
<td>40. 5</td>
<td>42. 2</td>
<td>44. 5</td>
<td></td>
</tr>
</tbody>
</table>
Fig. 3 – Spherical cap loaded by external pressure without radial constraint at edge. The $O(l)$ edge effect dominates both the displacement shown on the left and the stress shown on the right. The edge effect is significant in the decay distance $d$. ($S_{th OD} = $ stress in $q$ - direction on the outer surface, etc.)

$O(l)$ larger than those of the membrane solution. Using the coefficients (Eq. 6), the asymptotic result for the bending stress at such a discontinuity is in Steele and Skogh (Ref. 35). The one-term asymptotic approximation for the rotation and radial displacement at the intersection, with equal shell materials and thicknesses on the two sides of the discontinuity, is:

$$
\chi \approx 0
$$

(16)

$$
\eta \approx \frac{r \sqrt{2 \bar{c} \Delta \bar{H}_p}}{E tc \Delta g_2}
$$

and the direct stress at the intersection is:
\[ \sigma_{\theta D} = \frac{N_\theta}{t} \approx \frac{1}{g_2} \sqrt{\frac{R}{2c}} \Delta H_p \]  

(17a)

while the bending stress at the intersection is:

\[ \sigma_{s B} = \frac{6M_s}{t^2} \approx \sqrt{\frac{3}{1-\nu^2}} \frac{1}{g_2} \sqrt{\frac{R}{2c}} \frac{\Delta H_p}{t} \]  

(17b)

in which the discontinuity in slope enters into the factors:

\[ g_1 = \left(\sin \phi^{(1)}\right)^{1/2} + \left(\sin \phi^{(2)}\right)^{1/2} \]  

(18)

\[ g_2 = \left(\sin \phi^{(1)}\right)^{-1/2} + \left(\sin \phi^{(2)}\right)^{-1/2} \]  

and into the discontinuity of the membrane radial force \( D H_p \). For a pressure vessel, this term is:

\[ \Delta H_p = \frac{pR}{2} \left(\cot \phi^{(2)} - \cot \phi^{(1)}\right) \]  

(19)

where the \( \phi^{(1)} \) and \( \phi^{(2)} \) indicate the values of \( \phi \) on the two sides of the discontinuity and \( p \) is the pressure. Equations 16 and 17 are the one-term approximation. However, an interesting feature is that the second term correction to these is identically zero. Thus the error is \( O(l^{-2}) \).

An example of slope discontinuities is in the pressure vessel in Fig. 4. For simplicity, the thickness is taken to be constant in all the shell segments. The severe penalty of the slope discontinuities is evident. Similar examples are in many of the references. Usually, a stiffening ring or
a smooth knuckle is added to alleviate the stress at such points. Nevertheless, shells continue to be designed and built with such discontinuity points which are not sufficiently stiffened. Perhaps contributing to this is the lack of sufficient warning in design documents and the fact that a coarse mesh used in either a finite element or a finite difference calculation will grossly underestimate the peak stresses.

![Diagram of pressure vessel with meridional slope discontinuity at points C and D.](image)

**Fig. 4** — Pressure vessel with meridional slope discontinuity at points C and D. The stress distribution for a moderately thin shell \((R_s/t = 100)\) is on the right. The \(O(1)\) stress concentration at the points of slope discontinuity is clear, in comparison with the \(O(1)\) discontinuity at the points B and E of curvature discontinuity. Dimensions are: \(R_{s1} = 20, L_{c1} = 20, R_{s2} = 200, R_{s3} = 100, L_{c2} = 84, f_1 = \frac{p}{6}, f_2 = 0.1, E = 0.3 \times 10^8, n = 0.3, t = 1, p = 10.\)

**Table 3** — Meridional bending stress at points of slope discontinuity of pressure vessel in Fig.
4. The asymptotic result is reasonable even for the shallow region at point C.

<table>
<thead>
<tr>
<th>Thickness $t$</th>
<th>Pressure $p$</th>
<th>Point</th>
<th>2 - term</th>
<th>FAS T1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>C</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>49</td>
<td>83</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>10$^{-1}$</td>
<td>10$^{-1}$</td>
</tr>
<tr>
<td>0.1</td>
<td>1</td>
<td>C</td>
<td>0.7</td>
<td>0.7</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>89</td>
<td>83</td>
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<td></td>
<td></td>
<td></td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>10$^{-1}$</td>
<td>10$^{-1}$</td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>65</td>
<td>64</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>10$^{-5}$</td>
<td>10$^{-5}$</td>
</tr>
</tbody>
</table>

