Lecture Outline

Solving Second-Order Equations

Inhomogeneous Equations

Two Special Cases

Method of Undetermined Coefficients

Higher-Order Linear Equations with Constant Coefficients

Stationary States and Stability for Second-Order Equations
Second-Order Equations

A general second-order difference equation specifies the state $x_t$ at each time $t$ as a function $x_t = F_t(x_{t-1}, x_{t-2})$ of the state at two previous times.

We focus on linear equations in one variable with constant coefficients.

These take the form

$$x_{t+1} + ax_t + bx_{t-1} = f_t$$

where $a, b$ are scalars, and $f_t$ is the forcing term.

We assume that $b \neq 0$ because otherwise we have the first-order equation $x_{t+1} + ax_t = 0$. 
A Coupled Pair of First-Order Equations

The equation \( x_{t+1} + ax_t + bx_{t-1} = f_t \) can be converted into a coupled pair of first-order equations.

To do so, define \( y_t = x_{t-1} \), so the equations become

\[
\begin{align*}
x_{t+1} &= -ax_t - by_t + f_t \\
y_{t+1} &= x_t
\end{align*}
\]

In matrix form, these can be written as

\[
\begin{pmatrix}
x_{t+1} \\
y_{t+1}
\end{pmatrix} = 
\begin{pmatrix}
-a & -b \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
x_t \\
y_t
\end{pmatrix} + 
\begin{pmatrix}
f_t \\
0
\end{pmatrix}
\]

This kind of vector difference equation will be considered much more fully in part C.
The Homogeneous Case

Nevertheless, consider the homogeneous case when the vector equation is

\[
\begin{pmatrix}
  x_{t+1} \\
  y_{t+1}
\end{pmatrix} - \begin{pmatrix}
  -a & -b \\
  1 & 0
\end{pmatrix} \begin{pmatrix}
  x_t \\
  y_t
\end{pmatrix} = \begin{pmatrix}
  0 \\
  0
\end{pmatrix}
\]

The solution in matrix form is evidently

\[
\begin{pmatrix}
  x_t \\
  y_t
\end{pmatrix} = \begin{pmatrix}
  -a & -b \\
  1 & 0
\end{pmatrix}^t \begin{pmatrix}
  x_0 \\
  y_0
\end{pmatrix}
\]

for an arbitrary initial state \( \begin{pmatrix}
  x_0 \\
  y_0
\end{pmatrix} \).

Inspired by our earlier discussion of matrix powers, consider the case when \( \begin{pmatrix}
  x_0 \\
  y_0
\end{pmatrix} \) is an eigenvector, with corresponding eigenvalue \( \lambda \) — that is, suppose

\[
\begin{pmatrix}
  -a & -b \\
  1 & 0
\end{pmatrix} \begin{pmatrix}
  x_0 \\
  y_0
\end{pmatrix} = \lambda \begin{pmatrix}
  x_0 \\
  y_0
\end{pmatrix}
\]
Solving the Homogeneous Case

In case \[ \begin{pmatrix} -a & -b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \], the solution takes the form

\[
\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} -a & -b \\ 1 & 0 \end{pmatrix}^t \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \lambda^t \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}
\]

But for this to work, the initial vector \[ \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \] must be a non-zero solution of the matrix equation

\[
\begin{pmatrix} -a - \lambda & -b \\ 1 & -\lambda \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

For such a non-zero solution to exist, the matrix \[ \begin{pmatrix} -a - \lambda & -b \\ 1 & -\lambda \end{pmatrix} \] must be singular, implying that

\[
\left| \begin{array}{cc} -a - \lambda & -b \\ 1 & -\lambda \end{array} \right| = \lambda^2 + a\lambda + b = 0
\]
The Auxiliary Equation

Instead of treating the second-order equation as a coupled pair, consider directly the homogeneous second-order equation

\[ x_{t+1} + ax_t + bx_{t-1} = 0 \]

Inspired by our previous analysis using eigenvalues of a suitable matrix, we look for a solution of the form \( x_t = \lambda^t x_0 \), for suitable constants \( \lambda \) and \( x_0 \).

It is a solution provided that \( \lambda^{t+1} x_0 + a\lambda^t x_0 + b\lambda^{t-1} x_0 = 0 \).

Ignoring the trivial solutions when \( x_0 = 0 \) or \( \lambda = 0 \), cancel \( \lambda^{t-1} x_0 \) to obtain the auxiliary or characteristic equation

\[ \lambda^2 + a\lambda + b = 0 \]

This, of course, is the condition for \( \lambda \) to be an eigenvalue.
The Auxiliary Equation and Its Roots

The auxiliary equation $\lambda^2 + a\lambda + b = 0$ is quadratic.

It therefore has two roots $\lambda_1, \lambda_2$
satisfying $\lambda^2 + a\lambda + b = (\lambda - \lambda_1)(\lambda - \lambda_2)$.

In particular $\lambda_1 + \lambda_2 = -a$ and $\lambda_1\lambda_2 = b$.

The assumption that $b \neq 0$ implies that the two roots $\lambda_1, \lambda_2$ are both non-zero.

This leaves three cases:

1. two distinct real roots $\lambda_1, \lambda_2 \in \mathbb{R}$,
   which is true iff $a^2 > 4b$;

2. two complex conjugate roots $\lambda_1, \lambda_2 = re^{\pm i\theta} \in \mathbb{C}$,
   which is true iff $a^2 < 4b$;

3. two coincident real roots $\lambda = \lambda_1 = \lambda_2 \in \mathbb{R}$,
   which is true iff $a^2 = 4b$. 
Case 1: Two Distinct Real Roots

In this case $\lambda^2 + a\lambda + b = (\lambda - \lambda_1)(\lambda - \lambda_2)$,
where $\lambda_1, \lambda_2 = -\frac{1}{2}a \pm \frac{1}{2}\sqrt{a^2 - 4b}$.

Note that $a = \lambda_1 + \lambda_2$ and $b = \lambda_1\lambda_2$
with $a^2 - 4b = (\lambda_1 + \lambda_2)^2 - 4\lambda_1\lambda_2 = (\lambda_1 - \lambda_2)^2 > 0$.

