Package Auctions

Jonathan Levin
Econ 285, Market Design
February 2009
The problems begin…

- Two items for sale, A and B
- Two bidders with values:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>AB</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
</tbody>
</table>

- No item prices clear the market.
- Such prices must result in bidder 1 efficiently buying both: $p_A \geq 10$, $p_B \geq 10$, and $p_A + p_B \leq 12$!
Focus on “core” outcomes

- Core is the relevant generalization of comp. eqm.
- Roughly, an allocation is in the core if there is no set or “coalition” of players that could make a deal on their own from which all of them would benefit.

- Core in NTU and TU models
  - In matching theory, we focused on NTU case.
  - In transferable utility games, core is defined to be the set of imputed payoff profiles, or “imputations,” which correspond to core allocations.
Definitions

- $X$ is the set of feasible allocations, $x \in X$
- $N$ is the set of bidders and seller, $j \in N$
- $u_j(x_j)$ the valuation function for bidder
- Coalitional value function $w$, for coalition $S \subset N$
  \[
  w_u(S) = \begin{cases} 
  0 & \text{if seller } \notin S \\
  \max_{x \in X} \sum_{j \in S} u_j(x_j) \text{ subject to } x_S = 0 & \text{if seller } \in S
  \end{cases}
  \]
- $(N, w)$ is the coalitional game derived from trade between the seller and bidders.
- Core “imputations” are defined as follows
  \[
  Core(N, w) = \left\{ \pi \geq 0 \left| \sum_{j \in N} \pi_j = w(N), (\forall S) \sum_{j \in S} \pi_j \geq w(S) \right. \right\}
  \]
Auctions vs exchanges

- Same example: items A, B and bidders with values

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>AB</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
</tbody>
</table>

- With one seller, the imputation (10,2,0) is in the core.
- But if goods A and B belong to different sellers, then the core is empty, because…
  - Bidder 2 must get 0
  - Coalition of either seller and bidder 2 must get 10
  - So, each seller must get 10, but only 12 is available.
Package markets in practice

- Sellers often use *pay-as-bid package auctions*
  - London bus routes (Cantillon & Pesendorfer)
  - Sears truck routes (Ledyard et al)
  - Chilean school lunches (Epstein et al)
  - *British spectrum auctions to use new design…*
- Many decentralized markets also have package problems, and some evidence of market failures.
  - Compare cell phones in the US and Europe
  - Real estate development patterns in US
Why not Vickrey auction?

### Outcome of VCG mechanism

- Bidder 1 gets A, B, - as is efficient.
- Bidder 1 pays 10
- Payoffs (10, 2, 0) are in the core.

- But… VCG outcomes need not be in the core…
VCG and the Core

- Serious problem with VCG: outcomes may be outside the core with very low revenues.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>AB</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
</tbody>
</table>

- Bidders 2 and 3 win items at Vickrey price 2
- Seller payoff is just 4, so payoffs (4, 0, 8, 8)
- But bidder 1 and seller can get 12!
Vickrey, Core and Comp Eqm

- Bidder j’s Vickrey payoff \( v_j = w(N) - w(N-j) \).
  - Marginal contribution to the coalition of the whole.
  - Note: there is a core allocation in which \( j \) gets \( v_j \) -- other bidders gets zero, and seller gets \( w(N) - v_j \).

- Theorem (Bikchandani-Ostroy, Ausubel-Milgrom). A bidder’s Vickrey payoff is \( v_j = \max \{ r_j | r \in \text{Core}(N,w) \} \).

- Corollary 1. The Vickrey payoff vector is in the core if and only if there is a unique bidder optimal core imputation.

- Corollary 2. If the Vickrey payoff vector is not in the core, then for every core imputation \( v' \in \text{Core}(N,w) \), \( v_0 < v'_0 \).
  - Interpretation: When the Vickrey payoff is not in the core, the seller’s Vickrey revenue is uncompetitively low.
Scope of the problem

- Questions
  - When do core outcomes exist? (With one seller: Always!)
  - When are Vickrey outcomes in the core?
  - When do competitive equilibria exist?

- Two “positive” results:
  - **Theorem (Ausubel-Milgrom):** If goods are substitutes for all bidders, then Vickrey outcomes are core outcomes.
  - **Theorem (Milgrom, Gul-Stacchetti):** If goods are substitutes for all bidders, then competitive equilibria exist.

- So, the two problems vanish when goods are substitutes.
Core selecting auctions

- The most popular package auction alternatives to Vickrey are designs that result in core outcomes.

- Day and Milgrom (2008) analyze general class.
  - Bidders report preferences
  - Auctioneer selects a “suitable” core allocation

- Possible desiderata in design
  - Efficiency (maximize total value + participation)
  - Competitive revenue, discourage shills
  - Make “truthful” reporting optimal, or close to it.
Example 1: Pay-as-Bid

- **Canonical rules**
  - Each bidder bids a separate price for each package it may want to buy.
  - Seller may accept at most one bid per bidder
  - Seller may impose constraints, such as
    - Minimum quantity sold to minority-owned bidders
    - Maximum concentration ratio of sales
    - Procurement sales: geographic diversification of supply
  - Auctioneer selects the feasible combination of offers that is optimal according to some objective.