3.5 Dimpling of Shell

The analysis of large displacement of thin shells is generally a difficult problem. Guidance can be found in Love's "principle of applicable surfaces" (Ref. 17). Higher levels of strain energy are required for deformations involving extensional distortion of the reference surface, than are required for inextensional bending. Consequently, even elaborate post-buckling patterns are usually characterized by regions of inextensional deformation.
joined by lines of localized bending. Ashwell (Ref. 1) uses this concept for the problem of a spherical shell with a static radial point load, for which the geometry is quite simple. He showed that a good approximation can be obtained for the large displacement behavior. The procedure consists of assuming that a portion of the sphere, a "cap", undergoes a curvature reversal and becomes a "dimple". Continuity of stress and displacement is obtained by solutions of the linear equations for the inverted cap and the remaining portion of shell. Ashwell uses the Bessel function solutions of the shallow shell equations (Ref. 26). Ranjan and Steele (Ref. 24) show that a much simpler solution can be obtained by using the approximate edge stiffness coefficients (Eq. 6). Furthermore, the accuracy is increased by including the geometric nonlinearity from a perturbation expansion of the moderate rotation equations by Ranjan and Steele (Ref. 25).

3.5.1 Displacement Under Point Load. A spherical shell under a point load \( P \) has the displacement \( x \) according to the linear solution by Reissner (Ref. 26) given by:

\[
\frac{x}{t} = \frac{PR}{8Et^2c} = \frac{p^*}{t}
\]  

(20)

A simple solution is sought which will capture the significant features of the nonlinear behavior. Additional efforts are by Ranjan and Steele (Ref. 24) and Libai and Simmonds (Ref. 16) to improve the solution. A convenient form is obtained with the assumption that the edge angle \( a \) of the inverted cap, or dimple, shown in Fig. 1, is both "shallow":
and "steep":

\[ \sqrt{\frac{R}{c}} \alpha >> 1 \]  
(22)

When the shell is thin enough, there is a range of angles \( \alpha \) which satisfy both Eqs. 21 and 22.

Adding the inverted dimple modifies the potential energy. The dominant terms are external force potential and the strain energy of bending of the edges of the dimple and the remaining shell to obtain continuity of the meridional slope. Note that the angle of rotation \( c \) of each shell edge is equal to the original angle \( \alpha \), which violates the assumptions of the linear theory. Ignoring this contradiction for the moment, we obtain the approximate potential as

\[ U(\alpha) = 2 \pi E t c^2 \sqrt{\frac{2R}{c}} \alpha^3 - P R \alpha^2 \]  
(23)

For equilibrium, the potential must be stationary with respect to a change in the edge angle \( \alpha \). This
provides the relation between the edge angle and the load magnitude \( P \). Adding the linear result to the additional displacement because of the dimpling gives the total displacement:

\[
\frac{X}{l} = P^* + K \left( P^* \right)^2
\]  

(24)

where the constant is:

\[
K = \left( \frac{8}{3\pi} \right)^2 \sqrt{3 \left( 1 - \nu^2 \right)} = 1.19 \quad \text{for } \nu = 0.3
\]  

(25)

To at least partially relieve the contradiction of the large rotation, the nonlinear correction (Eq. 11) can be used. Integrating to obtain the energy, and setting the rotation equal to the angle \( \alpha \) provides a 4/5 reduction in the strain energy and an increase in the constant:

\[
K_{NL} = K \left( \frac{5}{4} \right)^2 = 1.86 \quad \text{for } \nu = 0.3
\]  

(26)

Using Eq. 26 in Eq. 24 gives a result which agrees well with the experiments of Penning and Thurston (Ref. 23) and the numerical results of Fitch (Ref. 9), all on very thin spherical shells for displacement magnitude up to about fifteen shell thicknesses. More remarkable are the results of Taber (Ref. 37), showing that the equivalent of Eq. 23 provides good agreement with experiments on a thick shell ( \( R/t = 7 \)). In addition, Taber considered the problem of a shell filled with an incompressible fluid, for which the strain energy of the wall extension because of the internal pressure must be added. The agreement between calculation and experiment is reasonable for
displacements in magnitude up to about half the radius.

Simo, et. al (Ref. 32), with resultant based finite elements, find excellent agreement with Taber's results when transverse shear deformation is neglected. In computation with shear deformation included (Simo, pers. comm.) the agreement is not as good. It is possible that Taber chose a value of the elastic modulus for the shell (a racket ball) to obtain the good agreement. The adjustment for shear deformation given by Eq. 7 is:

\[ K_{NL+\text{shear def}} = K \left( \frac{5}{4} \right)^2 \left( 1 + \beta \right)^2 \]

\[ \frac{1}{1 + \beta / 2} \]

\[ (27) \]

\[ = 2.18 \quad \text{for} \ n = 0.3, \ R / t = 7 \]