There are two degrees of freedom in the difference equation,
so we look for two linearly independent solutions $x_H^{(1)}(t)$ and $x_H^{(2)}(t)$
of the homogeneous difference equation $x_{t+1} + ax_t + bx_{t-1} = 0$.
— that is two solutions for which $Ax_H^{(1)}(t) + Bx_H^{(2)}(t) \equiv 0$
implies that the two scalars $A$ and $B$ satisfy $A = B = 0$. 
Two Linearly Independent Solutions

Note that $A\lambda_1^t + B\lambda_2^t = 0$ for both $t = 0$ and $t = 1$ if and only if

$$
\begin{pmatrix}
1 & 1 \\
\lambda_1 & \lambda_2
\end{pmatrix}
\begin{pmatrix}
A \\
B
\end{pmatrix} =
\begin{pmatrix}
0 \\
0
\end{pmatrix}
$$

This has a non-trivial solution in the two constants $A$ and $B$ iff $0 = \frac{1}{1} 1$, or if and only if $0 = \lambda_2 - \lambda_1$.

So when $\lambda_1 \neq \lambda_2$, the only solution is trivial, with $A = B = 0$.

Hence, the two functions $x_1^t = x_0\lambda_1^t$ and $x_2^t = x_0\lambda_2^t$ with $x_0 \neq 0$ are linearly independent solutions of $x_{t+1} + ax_t + bx_{t-1} = 0$.

There are two degrees of freedom in the difference equation.

Therefore, its general solution with these two degrees of freedom is $x_t = A\lambda_1^t + B\lambda_2^t$ for arbitrary real constants $A$ and $B$. 
Example: The Fibonacci Sequence

The Fibonacci sequence is

\[(x_t)_{t=0}^\infty = (0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots)\]

It is the unique solution with \(x_0 = 0\) and \(x_1 = 1\) of the Fibonacci difference equation \(x_{t+1} - x_t - x_{t-1} = 0\).

The characteristic equation is \(\lambda^2 - \lambda - 1 = 0\), with characteristic roots \(\lambda_1, 2 = -\frac{1}{2}(-1 \pm \sqrt{5})\).

Its two roots are:
the golden ratio \(\varphi := \lambda_1 = \frac{1}{2}(1 + \sqrt{5}) \approx 1.61803398875\);
and \(\lambda_2 = 1 - \lambda_1 = \frac{1}{2}(1 - \sqrt{5}) \approx -0.61803398875\).

The general solution of the Fibonacci difference equation
is \(x_t = A\lambda_1^t + B\lambda_2^t\) for arbitrary constants \(A\) and \(B\).

To obtain the Fibonacci sequence with \(x_0 = 0\) and \(x_1 = 1\) requires \(B = -A\) and \(1 = A(\lambda_1 - \lambda_2) = A\sqrt{5}\),
so \(B = -A = -\frac{1}{5}\sqrt{5}\).

Hence \(x_t = \frac{1}{5}\sqrt{5} \cdot 2^{-t} [(1 + \sqrt{5})^t - (1 - \sqrt{5})^t]\), so \(x_t \in \mathbb{N}\).
Case 2: Two Complex Conjugate Roots

In this case \( \lambda^2 + a\lambda + b = (\lambda - re^{i\theta})(\lambda - re^{-i\theta}) \) where

\[
a = re^{i\theta} + re^{-i\theta} = r(\cos \theta + i \sin \theta) + r(\cos \theta - i \sin \theta) = 2r \cos \theta
\]

and \( b = (re^{i\theta})(re^{-i\theta}) = r^2 \) with \( \sin \theta \neq 0 \).

It follows that \( a^2 - 4b = 4r^2 \cos^2 \theta - 4r^2 = -4r^2 \sin^2 \theta < 0 \).

Note that \( r = \sqrt{|b|} \) and \( \theta = \arccos \left( \frac{a}{2r} \right) = \arccos \left( \frac{1}{2} a|b|^{-\frac{1}{2}} \right) \).
Case 2: Oscillating Solutions

In the complex plane \( \mathbb{C} \), two possible solutions of the difference equation \( x_{t+1} + ax_t + bx_{t-1} = 0 \) with \( x_0 \neq 0 \) are

\[
x_t^{(1)} = x_0 (re^{i\theta})^t = x_0 r^t e^{i\theta t} = x_0 r^t (\cos \theta t + i \sin \theta t)
\]

and

\[
x_t^{(2)} = x_0 (re^{-i\theta})^t = x_0 r^t e^{-i\theta t} = x_0 r^t (\cos \theta t - i \sin \theta t)
\]

In the real line \( \mathbb{R} \), two possible solutions are

\[
x_t^{(1)} = r^t \cos \theta t \quad \text{and} \quad x_t^{(2)} = r^t \sin \theta t
\]

These are linearly independent because

\[
\begin{vmatrix}
x_0^{(1)} & x_0^{(2)} \\
x_1^{(1)} & x_1^{(2)}
\end{vmatrix}
= \begin{vmatrix}
1 & 0 \\
r \cos \theta & r \sin \theta
\end{vmatrix}
= r \sin \theta \neq 0
\]

The general solution is therefore \( x_t = r^t (A \cos \theta t + B \sin \theta t) \) for arbitrary real constants \( A \) and \( B \), where \( A = x_0 \).
Case 3: Two Coincident Roots

In this case \( \lambda^2 + a\lambda + b = (\lambda - \bar{\lambda})^2 \),
where \( b = \bar{\lambda}^2 \) with \( a^2 - 4b = 0 \).

Consider the perturbed equation \( x_{t+1} + ax_t + \tilde{b}x_{t-1} = 0 \)
where \( \tilde{a} = -2\bar{\lambda} \) and \( \tilde{b} = \bar{\lambda}^2 - \epsilon^2 \) with \( \epsilon \) a small positive number.

We consider the behaviour of its general solution as \( \epsilon \to 0 \).

The auxiliary equation \( \lambda^2 + a\lambda + \tilde{b} = 0 \)
can be written as \( \lambda^2 + 2\bar{\lambda}\lambda + \bar{\lambda}^2 - \epsilon^2 = 0 \).

Note that \( \lambda^2 + 2\bar{\lambda}\lambda + \bar{\lambda}^2 - \epsilon^2 = (\lambda + \bar{\lambda} + \epsilon)(\lambda + \bar{\lambda} - \epsilon) \).

So the perturbed auxiliary equation
has the two real roots \( \lambda = \bar{\lambda} \pm \epsilon \).
The Solution with Fixed Initial Conditions

Fix $\bar{x}_0$ and $\bar{x}_1$.

The general solution satisfying $x_0 = \bar{x}_0$ and $x_1 = \bar{x}_1$ is

$$x_t = A(\bar{\lambda} + \epsilon)^t + B(\bar{\lambda} - \epsilon)^t$$

where $\bar{x}_0 = A + B$ and $\bar{x}_1 = A(\bar{\lambda} + \epsilon) + B(\bar{\lambda} - \epsilon) = (A + B)\bar{\lambda} + (A - B)\epsilon$.