- **Each winning bidder pays the price it bid.**
Pay-as-Bid Theory

- **Definition.** A core allocation is “bidder optimal” if there is no other core allocation that is weakly preferred by every bidder and strictly preferred by at least one.

- **Theorem (Bernheim-Whinston).** The full-information, coalition-proof equilibrium outcomes of the pay-as-bid package auction are exactly the bidder optimal core allocations.
  - Special case: one good, Bertrand equilibrium
  - At each equilibrium, for every package, each bidder bids its value minus its core payoff (“profit target strategy”).
Pay-as-Bid Auction Example

- **Values**

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>AB</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>10*</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>10*</td>
<td>8</td>
<td>11</td>
</tr>
</tbody>
</table>

- **Equilibrium bids:** “constant profit targets” but no bid is less than zero. Equilibrium with \( \pi=(12,0,3,5) \).
Interpretation?

- **Theorem (Bernheim-Whinston).** The full-information, coalition-proof equilibrium outcomes of the pay-as-bid package auction are exactly the bidder optimal core allocations.

- How should we interpret this?
  - Competitive payoffs (Bertrand analogy)?
  - Why are these equilibria interesting?
    - Infeasibility of the full-information strategy
    - Equilibrium selection criterion
    - Still multiple equilibria!
  - Why is the design popular? Theorem is a poor answer.
Example 2: Ascending Proxy

- Several package extensions of ascending auction
- Ausubel-Milgrom proxy auction
  - Bidders report maximum bids to a proxy bidder.
  - Auction initiates with bids of zero by all bidders for all packages
  - Auctioneer “holds” its most preferred feasible collection of bids.
    - Typically, total sales revenue determines the preference
    - Tie-breaking rule makes auctioneer preferences strict
  - At each round,
    - Bidders with bids being held do nothing
    - For others, proxy bidders makes the most “profitable” new bid, or no bid if none is profitable.
  - Bids accumulate: the auctioneer may choose from all previously submitted bids.
  - Auction ends when there are no new bids.
Algorithm Property

- **Theorem (Ausubel-Milgrom).** The ascending proxy auction terminates at an efficient outcome (cf Parkes) and, what is more, at a core allocation, both with respect to the reported preferences.

- **Proof Idea (NTU argument).** At the termination of the algorithm,
  - Allocation is feasible (by construction)
  - Allocation is unblocked by any coalition $S$
    - Each bidder in $S$ has made every offer that she prefers.
    - The seller prefers the selected outcome to any other profile of offered bids.

- **Equilibrium?** Covered by more general analysis…
Direct Auction Mechanisms

- **Definition.** A direct auction mechanism is a pair \((f,P)\) – allocation and payment rule.
- Denote actual values by \(u = \{u_j\}_{j=1}^J\) and reported values by \(\hat{u} = \{\hat{u}_j\}_{j=1}^J\). Then,

\[
\text{Outcome } = (x,p) = (f(\hat{u}), P(\hat{u}))
\]

\[
f(\hat{u}) \in \arg\max_{x \in X} \sum_j \hat{u}_j(x_j)
\]

\[
\pi_0 = u_0(x_0) + \sum_{j \neq 0} p_j
\]

\[
\pi_j = u_j(x_j) - p_j \text{ for } j = 1, \ldots, J
\]
Vickrey Payoff is “Available”

- …at least with complete information.
- Given the reports of others, let $\pi_j = w(N) - w(N-j)$ be the payoff that $j$ would earn by reporting truthfully in a Vickrey auction.
- In any core selecting auction, $j$ can earn (within $\varepsilon > 0$ of) $\pi_j$ by reporting values of $u_j(x') - \pi_j + \varepsilon$ for all allocations $x'$, because
  - this report does not change the auction outcome $x$
  - $j$’s maximum core payment is $u_j(x) - \pi_j + \varepsilon$ leading to a payoff of $\pi_j - \varepsilon$. 
Details

- With the new report, the value maximizing allocation still requires $j$’s participation:

$$\hat{w}(N) - \hat{w}(N - j) = (\hat{w}(N) - w(N)) + (w(N) - w(N - j))$$

$$\geq -(\pi_j - \varepsilon) + \pi_j = \varepsilon > 0$$

- Since $j$’s reported profit is at least 0, its actual profit is nearly its Vickrey payoff $\pi_j$:

$$0 \leq \hat{u}_j(x_j) - p_j$$

$$= u_j(x_j) - p_j - (\pi_j - \varepsilon)$$
No Shills ⇔ Core-Selecting

- **Theorem** (Day-Milgrom). An efficient direct mechanism (1) has the property that no bidder can ever earn more than its Vickrey payoff by using shills if and only if (2) it is a core-selecting mechanism.
Proof

- We must show that no collection of shill bidders earns more than its Vickrey payoff if and only if the core inequalities are all satisfied.
- It is enough to show that for all collections of (shill) bidders \( S \):
  \[
  \sum_{j \in S} \pi_j \leq w(N) - w(N - S) \iff \sum_{j \in N - S} \pi_j \geq w(N - S)
  \]
- The implication holds because the total payoff to all participants in an efficient mechanism is \( w(N) \).
Truncations

- A “truncation report” in matching is a report that preserves the ordering of all outcomes except the unmatched outcome, which is reported to be ranked higher than its real rank.
- A “$\pi_j$-truncation report” in an auction is one in which all reported values for non-null packages are reduced by $\pi_j$.
- Other names for truncation reports include “profit target bids” and “truthful bids.”
Tie-Breaking

- Tie-breaking is always an issue in full-information auction equilibrium analysis. We suppress it in the classroom, allowing ourselves the freedom to work with simple formulations and declare the winner any way we wish in the event of ties among the bids.