Thus, the shear deformation for such a thick shell should increase the displacement by 17 percent. However, for the thick shells the conditions in Eqs. 21 and 22 cannot be satisfied. So any precise agreement with reality should not be expected from Eq. 24. Another objection to this treatment is that in the experiments and numerical computation by Fitch (Ref. 9), bifurcations to nonsymmetric patterns occur. However, a substantial load loss with such bifurcations apparently does not take place, so the symmetric solution remains a good approximation. This analysis is restricted to dimples more than a decay distance from any boundary, for which the displacement increases with the load. When the decay distance encounters a ring-stiffened edge, snap-through buckling can occur, as discussed by Penning and Thurston (Ref. 23).
3.5.2 Displacement Under Pressure. With external pressure loading \( p \), the volume displacement is needed for the total potential change:

\[
U(\alpha) = \frac{4}{5} 2\pi E t c^2 \sqrt{\frac{2R}{c}} \alpha^3 - p \frac{\pi}{4} R^3 \alpha^4
\]  

(28)

in which the nonlinear reduction factor of \( 4/5 \) is included. Because the potential now has a higher power of the angle in the load term, the derivative of this with respect to the angle gives an angle which is inversely proportional to the pressure. This dimple gives, therefore, an unstable equilibrium curve. The volume displacement \( D V \) is:

\[
\frac{\Delta V}{8pR^2c} = (1 - \nu) \rho + \frac{c}{R} \left( \frac{3\sqrt{2}}{5} \rho \right)^4
\]  

(29)

in which the prestress factor (Eq. 8) is

\[
\rho = - \frac{p R^2}{4 E t c}
\]  

(30)

The classical bifurcation occurs at \( r = -1 \); Eq. 29 gives the post-buckling curve for values of \( r \) small in comparison with unity.

3.5.3 Displacement Under Line Load. With a line load of intensity \( q \) the cross-sectional area displacement is needed for the total potential:

\[
U(\alpha) = \frac{4}{5} 2\pi E t c^2 \sqrt{\frac{2R}{c}} \alpha^3 - q \frac{4}{3} R^2 \alpha^3
\]  

(31)

For the line load, the same power of the angle \( a \) occurs in the strain energy and load terms. Thus, the
A nontrivial solution is a state of neutral equilibrium. Setting Eq. 31 to zero gives the magnitude of the critical line load to be:

\[ q_{cr} = \frac{4}{5} \frac{3 \pi E t c}{2} \sqrt{\frac{2c}{R}} \]  

(32)

It is of interest to calculate the linear solution for the concentrated line load at the equator of the sphere. The bifurcation estimate obtained by setting Eq. 8 equal to unity is only 6 percent higher than Eq. 32. The conclusion is that the line load will not be imperfection sensitive, because neutral equilibrium is maintained as the dimples form when the load is near the classical bifurcation value. The significant condition on the edge effect solution is a radial line load in such problems as the shell without radial constraint in Fig. 3 and the regions of slope discontinuity in Fig. 4. The conclusion is that the local instability in such problems also will not be imperfection sensitive, so that the classical bifurcation load will be a good indication of the actual load capability.

Fig. 6 – Dimple in Spherical Shell with Concentrated Moment.
Fig. 7 — Large-amplitude behavior indicated by analysis of dimple formation in a spherical shell with different types of loading. The unstable equilibrium path for pressure indicates high imperfection sensitivity.

3.5.4 Rotation Under Concentrated Moment. With the moment of intensity $M$ acting at a point on the spherical shell, the dimple will form as indicated in Fig. 6. The total potential is:

$$U(\alpha) = \frac{4}{5} 2 \pi E t c^2 \sqrt{\frac{2 R}{c}} \alpha^3 - M 2 \alpha$$

which yields the rotation in terms of the applied moment:

$$2 \alpha = \left(\frac{5 M}{3 p E t c \sqrt{2 R c}}\right)^{1/2}$$

which is a stiffening nonlinearity. A summary of the various types of response is shown in Fig. 7.

3.5.5 Nonlinear Material Behavior. The edge moment calculated at the edge of the dimple region
from Eq. 5 indicates that the yield stress of a metallic shell with moderate \( R/t \) values will be exceeded before the dimple becomes very large. For a rough indication of the behavior, an elastic-plastic, moment-curvature relation can be considered. As the dimple increases, each point on the sphere wall will experience, first, a substantial curvature increase and then, a reversed bending to reach the final curvature of opposite sign in the inner dimple region. If a reversible, nonlinear elastic material is considered, the analysis follows the same line of reasoning as used in the preceding sections. The significant strain energy is just that to bend the edge of the dimple and the edge of the outer portion of the shell, each through the angle of \( \alpha \). The total potential for the point load is just:

\[
U(\alpha) = M_u 2 \pi R \alpha (2 \alpha) - PR \alpha^2
\]

(35)

in which \( M_u \) is the yield moment of the shell wall. Now both terms have the same power of \( \alpha \), so the equilibrium is neutral, with the critical collapse value of the load:

\[
P_{cr} = 4 \pi M_u
\]

(36)

If the total work \( W \) of the external force is prescribed, the maximum displacement is:

\[
x = \frac{W}{4 \pi M_u}
\]

(37)

Following the similar analysis for the other load cases indicates that both the line load and pressure load will have unstable post-buckling equilibrium paths. Experiments and computations on the residual
permanent deformation because of impact of spherical shells are reported by Witmer, et. al (Ref. 40). The estimate in Eq. 37 exceeds their values by a factor of 3, which is not surprising because Eq. 35 does not consider the actual plastic work. However, it is curious that Eqs. 36 and 37 are independent of the shell geometry. More extensive study of this problem using the dimple may be warranted.