Hence $A + B = \bar{x}_0$ and $A - B = (1/\epsilon)(\bar{x}_1 - \bar{x}_0 \bar{\lambda})$, implying that $A = \frac{1}{2} \left[ \bar{x}_0 + (1/\epsilon)(\bar{x}_1 - \bar{x}_0 \bar{\lambda}) \right]$ and $B = \frac{1}{2} \left[ \bar{x}_0 - (1/\epsilon)(\bar{x}_1 - \bar{x}_0 \bar{\lambda}) \right]$.

The solution for fixed $\epsilon$ is therefore

$$x^\epsilon_t = \frac{1}{2} \left[ \bar{x}_0 + (1/\epsilon)(\bar{x}_1 - \bar{x}_0 \bar{\lambda}) \right] (\bar{\lambda} + \epsilon)^t$$

$$+ \frac{1}{2} \left[ \bar{x}_0 - (1/\epsilon)(\bar{x}_1 - \bar{x}_0 \bar{\lambda}) \right] (\bar{\lambda} - \epsilon)^t$$

which can be rewritten as

$$x^\epsilon_t = \frac{1}{2} \bar{x}_0 \left[ (\bar{\lambda} + \epsilon)^t + (\bar{\lambda} - \epsilon)^t \right] + \frac{1}{2} (\bar{x}_1 - \bar{x}_0 \bar{\lambda}) \left[ (\bar{\lambda} + \epsilon)^t - (\bar{\lambda} - \epsilon)^t \right] / \epsilon$$
The Limiting Solution as $\epsilon \to 0$

The limit of $x_t^\epsilon$ as $\epsilon \to 0$ takes the form

$$\bar{x}_0 \bar{\lambda}^t + \frac{1}{2}(\bar{x}_1 - \bar{x}_0 \bar{\lambda}) \lim_{\epsilon \to 0} \frac{[(\bar{\lambda} + \epsilon)^t - (\bar{\lambda} - \epsilon)^t]}{\epsilon}$$

To evaluate the last limit, apply l’Hôpital’s rule to obtain

$$\lim_{\epsilon \to 0} \frac{[(\bar{\lambda} + \epsilon)^t - (\bar{\lambda} - \epsilon)^t]}{\epsilon} = \lim_{\epsilon \to 0} \frac{[t(\bar{\lambda} + \epsilon)^{t-1} + t(\bar{\lambda} - \epsilon)^{t-1}]}{1} = 2t \bar{\lambda}^t$$

Two linearly independent possible solutions of the difference equation $x_{t+1} + ax_t + bx_{t-1} = 0$ with $x_0 \neq 0$ are $x_t^{(1)} = x_0 \lambda^t$ and $x_t^{(2)} = x_0 t \lambda^t$.

There are two degrees of freedom in the difference equation.

Its general solution is $x_t = (C + Dt)\lambda^t$

for arbitrary real constants $C$ and $D$. 
A Simpler Approach, 1

We are trying to solve the homogeneous second-order difference equation with a repeated root $\lambda$, taking the form

$$x_{t+1} - 2\lambda x_t + \lambda^2 x_{t-1} = 0$$

We know that one solution is $x_t = x_0 \lambda^t$ for arbitrary $x_0$.

To find a second linearly independent solution that we know must exist, try putting $x_t = \lambda^t y_t$.

Substituting into the original equation gives

$$\lambda^{t+1} y_{t+1} - 2\lambda^{t+1} y_t + \lambda^{t+1} y_{t-1} = 0$$

Disregarding the trivial case when $\lambda = 0$, one has $y_{t+1} - 2y_t + y_{t-1} = 0$. 
To solve $y_{t+1} - 2y_t + y_{t-1} = 0$, try introducing yet another new variable $z_t = y_{t+1} - y_t$.

This leads to the new difference equation $z_t - z_{t-1} = 0$ whose solution is obviously $z_t = z_0$ for all $t = 1, 2, \ldots$.

Then $y_{t+1} - y_t = z_0$ for all $t$, implying that $y_t = y_0 + z_0 t$.

It follows that $x_t = \lambda^t y_t = (y_0 + z_0 t) \lambda^t$.

To conclude, two solutions are $x_t^{(1)} = \lambda^t$ and $x_t^{(2)} = t \lambda^t$.

These are linearly independent because

$$\begin{vmatrix} x_0^{(1)} & x_0^{(2)} \\ x_1^{(1)} & x_1^{(2)} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \lambda & \lambda \end{vmatrix} = \lambda \neq 0$$

The general solution is therefore $x_t = (A + Bt) \lambda^t$ for arbitrary real constants $A$ and $B$, where $A = x_0$. 
Solving Second-Order Equations

Inhomogeneous Equations

Two Special Cases

Method of Undetermined Coefficients

Higher-Order Linear Equations with Constant Coefficients

Stationary States and Stability for Second-Order Equations
From Particular to General Solutions

The homogeneous equation with constant coefficients takes the form

\[ x_{t+1} + ax_t + bx_{t-1} = 0 \]

An associated inhomogeneous equation takes the form

\[ x_{t+1} + ax_t + bx_{t-1} = f_t \]

for a general forcing term \( f_t \) on the RHS.

Let \( x_t^P \) denote a particular solution, and \( x_t^G \) any alternative general solution, of the inhomogeneous equation.
Characterizing the General Solution

Our assumptions imply that, for each \( t = 1, 2, \ldots \), one has

\[
x_{t+1}^P + ax_t^P + bx_{t-1}^P = f_t
\]
\[
x_{t+1}^G + ax_t^G + bx_{t-1}^G = f_t
\]

Subtracting the first equation from the second implies that

\[
x_{t+1}^G - x_{t+1}^P + a(x_t^G - x_t^P) + b(x_{t-1}^G - x_{t-1}^P) = 0
\]

This shows that \( x_t^H := x_t^G - x_t^P \) solves the homogeneous equation.