4. NONSYMMETRIC LOADING EXAMPLES

For general loading of a general surface, the inextensional solution in column 2 (and column 5) of Table 1 plays a significant role. The membrane and edge effect solutions, demonstrated in the axisymmetric problem, retain their importance in the general problem. The following examples illustrate some main features for the linear solutions without transverse shear deformation.

4.1 Spherical Shell With Edge Loading

For one term of the Fourier series expansion (Eq. 1), the edge loading is similar to that in Fig. 8a. Adding the terms of the series provides the solution for general loading, such as the point loads in Fig. 8b.
The coefficients of the force and displacement which can be prescribed on an edge may be placed into the "force" and "displacement" vectors:

\[
F = \begin{bmatrix}
M_{sn}/Et \\
\lambda Q_{sn}/Et \\
N_{sn}/Et \\
N_{s\theta n}/Et
\end{bmatrix}
\]

\[
D = \begin{bmatrix}
\chi_n/\lambda \\
w_n/r_2 \\
\lambda u_n/r_2 \\
\lambda v_n/r_2
\end{bmatrix}
\]

which are related by the edge stiffness matrix:
The stiffness matrix provides the solution for the edge tractions because of prescribed edge displacements. The result in Steele (Ref. 34) for this stiffness matrix for a spherical cap is:

\[
K = \begin{bmatrix}
\sqrt{\gamma} & \bullet & \bullet & \bullet \\
1 & \sqrt{\gamma} & \bullet & \bullet \\
(f / 2 + \eta) \Omega & (f / 2 + \eta) \Omega \sqrt{\gamma} & \frac{f \Omega}{2(1 + \nu)} & \bullet \\
(1 - f / 2) \Omega & (1 - f / 2) \Omega \sqrt{\gamma} & -\frac{f \Omega}{2(1 + \nu)} & \frac{f \Omega}{2(1 + \nu)}
\end{bmatrix}
\]

in which only the lower off-diagonal coefficients are shown because this is must be a symmetric matrix, and the terms not previously defined are:

\[
\eta = \frac{\cos \phi}{n} \quad \Omega = \frac{n \sqrt{\gamma / r}}{c} \\
f(\phi, n) = \frac{\tan^{2n}(\phi/2)}{2n \sin^2 \phi} g(\phi, n) = \frac{2(n^2 - 1)}{2n(n + \cos \phi) - \sin^2 \phi} \\
g(\phi, n) = \int_{0}^{\phi} \frac{\tan^{2n}(\phi/2)}{\sin^3 \phi} d\phi = \frac{2n(n + \cos \phi) - \sin^2 \phi}{4n(n^2 - 1) \sin^2 \phi}
\]

(The integral was obtained by Rayleigh.) Equation 40 is an asymptotic approximation, valid for the low harmonics when \( \bar{w} < 0.3 \). As indicated by the first three columns of Table 1, the edge effect has little effect on the direct stress components, the third and fourth components of \( \mathbf{F} \). Therefore, the lower right coefficients in Eq. 40 provide an
excellent check on the proper computation of the membrane behavior. The upper left coefficients in Eq. 40 are primarily because of the edge effect, while the off-diagonal blocks of coefficients depend on all three types of solutions.

If the edge stress resultants are prescribed for a spherical cap, then the edge displacements are given by the flexibility matrix $E$:

$$D = E \cdot F$$

(41)

The asymptotic approximation for $W < 0.3$, obtained using only the inextensional solution, is:

$$E = f_5$$

$$= \begin{bmatrix}
(1 + \eta)^2 \Omega^{-1} & \cdot & \cdot & \cdot \\
-(1 + \eta)^2 \Omega^{-2} & (1 + \eta)^2 \Omega^{-3} & \cdot & \cdot \\
(1 + \eta) \Omega^{-3} & -(1 + \eta) \Omega^{-4} & \Omega^{-5} & \cdot \\
(1 + \eta) \Omega^{-3} & -(1 + \eta) \Omega^{-4} & \Omega^{-5} & \Omega^{-5}
\end{bmatrix}$$

(42)

in which

$$f_5 = f_5(\phi, n) = \frac{f(\phi, n)}{2(1 - \nu)(1 - n^{-2})^2}$$

As previously mentioned and indicated in Table 1, columns 3-6, the inextensional solution dominates the stress boundary value problem. The maximum edge stress is due to circumferential bending, which is given by:

$$\frac{M_{\theta n}}{Et} = (1 - \nu^2) \Omega^2 \frac{1 - n^{-2}}{1 + \eta} \left( \frac{w_n}{r_2} \right)$$

(43)
The asymptotic approximations (Eqs. 40 and 42) form singular matrices. The exact stiffness and inverse flexibility matrices have small additional terms so that the matrices are the inverse of each other.