So the general solution \( x_t^G \)

of the inhomogeneous equation \( x_{t+1} + ax_t + bx_{t-1} = f_t \)

with forcing term \( f_t \) is the sum \( x_t^P + x_t^H \) of

- any particular solution \( x_t^P \) of the inhomogeneous equation;
- the general solution \( x_t^H \)

of the homogeneous equation \( x_{t+1} + ax_t + bx_{t-1} = 0 \).
Linearity in the Forcing Term

Theorem

Suppose that \( x_t^P \) and \( y_t^P \) are particular solutions of the two respective difference equations

\[
x_{t+1} + ax_t + bx_{t-1} = d_t \quad \text{and} \quad y_{t+1} + ay_t + by_{t-1} = e_t
\]

Then, for any scalars \( \alpha \) and \( \beta \), the linear combination \( z_t^P := \alpha x_t^P + \beta y_t^P \) is a particular solution of the equation \( z_{t+1} + az_t + bz_{t-1} = \alpha d_t + \beta e_t \).

Proof.

Routine algebra.

Consider any equation of the form \( x_{t+1} + ax_t + bx_{t-1} = f_t \) where \( f_t \) is a linear combination \( \sum_{k=1}^{n} \alpha_k f_t^k \) of \( n \) forcing terms.

The theorem implies that a particular solution is the corresponding linear combination \( \sum_{k=1}^{n} \alpha_k x_t^{P_k} \) of particular solutions to the equations \( x_{t+1} + ax_t + bx_{t-1} = f_t^k \).
Deriving an Explicit Particular Solution, I

In part A we were able to derive an explicit solution to the general first-order linear equation \( x_t - a_t x_{t-1} = f_t \).

Here, for the special case of constant coefficients, we derive an explicit particular solution satisfying \( x_0 = x_1 = 0 \) to the general second-order linear equation \( x_{t+1} + ax_t + bx_{t-1} = f_t \).

Indeed, suppose that \( \lambda^2 + a\lambda + b = (\lambda - \lambda_1)(\lambda - \lambda_2) \) because \( \lambda_1 \) and \( \lambda_2 \) are the roots (possibly coincident, or possibly complex conjugates) of the auxiliary equation \( \lambda^2 + a\lambda + b = 0 \).

Introduce the new variable \( y_t = x_t - \lambda_1 x_{t-1} \), implying that

\[
y_{t+1} - \lambda_2 y_t = x_{t+1} - \lambda_1 x_t - \lambda_2 x_t + \lambda_1 \lambda_2 x_{t-1} = x_{t+1} - (\lambda_1 + \lambda_2)x_t + \lambda_1 \lambda_2 x_{t-1} = x_{t+1} + ax_t + bx_{t-1} = f_t
\]
Deriving an Explicit Particular Solution, II

Instead of the second-order equation $x_{t+1} + ax_t + bx_{t-1} = f_t$, we have the recursive pair of first-order equations

$$ x_t - \lambda_1 x_{t-1} = y_t \quad \text{and} \quad y_{t+1} - \lambda_2 y_t = f_t \quad (\text{for } t = 1, 2, \ldots) $$

where $\lambda_1$ and $\lambda_2$ are the roots of $\lambda^2 + a\lambda + b = 0$.

Given the initial conditions $x_0 = x_1 = 0$ and so $y_1 = 0$, the explicit solutions like those derived in Part A are the sums

$$ y_t = \sum_{k=1}^{t-1} \lambda_2^{t-k-1} f_k \quad \text{and} \quad x_t = \sum_{s=2}^{t} \lambda_1^{t-s} y_s \quad \text{for } t = 1, 2, \ldots $$

Substituting the first equation in the second yields the double sum

$$ x_t = \sum_{s=2}^{t} \lambda_1^{t-s} \sum_{k=1}^{s-1} \lambda_2^{s-k-1} f_k $$

which we would like to reduce to $x_t = \sum_{k=1}^{t-1} \xi_{t-k-1} f_k$ — i.e., a linear combination of the forcing terms $(f_1, f_2, \ldots, f_{t-1})$. 
We begin by introducing the mapping \( \mathbb{N} \times \mathbb{N} \ni (k, s) \mapsto 1_{ks}\{k < s\} \in \{0, 1\} \) defined by

\[
1_{ks}\{k < s\} := \begin{cases} 
1 & \text{if } k < s \\
0 & \text{if } k \geq s
\end{cases}
\]

Then we can rewrite \( x_t = \sum_{s=2}^{t} \lambda_1^{t-s} \sum_{k=1}^{s-1} \lambda_2^{s-k-1} f_k \) as the double sum \( x_t = \sum_{s=2}^{t} \sum_{k=1}^{t-1} 1_{ks}\{k < s\} \lambda_1^{t-s} \lambda_2^{s-k-1} f_k \).

Interchanging the order of summation gives

\[
x_t = \sum_{k=1}^{t-1} \sum_{s=2}^{t} 1_{ks}\{k < s\} \lambda_1^{t-s} \lambda_2^{s-k-1} f_k \\
= \sum_{k=1}^{t-1} \left( \sum_{s=k+1}^{t} \lambda_1^{t-s} \lambda_2^{s-k-1} \right) f_k \\
= \sum_{k=1}^{t-1} \left( \lambda_1^{t-k-1} + \lambda_1^{t-k-2} \lambda_2 + \ldots + \lambda_1 \lambda_2^{t-k-2} + \lambda_2^{t-k-1} \right) f_k
\]

This reduces to \( x_t = \sum_{k=1}^{t-1} \xi_{t-k-1} f_k \) where \( \xi_m := \sum_{j=0}^{m} \lambda_1^{m-j} \lambda_2^{j} \).
Deriving an Explicit Particular Solution: IV

The value of the sum $\xi_m = \sum_{j=0}^{m} \lambda_1^{m-j} \lambda_2^j$ depends on whether:

- $\lambda_1 \neq \lambda_2$ in the general case;
- $\lambda_1 = \lambda_2 = \lambda$ in the degenerate case.

In the general case one has

$$(\lambda_1 - \lambda_2) \xi_m = \sum_{j=0}^{m} \left( \lambda_1^{m+1-j} \lambda_2^j - \lambda_1^{m-j} \lambda_2^{j+1} \right) = \lambda_1^{m+1} - \lambda_2^{m+1}$$

implying the particular solution

$$x_t^p = \frac{1}{\lambda_1 - \lambda_2} \sum_{k=1}^{t-1} \left( \lambda_1^{t-k} - \lambda_2^{t-k} \right) f_k$$

In the degenerate case one has $\xi_m = (m + 1)\lambda^m$, implying the particular solution

$$x_t^p = \sum_{k=1}^{t-1} (t - k)\lambda^{t-k} f_k$$
Lecture Outline

Solving Second-Order Equations

Inhomogeneous Equations

Two Special Cases

Method of Undetermined Coefficients

Higher-Order Linear Equations with Constant Coefficients

Stationary States and Stability for Second-Order Equations
First Special Case with Distinct Real Roots, I

Consider the equation $x_{t+1} + ax_t + bx_{t-1} = f_t$ in the first special case when $f_t$ is the power function $\mu^t$ with $\mu \neq 0$.