The problem of a spherical cap with concentrated pinch loads (Fig. 8b) is used by Morley and Morris (Ref. 19), MacNeal and Harder (Ref. 18), and Simo, et al. (Ref. 31) to validate various finite elements. This is a case of pure edge loading, for which the normal displacement can be approximated by using the edge flexibility coefficient $e_{22}$ in Eq. 42. Adding the contributions of the Fourier harmonics yields the result for the total displacement under the load in the case of no crown opening ($f_1 = 0$, $f_2 = p/2$):

$$w = \frac{4 P R^2}{(1 - V) \pi E t c^2} \sum_{n=2,6,10,\ldots}^{\infty} \frac{n}{(n^2 - 1)(2n^2 - 1)}$$

(44)

For general angles of the crown opening and the loaded edge, the flexibility coefficients are modified by adding the values for positive and negative $n$, and taking the integral over the shell meridian, with the result:

$$e_{ij} \rightarrow \frac{e_{ij}(\phi_2, n)}{1 - g(\phi_1, n)} + \frac{e_{ij}(\phi_2, -n)}{1 - g(\phi_1, -n)}$$

(45)

The results for the case of no opening (Eq. 44), and the modification for various values of $f_1$ with $f_2 = p/2$ are in Table 3. The value for $f_1 = 18^\circ$ is a few percent less than that used by MacNeal and Harder.
(Ref. 18), and Simo, et al. (Ref. 31). The opening has little effect until the angle becomes rather large, which is because of the elliptic nature of the inextensional equations for the shell of positive curvature. Also clear from the rapid convergence of the series (Eq. 44) is that the main
Table 3 — Displacement of hemisphere with open crown loaded by concentrated forces at equator, as in Fig. 8b. The displacement is the normal component directly at the point loads as computed from the inextensional solution, which should approximate the exact solution with an error $O(l^{-2})$. (Radius $R = 10$, thickness $t = 0.04$, $E = 6.825 \times 10^7$, $n = 0.3$, $P = 2$)

<table>
<thead>
<tr>
<th>Crown angle angle (deg)</th>
<th>Disp w</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.0888</td>
</tr>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>0.0898</td>
</tr>
<tr>
<td></td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>0.0906</td>
</tr>
<tr>
<td></td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>0.0920</td>
</tr>
<tr>
<td></td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>0.0947</td>
</tr>
<tr>
<td></td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>0.0998</td>
</tr>
<tr>
<td></td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>0.1098</td>
</tr>
<tr>
<td></td>
<td>0.1301</td>
</tr>
<tr>
<td></td>
<td>0.1772</td>
</tr>
<tr>
<td></td>
<td>0.3321</td>
</tr>
</tbody>
</table>

contribution to the displacement is from the first term $n = 2$. Therefore, this is an excellent problem for verifying the slowly-varying, inextensional bending behavior of elements. This is not a good
example for stress, because the series is divergent, giving the logarithmic singularity at the point load. A suggestion is to use a distribution of load, such as in Fig. 8a, particularly the harmonic $n = 2$. For this, the result (Eq. 42) should be a good comparison.

For a thorough verification, all the coefficients in the matrices (Eqs. 40 and 42) should be confirmed for the harmonic $n = 2$. A coarse mesh should be adequate for the flexibility matrix, which verifies the inextensional performance, and for the lower, right-hand block in the stiffness matrix, which verifies the membrane performance. The upper, left-hand block and the off-diagonal blocks in the stiffness matrix depend on the edge effect. For these, a fine mesh is necessary, similar to that discussed for the axisymmetric shell.

Because the columns in the flexibility matrix (Eq. 42) are proportional, certain combinations of edge load components will produce zero displacement, according to this asymptotic approximation. In fact, such a combination of loads is a special case which will satisfy the boundary conditions of the membrane and edge effects exactly. To demonstrate this, consider the following result for the meridional displacement of the hemisphere because of prescribed values of the direct stress resultants at the edge:
\[ N_s = N_{sn} \cos n\theta \quad N_{s\theta} = N_{s\theta n} \sin n\theta \tag{46} \]

\[
\frac{E t u_n}{R} = 1 + \nu \frac{1}{2n} \left[ N_{sn} - N_{s\theta n} + \frac{2\lambda^4 (N_{sn} + N_{s\theta n})}{(1 - \nu^2) (n^2 - 1)} \right] \]

For \( N_{sn} = -N_{sn} \), the boundary conditions of the membrane solution are satisfied. However, if \( N_{sn} \) deviates slightly from that value, the \( O(l^4) \) second term becomes large and the response in inextensional.

4.2 Shell of Negative Gaussian Curvature With Edge Loading

Much of the interesting shell behavior is encountered for geometries other than the sphere or cylinder. The approximate membrane and inextensional behavior is governed by partial differential equations which are classified as elliptic, parabolic, or hyperbolic, depending on whether the surface has a Gaussian curvature which is positive, zero, or negative, respectively. If the direct stress boundary conditions are prescribed in such a way as to make the membrane boundary-value problem well posed, then generally, the stress and displacement are well behaved, and the relative magnitudes in Table 1, columns 1-3 hold. Equation 46 shows the consequence of prescribing both direct stress components at one edge of a positive curvature shell. A small change in the boundary data results in a large change in the solution. Interesting behavior is also encountered for a negative curvature shell. Because the membrane equations are hyperbolic, when both direct stress components are prescribed at one edge, the problem is well posed. This is not the case when "diaphragm", i.e., simply-supported, edges are prescribed for the
negative curvature shell in Fig. 9. In Fig. 10 (from Steele (Ref. 34)) is the meridional edge displacement caused by a meridional edge direct stress for a hyperboloid with \( r_1 = -r_2 \). The ratio of displacement to edge stress resultant and the length parameters are given by:

\[
U = \frac{E t \lambda u_n}{r_2 N_{sn}} \quad \quad L = \frac{n d}{\pi r}
\]

in which \( d \) is the distance between the edges.
For a low circumferential harmonic of the edge load, e.g., $W = 0.1$, the displacement changes two orders of magnitude as the length changes slightly. For high harmonics of edge loading, $W > 0.3$, the elliptic bending operator of the complete shell equations dominates and more typical elastic behavior occurs.

Fig. 9 – Hyperboloid With Direct Stress $N$, on Edges.
Fig. 10 — Meridional deflection of a shell of negative Gaussian curvature with parallel, diaphragm-supported edges, loaded by a static, meridional direct resultant. For critical values of the length parameter $L$, the boundary conditions cannot be satisfied by a membrane solution and the shell responds with large amplitude, inextensional bending. For high harmonics, $W > 0.3$, the effect is diminished.

4.3 Cylindrical Panel With Free and Supported Edges

The cylindrical panel discussed by Scordelis and Lo (Ref. 29), MacNeal and Harder (Ref. 18), and Simo, et al. (Ref. 31), is an interesting case. The two ends are diaphragm supported while the two lateral sides are completely free, while the loading corresponds to body weight. For this shell of zero Gaussian curvature, the two diaphragm supports
eliminate the possibility of a dominant inextensional deformation, which would give the relations in Table 1, columns 4-6. However, it is impossible to obtain a membrane solution, which would give the relations in Table 1, columns 1-3, because of the free lateral edge condition. This is a case of the boundary which is tangent to an asymptotic line on the surface, discussed by Gol'denweiser (Ref. 12). On the lateral edges \( r \to \infty \), and the decay distance of the edge effect from Eq. 4 is infinite. This is a degenerate situation in which all solutions are slowly varying over the shell, requiring only a coarse mesh of elements. To see the behavior, the asymptotic approximation for the flexibility coefficient \( e_{23} \), is extracted from Steele (Ref. 34), which yields the approximation for the normal displacement at the center of the free edge:

\[
\text{disp} = 5.82 \frac{L^2}{\pi^2 R c} \left( \frac{p R^2}{E t} \right)
\]

This approximation is valid for the very thin, shallow panel. From Scordelis and Lo (Ref. 29) the values \( p = 90, R = 25, L = 50, t = 0.25, E = 4.32 \times 10^8, \nu = 0 \), with the edge angle of 40°, give the displacement 0.309, while Eq. 48 gives the displacement 0.43. Because the panel is not shallow and not exceptionally thin, the approximation is reasonable. More important is that Eq. 48 shows that the displacement is \( O(L^2) \) in comparison with the membrane solution, halfway between the two stress states in Table 1.

The conclusion is that the two shell problems used by MacNeal and Harder, (Ref. 18), consisting of the cylindrical panel and the open hemisphere with the point loads (Fig. 8b), are excellent problems but do not cover the full range of shell behavior. In both,
the dominant state of deformation is slowly varying and can be handled by a coarse mesh. A serious objection is that no consideration of the stress is made. In the panel problem, the stress is also slowly varying, so accurate results should be obtainable with the coarse mesh. In the point load problem, however, the interesting stress occurs exactly at the point load and has a logarithmic singularity, as in the flat plate bending problem. As previously mentioned, a thorough verification requires consideration of a single, low harmonic of boundary conditions to ensure the accuracy of both membrane and inextensional displacements and stress.

![Diagram](image)

**Fig. 11** — Instability of measure band with moment load $M$. The buckled region has the same radius of curvature $R$ in the transverse direction as the original curvature in the longitudinal direction.