In the general case when the two roots $\lambda_1$ and $\lambda_2$ of the auxiliary equation $\lambda^2 + a\lambda + b = 0$ are distinct, the particular solution with $x_0^P = x_1^P = 0$ is

$$x_t^P = \frac{1}{\lambda_1 - \lambda_2} \sum_{k=1}^{t-1} \left( \lambda_1^{t-k} - \lambda_2^{t-k} \right) \mu^k$$

But $(\lambda - \mu) \sum_{k=1}^{t-1} \lambda^{t-k} \mu^k = \sum_{k=1}^{t-1} \left( \lambda^{t-k+1} \mu^k - \lambda^{t-k} \mu^{k+1} \right)$, so

$$x_t^P = \frac{1}{\lambda_1 - \lambda_2} \left( \frac{\lambda_1^t \mu - \lambda_1 \mu^t}{\lambda_1 - \mu} - \frac{\lambda_2^t \mu - \lambda_2 \mu^t}{\lambda_2 - \mu} \right)$$

in case $\mu \notin \{\lambda_1, \lambda_2\}$.

Disregarding the terms in $\lambda_1^t$ and $\lambda_2^t$ that solve the corresponding homogeneous equation, the solution reduces to $x_t^P = \alpha \mu^t$ for a suitable constant $\alpha$. 
First Special Case with Distinct Real Roots, II

The degenerate case when \( \mu \in \{ \lambda_1, \lambda_2 \} \) is more complicated.

In case \( \lambda_1 \neq \lambda_2 = \mu \), the particular solution with \( x_0^p = x_1^p = 0 \) is still

\[
x_t^p = \frac{1}{\lambda_1 - \lambda_2} \sum_{k=1}^{t-1} \left( \lambda_1^{t-k} - \lambda_2^{t-k} \right) \mu^k
\]

Because \( \lambda_2 = \mu \), this reduces to

\[
x_t^p = \frac{1}{\lambda_1 - \mu} \sum_{k=1}^{t-1} \left( \lambda_1^{t-k} \mu^k - \mu^t \right)
= \frac{1}{\lambda_1 - \mu} \left[ \frac{\lambda_1^t \mu - \lambda_1 \mu^t}{\lambda_1 - \mu} - (t - 1) \mu^t \right]
\]

Disregarding the terms in \( \lambda_1^t \) and in \( \lambda_2^t = \mu^t \) that solve the corresponding homogeneous equation, the solution reduces to \( x_t^p = \alpha t \mu^t \) for a suitable constant \( \alpha \).
First Special Case with Coincident Real Roots

Consider now the degenerate case with coincident real roots $\lambda_1 = \lambda_2 = \lambda$.

So the inhomogeneous equation is $x_{t+1} - 2\lambda x_t + \lambda^2 x_{t-1} = \mu^t$.

As before, put $y_t = x_t - \lambda x_{t-1}$ so that

$$y_{t+1} - \lambda y_t = x_{t+1} - \lambda x_t - \lambda x_t + \lambda^2 x_{t-1} = \mu^t$$

We consider again the particular solution with $x_0 = x_1 = 0$ and so $y_1 = 0$. 
First Special Case with Coincident Real Roots: \( \lambda \neq \mu \)

Provided that \( \lambda \neq \mu \), for \( t = 2, 3, \ldots \) one has

\[
y_t^P = \sum_{k=2}^{t} \lambda^{t-k} \mu^{k-1} = \frac{\mu(\lambda^{t-1} - \mu^{t-1})}{\lambda - \mu}
\]

and then

\[
x_t^P = \sum_{k=2}^{t} \lambda^{t-k} y_k^P = \sum_{k=2}^{t} \lambda^{t-k} \mu \frac{\lambda^{k-1} - \mu^{k-1}}{\lambda - \mu}
\]

\[
= \sum_{k=2}^{t} \frac{\mu \lambda^{t-1} - \lambda^{t-k} \mu^k}{\lambda - \mu}
\]

\[
= \frac{\mu(t - 1)\lambda^{t-1}}{\lambda - \mu} - \frac{\lambda^{t-1} \mu^2 - \mu^{t+1}}{(\lambda - \mu)^2}
\]

Hence \( x_t^P = (\alpha + \beta t)\lambda^t + \gamma \mu^t \) for suitable constants \( \alpha, \beta \) and \( \gamma \) that depend on \( \lambda \) and \( \mu \), but not on \( t \).

Because \( (\alpha + \beta t)\lambda^t \) is a complementary solution of the homogeneous equation, the particular solution can be reduced to \( x_t^P = \gamma \mu^t \).
First Special Case with Coincident Real Roots: \( \lambda = \mu \)

In case \( \lambda = \mu \), however, for \( t = 2, 3, \ldots \) one has

\[
y_t^P = \sum_{k=2}^{t} \lambda^{t-k} \mu^{k-1} = (t-1)\lambda^{t-1}
\]

and then

\[
x_t^P = \sum_{k=2}^{t} \lambda^{t-k} y_k^P = \sum_{k=2}^{t} \lambda^{t-k}(k-1)\lambda^{k-1}
\]

\[
= \sum_{k=2}^{t} (k-1)\lambda^{t-1} = \frac{1}{2} t(t-1)\lambda^{t-1}
\]

Hence \( x_t^P = (\alpha t + \beta t^2)\lambda^t \) for suitable constants \( \alpha \) and \( \beta \) that depend on \( \lambda = \mu \), but not on \( t \).

Because \( \alpha t\lambda^t \) is a complementary solution of the homogeneous equation, the particular solution can be reduced to \( x_t^P = \beta t^2 \mu^t \).
Second Special Case: Theorem

Consider next the equation \( x_{t+1} + ax_t + bx_{t-1} = f_t \)
in the second special case when \( f_t = t^r \mu^t \) with \( \mu \neq 0 \) and \( r \in \mathbb{N} \).

As before, let \( \lambda_1 \) and \( \lambda_2 \) denote the roots
of the auxiliary equation \( \lambda^2 + a\lambda + b = 0 \).

Theorem

The difference equation \( x_{t+1} + ax_t + bx_{t-1} = t^r \mu^t \)
has a particular solution of the form \( x^P_t = \xi^P(t)\mu^t \)
where \( \xi^P(t) = \sum_{j=0}^{d} \xi_{ij} t^j \) is a polynomial in \( t \) which has degree:

- \( d = r \) in case \( \mu \notin \{\lambda_1, \lambda_2\} \);
- \( d = r + 2 \) in case \( \mu = \lambda_1 = \lambda_2 \);
- \( d = r + 1 \) otherwise.