The buckling of a long cylinder with free lateral edges according to Calladine (Ref. 5) is one problem which Love did not solve, but which is resolved by Rimrott (Ref. 28). The interesting feature is that the buckled region in Fig. 11, which must remain as a surface of zero Gaussian curvature, pops from a surface with the initial radius of curvature $R$ in the one direction to a surface with the radius of
curvature $R$ in the orthogonal direction. This seems to be the equivalent of the dimple for a surface of positive curvature. The potential for this is:

$$U = Etc^2 (1 - \nu) \beta \alpha - M \alpha$$

(49)

Because the strain energy of the buckled region and the potential of the external moment are both proportional to the angle $\alpha$, the solution gives neutral equilibrium with the critical moment:

$$M_{cr} = Etc^2 (1 - \nu) \beta$$

(50)

The classical buckling load, obtained from the membrane solution and the local instability condition Eq. 10, is:

$$M_{cl} = \frac{8}{15} Etc R \beta^3$$

(51)

which is $O(L^1)$ higher. Thus, Eq. 50 gives a low, post-buckling plateau in the moment-angle relationship. Returning to the problem of the panel loaded by weight and with diaphragm supports at the ends, one may consider the post-buckling behavior with a "yield hinge" of magnitude (Eq. 50) at the center, which yields the critical weight magnitude of:

$$p_{cr} = \frac{2 Etc^2 (1 - \nu)}{RL^2}$$

(52)

For the dimensions of Scordelis and Lo (Ref. 29), this gives $p_{cr} = 18$. It will be interesting to see the post-buckling behavior for this problem from a direct,
nonlinear, finite element calculation for displacement large in comparison with the shell thickness.

4.4 Nearly Concentrated Line Load

Concentrated effects in shells are important. To avoid the singularity of the point load, however, a localized line load such as those shown in Fig. 12 can be used. The distribution of the line load intensity in the circumferential direction is:

\[ q(\theta) = \frac{q_{\text{max}}}{1 + (3\theta/\theta_q)^2} \]

in which \( q_q \) is the angle at which the intensity is 10 percent of the maximum value. The arc length between the 10 percent points is \( d = 2r q_q \). The total force is:

\[ P = q_{\text{max}} \pi d / 6 \]

In unpublished work based on Steele (Ref. 34), Fourier integrals were used for the following results. For a region of the surface of positive Gaussian curvature, the maximum displacement under the load is given by:

\[ w_{\text{max}} = \frac{Pr_m}{8Et_c} F_{\text{disp}}(\Omega, \nu) \]
Fig. 12 — Shell with line load of bell-shaped distribution in region of positive curvature and in region of negative curvature. The width of the distribution is $d$, measured to the points at which the intensity is 0.1 of the maximum value.

in which the mean radius of curvature is:

$$
\frac{1}{r_m} = \frac{1}{2} \left( \frac{1}{r_1} + \frac{1}{r_2} \right)
$$

(56)

and the factor $F_{\text{disp}}$ depends on Poisson's ratio and the parameter:

$$
\Omega = \frac{6 \sqrt{c r_2}}{d}
$$

(57)

The limiting behavior is:
\[ F_{\text{disp}} \rightarrow \left( \frac{r_2}{r_1} + 1 \right) \frac{\Omega}{\pi} \quad \text{for } \Omega \rightarrow 0 \]
\[ \rightarrow 1 \]

for \( \Omega \rightarrow \infty, \frac{r_2}{r_1} = 1 \) \( \quad (58) \)

The transition between the small and large values of \( W \) is in Fig. 13a for \( n = 0.3 \). It is clear that when the width of the load distribution is small compared with the decay distance, the point load result (Eq. 20) is approached. When the width of the load distribution is large compared with the decay distance, the displacement is substantially smaller. Such compact results can be obtained only for the regions of positive Gaussian curvature. Otherwise, the displacement is not local, i.e., the displacement under the load depends on the global boundary conditions.

For the stress, however, the maximum values occur at the maximum load point and are insensitive to the global boundary conditions for all shells. The meridional bending moment is:

\[ M_{\text{max}} = \frac{P}{2 \pi} F_{\text{moment}}(\Omega, \nu) \quad (59) \]

The limiting values for the moment factor are:

\[ F_{\text{moment}} \rightarrow \Omega / \sqrt{\Omega} \quad \text{for } \Omega \rightarrow 0, \frac{r_2}{r_1} \neq 0 \]
\[ \rightarrow (1 + \nu) \Omega / \sqrt{\Omega} \quad \text{for } \Omega \rightarrow 0, \frac{r_2}{r_1} = 0 \]
\[ \rightarrow \frac{(1 + \nu)}{2} \log \Omega \quad \text{for } \Omega \rightarrow \infty \quad (60) \]

The transition between these limits is in Fig. 13b for \( n = 0.3 \). The bending moment in the circumferential direction is smaller for small \( W \) and equal for large \( W \).
The circumferential direct stress is:

\[ N_{\theta_{\max}} = \frac{P}{2\pi t c} F_{\text{direct}}(\Omega, \nu) \]  

(61)

The limiting values for the direct stress factor are:

\[ F_{\text{direct}} \rightarrow \Omega / \sqrt{T} \quad \text{for } \Omega \rightarrow 0 \]

\[ \rightarrow \text{constant} \quad \text{for } \Omega \rightarrow \infty \]  

(62)

The transition between these limits is in Fig. 13c for \( n = 0.3 \). The direct stress in the meridional direction is smaller for small \( \dot{W} \) and equal for large \( \dot{W} \). The transition in Figs. 13a-c from the elementary shell solutions estimated in Table 1 to the local point load is an important part of shell behavior, which should be properly represented in a numerical computation.

5. COMPUTER IMPLEMENTATION

The approximate solutions in the preceding sections are "closed form" or obtained with a relatively little amount of computation, as for Fig. 13. Generally, such solutions can be obtained from a formal asymptotic expansion procedure. As a rule, the asymptotic expansion is designed to take advantage of the feature of the problem which makes direct numerical computation difficult. A natural question is whether or not the asymptotic approach can be used in a numerical procedure. The difficulty for shells is that such a battery of different asymptotic techniques are used for the variety of problems. Limits of applicability and/or error estimates are often difficult to obtain or are too conservative. It
is no wonder that the overwhelming emphasis in the last years has been on developing finite element techniques sufficiently robust so that a user can solve real shell problems without the prerequisite of having to spend years learning the information in the references. However, shells present difficulties to such an approach as well. Shell problems are of such a nature that all the information and techniques which are at our disposal should be used, including finite element and asymptotic methods.

Fig. 13a — Displacement factor for nearly concentrated line load. The heavy, dashed lines show the asymptotic behavior for the load distributed in a distance which, when compared with the decay distance, is small ($\bar{W} \gg 1$) and large ($\bar{W} < 0.3$).
Fig. 13b — Moment factor for nearly concentrated line load. The heavy, dashed lines show the asymptotic behavior. When the load is highly concentrated ($W >> 1$), the bending moment at the center of the load increases logarithmically.
Fig. 13c – Direct stress factor for nearly concentrated line load. The heavy dashed line shows the asymptotic behavior. When the load is highly concentrated ($W >> 1$), the direct stress at the center of the load approaches a constant.

Consequently, our recent effort has been directed toward developing asymptotic analysis into a more general purpose computer program. A beginning was made on the problem of the intersection of two cylindrical shells. Because the intersection curve is tangent at one point to an asymptotic line on the surface, there is a mix of the membrane, inextensional and edge effect types of solution which makes any simple closed-form result impractical. However, the exact solutions for the complete cylinder, shallow shell solutions, and techniques for matching are used in the program 'FAST2', reported in Steele and Steele (Ref. 36). Compared to existing finite element programs, the user preparation time is reduced for a given geometry for a reinforced nozzle, from days to a few minutes, and the CPU time reduced by a factor of 500-1000. This is, however, a special purpose program, but it does indicate that analytic considerations should not be neglected in shell computation. More recent is the development of 'FAST1', a program for a general, axisymmetric shell structure, which represents a step toward a general purpose program. The procedure is to use "very large finite elements" (VLFEM) in which one element is, e.g., a complete hemisphere or a long, cylindrical shell with a hole in the side wall. For such an "element", the shape function cannot be a polynomial of low or high order but is obtained from exact
solutions or asymptotic approximations properly corrected.

6. CONCLUSION

The thin shell is an intriguing device with many contradictions. Compared to flat plate and beam structures, great material savings are possible. On the other hand, the potential for great catastrophe is always present. Extremely simple approximate theory and solutions have provided the basis for shell design for more than 100 years. On the other hand, a satisfactory first approximation theory is relatively recent. The general equations for static loading are elliptic, so St. Venant's principle holds. On the other hand, the reduced equations for the membrane and inextensional behavior are hyperbolic for shells of negative curvature, so St. Venant's principle may not hold. A requirement for a well-posed problem of mathematical physics is that the solution be continuously dependent upon the parameters. On the other hand, as indicated for the spherical shell with stress boundary conditions (Eq. 46) and the shell of negative curvature with diaphragm-supported edges in Fig. 10, the change in the solution because of a small change in the parameters can be quite large.

To capture such possibilities, it appears that a validation of a numerical program for analysis of shells should be subjected to a substantially more demanding array of benchmark problems than has been used in the past. The problems described in this paper represent a minimal beginning.

That approximate results for special problems can be obtained with minimal computation should not be ignored in future development. Even with future generations of computers, problems involving
optimization and dynamic response of complex shell structures will remain of excessive cost with the currently available shell elements and mesh generation techniques. We may look forward to numerical procedures in which the best features of current finite elements and asymptotic analysis are exploited fully. The thin shell is sufficiently demanding that all the techniques at our disposal, analytical and numerical, can be fruitfully employed.

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