We begin the proof by introducing, as before,
the new variable \( y_t := x_t - \lambda_1 x_{t-1} \), implying that

\[
y_{t+1} - \lambda_2 y_t = x_{t+1} - \lambda_1 x_t - \lambda_2 x_{t-1} + \lambda_1 \lambda_2 x_{t-1} = x_{t+1} + ax_t + bx_{t-1} = t^r \mu^t
\]
Continuing the Proof

By the result in part A, the first-order equation $y_{t+1} - \lambda_2 y_t = t^r \mu^t$ has a particular solution of the form $y_t = Q(t)\mu^t$, where $Q(t) = \sum_{j=0}^{d} q_{rj} t^j$ is a polynomial in $t$ which has degree:

(i) $d = r$ in case $\mu \neq \lambda_2$; (ii) $d = r + 1$ in case $\mu = \lambda_2$.

By the linearity property of particular solutions, the equation

$$x_t - \lambda_1 x_{t-1} = y_t = Q(t)\mu^t = \sum_{j=0}^{d} q_{rj} t^j \mu^t$$

has a particular solution $x_t^P = \xi^P(t)\mu^t$ where

$$x_t^P = \xi^P(t)\mu^t = \sum_{j=0}^{d} q_{rj} P_j(t)\mu^t$$

is the appropriate linear combination of the particular solutions $x_t = P_j(t)\mu^t$ ($j = 0, 1, 2, \ldots, d$) of the $d + 1$ first-order equations $x_t - \lambda_1 x_{t-1} = t^j \mu^t$. 
Ending the Proof

Again, using the result in part A, for each \( j = 0, 1, 2, \ldots, r \), the solution \( x_t = P_j(t)\mu^t \) of the first-order difference equation \( x_t - \lambda_1 x_{t-1} = t^j \mu^t \) involves a polynomial \( P_j(t) \) in \( t \) which has degree:

(i) \( j \) in case \( \mu \neq \lambda_1 \); (ii) \( j + 1 \) in case \( \mu = \lambda_1 \).

So the degree of the highest order polynomial \( P_d(t) \) is

(i) \( d \) in case \( \mu \neq \lambda_1 \); (ii) \( d + 1 \) in case \( \mu = \lambda_1 \).

Combined with our previous result on whether \( d = r \) or \( d = r + 1 \), the degree of \( \xi^P(t) \) is now easily seen to be

- \( r \) in case \( \mu \not\in \{\lambda_1, \lambda_2\} \);
- \( r + 2 \) in case \( \mu = \lambda_1 = \lambda_2 \);
- \( r + 1 \) otherwise.

\[\Box\]
Lecture Outline

Solving Second-Order Equations

Inhomogeneous Equations

Two Special Cases

Method of Undetermined Coefficients

Higher-Order Linear Equations with Constant Coefficients

Stationary States and Stability for Second-Order Equations
First Special Case: A Simpler Approach

We have proved that the second-order difference equation

\[ x_{t+1} + ax_t + bx_{t-1} = \mu^t \]

has a particular solution of the form \( x_t^P = \alpha \mu^t \).

But there is a much easier way to find \( x_t^P \), treating the parameter \( \alpha \) as an undetermined coefficient.

Indeed, for \( x_t = \alpha \mu^t \) to be a solution, one needs

\[ \alpha \mu^{t+1} + a\alpha \mu^t + b\alpha \mu^{t-1} = \mu^t. \]

Dividing each side by \( \mu^{t-1} \) yields the equation

\[ \alpha (\mu^2 + a\mu + b) = \mu. \]

In the non-degenerate case when \( \mu^2 + a\mu + b \neq 0 \) because \( \mu \) is not a root of the characteristic equation \( \lambda^2 + a\lambda + b = 0 \), one has

\[ \alpha = \mu (\mu^2 + a\mu + b)^{-1}. \]

Hence, a particular solution is

\[ x_t^P = (\mu^2 + a\mu + b)^{-1} \mu^{t+1}. \]
Degenerate Case When $\mu$ is a Characteristic Root

The simple degenerate case occurs when $\mu^2 + a\mu + b = 0$ because $\mu$ equals one of the two distinct roots $\lambda_1$ and $\lambda_2$ of the characteristic equation $\lambda^2 + a\lambda + b = 0$.

Then we have proved that the second-order difference equation

$$x_{t+1} + ax_t + bx_{t-1} = \mu^t$$

has a particular solution of the form $x_t^P = \alpha t \mu^t$.

To determine the undetermined coefficient $\alpha$, we must solve

$$\alpha(t + 1)\mu^{t+1} + a\alpha t \mu^t + b\alpha(t - 1)\mu^{t-1} = \mu^t$$

Dividing each side by $\mu^{t-1}$ and gathering terms yields the equation $\alpha t (\mu^2 + a\mu + b) + \alpha (\mu^2 - b) = \mu$.

Provided that $\mu^2 \neq b$, this reduces to $\alpha = (\mu^2 - b)^{-1} \mu$. 
Doubly Degenerate Case

When \( \mu^2 = b \), however, the degenerate case is more complicated.

Indeed, the equation \( \mu^2 + a\mu + b = 0 \) implies that \( a\mu + 2b = 0 \).
Hence \( \mu = -2b/a \), so \( \mu^2 = b = 4b^2/a^2 \) implying that \( a^2 = 4b \).
Then the characteristic equation \( \lambda^2 + a\lambda + b = 0 \)
reduces to \( (\lambda - \mu)^2 = 0 \), with \( \mu \) as its repeated root.

Inspired by the earlier theorem,
we look for a particular solution of the form \( x_t^P = \alpha t^2 \mu^t \).

To determine the undetermined coefficient \( \alpha \), we must solve

\[
\alpha(t + 1)^2\mu^{t+1} + a\alpha t^2\mu^t + b\alpha(t - 1)^2\mu^{t-1} = \mu^t
\]

Dividing each side by \( \mu^{t-1} \) and gathering terms yields

\[
\alpha t^2(\mu^2 + a\mu + b) + \alpha(2t + 1)\mu^2 + \alpha b(-2t + 1) = \mu
\]

Because \( \mu^2 + a\mu + b = 0 \) and \( b = \mu^2 \), one has \( \alpha = 1/2\mu \).
Second Special Case

Again, inspired by earlier theorems, we can apply the method of undetermined coefficients to the equation

\[ x_{t+1} + ax_t + bx_{t-1} = \sum_{k=1}^{m} \sum_{j=1}^{r_k} \alpha_{kj} t^j \mu_k^t \]

where we naturally assume that the constants \( \mu_k \) \((k = 1, 2, \ldots, m)\) are all different.

A particular solution takes the form

\[ x_t^P = \sum_{k=1}^{m} \sum_{j=1}^{d_k} \beta_{kj} t^j \mu_k^t \]

where the degree \( d_k \) of each polynomial \( \sum_{j=1}^{d_k} \beta_{kj} t^j \) with undetermined coefficients \( \langle \langle \beta_{kj} \rangle_{j=1}^{d_k} \rangle_{k=1}^{m} \) is

- \( r_k \) in case \( \mu_k \not\in \{ \lambda_1, \lambda_2 \}; \)
- \( r_k + 2 \) in case \( \mu_k = \lambda_1 = \lambda_2; \)
- \( r_k + 1 \) otherwise.
Determining the Coefficients

The coefficients $\langle \beta_{kj} \rangle_{j=1}^{d_k}_{k=1}^{m}$ of the particular solution

$$x_t^P = \sum_{k=1}^{m} \sum_{j=1}^{d_k} \beta_{kj} t^j \mu_k^t$$

can be found (in principle!) by solving, for $k = 1, 2, \ldots, m$, the $m$ independent systems of linear equations that result from equating coefficients of powers of $t$ in the expansions

$$\sum_{j=1}^{d_k} \beta_{kj} [(t+1)^j \mu_k^2 + at^j \mu_k^t + b(t-1)^j] = \sum_{j=1}^{r_k} \alpha_{kj} t^j \mu_k$$
Lecture Outline

Solving Second-Order Equations

Inhomogeneous Equations

Two Special Cases

Method of Undetermined Coefficients

Higher-Order Linear Equations with Constant Coefficients

Stationary States and Stability for Second-Order Equations
Higher-Order Linear Equations with Constant Coefficients

An \textit{nth order linear equation with constant coefficients}
takes the form
\[ x_t + \sum_{r=1}^{n} a_r x_{t-r} = f_t \]
in the inhomogeneous case, and
\[ x_t + \sum_{r=1}^{n} a_r x_{t-r} = 0 \]
in the homogeneous case.

The corresponding \textit{auxiliary equation} is \[ \lambda^n + \sum_{r=1}^{n} a_r \lambda^{n-r} = 0. \]
The auxiliary equation can be written as $p_n(\lambda) = 0$ whose LHS is the polynomial $\lambda^n + \sum_{r=1}^{n} a_r \lambda^{t-r}$ of degree $n$.

By the fundamental theorem of algebra, this equation has at least one root $\lambda_1$, which may be complex.

Then $p_n(\lambda)$ can be factored as $p_n(\lambda) \equiv (\lambda - \lambda_1)p_{n-1}(\lambda)$.

But now the equation $p_{n-1}(\lambda) = 0$ also has at least one root $\lambda_2$, which may also be complex.

Repeating this argument $n$ times, the auxiliary equation $p_n(\lambda) = 0$ has $n$ roots $\lambda_1, \lambda_2, \ldots, \lambda_n$, some of which may be repeated.

In particular, $p_n(\lambda) \equiv \prod_{i=1}^{n} (\lambda - \lambda_i)$.
Solving the Homogeneous Equation

**Theorem**

Consider the homogeneous equation \( x_t + \sum_{r=1}^{n} a_r x_{t-r} = 0 \), and suppose that the auxiliary equation can be written as

\[
0 = \lambda^n + \sum_{r=1}^{n} a_r \lambda^{t-r} = \prod_{j=1}^{k} (\lambda - \rho_j)^{m_j}
\]

with \( k \) distinct roots \( \rho_j \) (\( j = 1, 2, \ldots, k \)) whose respective multiplicities \( m_j \) satisfy \( \sum_{j=1}^{k} m_j = n \).

Then the general solution of the homogeneous equation takes the form

\[
x_t = \sum_{j=1}^{k} \sum_{h=1}^{m_j} \alpha_{jh} t^{h-1} \rho_j^t
\]

for \( n \) arbitrary constants \( \alpha_{jh} \) (\( h = 1, 2, \ldots, m_j \) and \( j = 1, 2, \ldots, k \)).

That is, the general solution is an arbitrary linear combination of the functions \( t^{h-1} \rho_j^t \) (\( h = 1, 2, \ldots, m_j \) and \( j = 1, 2, \ldots, k \)).
Theorem

The general solution of the inhomogeneous equation

\[ x_t + \sum_{r=1}^{n} a_r x_{t-r} = \sum_{h=1}^{i} \sum_{j=1}^{q_h} \alpha_{hj} t^j \mu_h^t \]

is the sum of the general complementary solution of the corresponding homogeneous equation \( x_t + \sum_{r=1}^{n} a_r x_{t-r} = 0 \) and any particular solution.

One particular solution takes the form

\[ x_t^P = \sum_{h=1}^{i} \sum_{j=1}^{d_h} \beta_{hj} t^j \mu_h^t \]

where the degree \( d_h \) of each polynomial \( \sum_{j=1}^{d_h} \beta_{hj} t^j \) with undetermined coefficients \( \prod_{j=1}^{d_h} \beta_{hj} t^j \) is

- \( q_h \) in case \( \mu_h \notin \{ \rho_1, \rho_2, \ldots, \rho_k \} \);
- \( q_h + m_j \) in case \( \mu_h = \rho_j \), a root of multiplicity \( m_j \).
Solving Second-Order Equations

Inhomogeneous Equations

Two Special Cases

Method of Undetermined Coefficients

Higher-Order Linear Equations with Constant Coefficients

Stationary States and Stability for Second-Order Equations
Consider the second-order equation $x_{t+1} + ax_t + bx_{t-1} = f$
for a constant forcing term $f \in \mathbb{R}$.

Here a stationary state $x^* \in \mathbb{R}$ has the defining property that
$x_{t-1} = x_t = x^* \implies x_{t+1} = x^*$.

This is satisfied if and only if $x^* + ax^* + bx^* = f$, or equivalently, if and only if $(1 + a + b)x^* = f$.

In case $a + b = -1$, there is:

- no stationary state unless $f = 0$;
- the whole real line $\mathbb{R}$ of stationary states if $f = 0$.

Otherwise, if $a + b \neq -1$, the only stationary state is $x^* = (1 + a + b)^{-1}f$. 

Stability of a Linear Equation

If $a + b \neq -1$, let us denote by $y_t := x_t - x^*$ the deviation of state $x_t$ from the stationary state $x^* = (1 + a + b)^{-1}f$. Then

$$y_{t+1} = x_{t+1} - x^* = -ax_t - bx_{t-1}f - x^*$$

$$= -a(y_t + x^*) - b(y_{t-1} + x^*) + f - x^* = -ay_t - by_{t-1}$$

Thus $y_t$ solves the homogenous equation $x_{t+1} + ax_t + bx_{t-1} = 0$.

As already seen, the solution to this homogeneous equation depends on the roots $\lambda_{1,2} = -\frac{1}{2}a \pm \frac{1}{2}\sqrt{a^2 - 4b}$ of the quadratic characteristic equation

$$f(\lambda) \equiv \lambda^2 + a\lambda + b \equiv (\lambda - \lambda_1)(\lambda - \lambda_2) = 0$$

There are three cases to consider:

1. two distinct real roots because $a^2 - 4b > 0$;
2. two complex conjugate roots because $a^2 - 4b < 0$;
3. two coincident real roots because $a^2 - 4b = 0$. 
Stability Condition

With two distinct roots $\lambda_1$ and $\lambda_2$, real or complex, the general solution of the homogeneous equation is $x_t = A\lambda_1^t + B\lambda_2^t$.

Stability is satisfied if and only if for all $A, B \in \mathbb{R}$ one has $x_t \to 0$ as $t \to \infty$.

This is true if and only if the absolute values of both roots satisfy $|\lambda_1| < 1$ and $|\lambda_2| < 1$.

With two coincident roots $\lambda_1 = \lambda_2 = -\frac{1}{2}a = \sqrt{b}$, the general solution of the homogeneous equation is $x_t = (A + Bt)\lambda^t$.

Again, stability is satisfied if and only if for all $A, B \in \mathbb{R}$ one has $x_t \to 0$ as $t \to \infty$.

This is true if and only if the absolute value of the double root satisfies $|\lambda| < 1$. 
Two Distinct Real Roots

Here $\lambda^2 + a\lambda + b = (\lambda - \lambda_1)(\lambda - \lambda_2)$
where $\lambda_1$ and $\lambda_2$ are both real.

Note that the quadratic function $f(\lambda) \equiv \lambda^2 + a\lambda + b$ is convex
and satisfies $f(\lambda) \to +\infty$ as $\lambda \to \pm\infty$.

So the real roots of $f(\lambda) = 0$ satisfy $|\lambda_1| < 1$ and $|\lambda_2| < 1$ iff

$$f(-1) > 0 \text{ and } f(1) > 0 \text{ with } f'(-1) < 0 \text{ and } f'(1) > 0$$

These conditions are equivalent to

$$1 - a + b > 0 \text{ and } 1 + a + b > 0 \text{ with } -2 + a < 0 \text{ and } 2 + a > 0$$

or to $|a| < 2$ and $|a| < 1 + b$.

Together with the condition $a^2 > 4b$
for the equation $f(\lambda) = 0$ to have two distinct real roots,
these inequalities are equivalent to $|a| - 1 < b < 1$. 
Two Complex Conjugate Roots

With two complex conjugate roots because $a^2 - 4b < 0$, one has

$$\lambda_{1,2} = -\frac{1}{2}a \pm \frac{1}{2}i\sqrt{4b - a^2} = r \, e^{\pm i\theta} = r(\cos \theta \pm i \sin \theta)$$

where $r = \sqrt{b}$ and $\theta = \arccos(a/2\sqrt{b})$

Then the general solution of the homogeneous equation can be written as $x_t = r^t(A \cos \theta t + B \sin \theta t)$.

Stability is satisfied if and only if for all $A, B \in \mathbb{R}$ one has $x_t \to 0$ as $t \to \infty$.

This is true if and only if $b < 1$, as well as $a^2 - 4b < 0$. 
With two coincident real roots both equal to \(-\frac{1}{2}a\), the general solution of the homogeneous equation is \(x_t = (A + Bt)(-\frac{1}{2}a)^t\).

Stability is satisfied if and only if for all \(A, B \in \mathbb{R}\) one has \(x_t \to 0\) as \(t \to \infty\).

This is true if and only if the modulus of the repeated root \(\lambda = -\frac{1}{2}a\) satisfies \(|\lambda| < 1\), and so if and only if \(|a| < 2\).
Theorem

The linear autonomous equation \( x_{t+1} + ax_t + bx_{t-1} = f \)
is stable, both locally and globally, if and only if \( |a| < 1 + b < 2 \).

Proof.

Stability requires one of the following three to hold:

1. distinct real roots because \( a^2 > 4b \), with \( |a| - 1 < b < 1 \);
2. complex conjugate roots because \( a^2 < 4b \), with \( b < 1 \);
3. a repeated real root because \( a^2 = 4b \), with \( |a| < 2 \).

A diagram in the \((a, b)\)-plane shows that one of these three holds if and only if \( |a| < 1 + b < 2 \).
The stable region occurs where $|a| - 1 < b < 1$, in the interior of an isosceles right-angled triangle with corners at $(a, b) = (0, -1)$ and $(a, b) = (\pm 2, 1)$. 
A Variable Forcing Term

Consider now the second-order equation \( x_{t+1} + ax_t + bx_{t-1} = f_t \) for a variable forcing term \( f_t \).

The general solution takes the form \( x_t^G = x_t^H + x_t^P \) where:

- \( x_t^P \) is one particular solution of \( x_{t+1} + ax_t + bx_{t-1} = f_t \);
- \( x_t^H \) is any one of a two-dimension continuum of solutions of the homogeneous equation \( x_{t+1} + ax_t + bx_{t-1} = 0 \).

The stability condition \(|a| < 1 + b < 2\) is necessary and sufficient for any solution of the homogeneous equation to satisfy \( x_t^H \rightarrow 0 \) as \( t \rightarrow \infty \).

It is therefore also necessary and sufficient for the difference between any two solutions \( x_t^{(1)} \) and \( x_t^{(2)} \) of the inhomogeneous equation \( x_{t+1} + ax_t + bx_{t-1} = f_t \) to satisfy \( x_t^{(1)} - x_t^{(2)} \rightarrow 0 \) as \( t \rightarrow \infty \).

In the long run, this means that there is an asymptotically unique solution to \( x_{t+1} + ax_t + bx_{t-1} = f_t \).