- Careful statements of the following theorems are found in the Day-Milgrom paper.
Best Replies

- **Theorem.** For any core-selecting auction, suppose the bidders besides \( j \) report values \( \hat{u}_{-j} \). Then a best reply for \( j \) is to make the truncation report with the Vickrey profit target, as follows:
  \[
  \bar{\pi}_j = w_{\hat{u}}(N) - w_{\hat{u}}(N - j).
  \]

- **Proof.** We have shown above that this report achieves the Vickrey profit and that, in a core-selecting auction, no higher profit is achievable.
Bidder Optimality

- **Definition.** An imputation \( \pi \in \text{Core}(N,w) \) is *bidder optimal* if there is no other imputation \( \pi' \in \text{Core}(N,w) \) with \( \pi'_0 > \pi_0 \) (meaning weakly larger in all components and strictly larger in at least one).

- **Theorem.** For every core-selecting auction, every valuation profile \( u \), and every bidder optimal core imputation \( \pi \), the \( \pi \)-truncation strategies form a Nash equilibrium and each bidder \( j \) earns profits of exactly \( \pi_j \).
Proof Sketch

• Step 1: no goods assignment leads to value exceeding $\pi_0$. For suppose not and let $S$ be the smallest coalition to contradict. Then,

$$\pi_0 < \max_{x \in X, x - S = \emptyset} u_0(x) + \sum_{j \in S - 0} (u_j(x) - \pi_j)$$

$$\leq w_u(S) - \sum_{j \in S - 0} \pi_j$$

which contradicts the core.

• Step 2: For any bidder $j$, there is a coalition excluding $j$ that offers value $\pi_0$ (else $\pi$ is not bidder optimal).

• Step 3: Apply the preceding theorem.
Optimal Incentives

- Definitions.
  1. The *incentive profile* for a core-selecting auction $P$ at $u$ is the vector of $\varepsilon_j^P(u)$ which represent the maximum gains from deviating from truthful reporting.
  2. Incentives are optimal if no other core-selecting auction leads to a smaller incentive profile.
Optimal Incentives

- **Theorem.** A core-selecting auction provides optimal incentives for all value profiles $u$ if and only if for every report, the auction chooses a bidder-optimal core allocation.

- **Proof Sketch.** A bidder’s maximum payoff is its Vickrey payoff. The incentive to deviate is the difference between that and the assigned payoffs. Incentives are best when assigned payoffs are maximized.
Min Rev Core-Selecting Auctions

- If there is but a single good for sale, then the unique minimum revenue core-selecting auction is the second-price auction.

- In an auction with multiple winners, there can be multiple minimum revenue points in the core, so there are multiple minimum-revenue core-selecting auctions.

<table>
<thead>
<tr>
<th>Bidder</th>
<th>A</th>
<th>B</th>
<th>AB</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>

Min Revenue Possibilities

\[ p_A + p_B = 12 \]

\[ 2 \leq p_A, p_B \leq 10 \]
Theorem. For every bidder optimal core-selecting mechanism, if goods are substitutes for all bidders, then truthful reporting of values is an ex post equilibrium.

- This means that, even if it were to learn others’ bids, no bidder could ever profit by then changing her own bids.
- Proof: Truthful reporting leads to the unique bidder optimal point, with Vickrey profits. No higher profit is available.

Corollary. For every minimum-revenue core-selecting mechanism, if goods are substitutes for all bidders, then truthful reporting of values is an ex post equilibrium.
British WiMax Auction

- First large-scale implementation of a min-rev core-selecting auction, with initial phase.

- Primary rounds: an ascending clock auction.
- Supplementary round: a direct mechanism which finds the total bid maximizing allocation and sets base prices equal to be the *Vickrey-nearest minimum-revenue core prices*.

- Activity rule in clock phase is crucial to the design – initially problematic and would have lead to serious problems…
Minimum Revenue Core Prices

- A bidder can have a strategic incentive to mark-down its bid to less than its value for packages that it expects to win.
  - Importantly… not for other packages!
  - …and not if the goods are substitutes
  - …and not if the relevant package is the package of the whole.

- A \textit{minimum-revenue} core-selecting auction
  - among core-selecting auctions, minimizes the total amount that all bidders can gain by marking down bids
  - ensures that total revenue is a non-decreasing function of the total amounts bid.
Vickrey-Nearest Core Prices

- Select the minimum revenue core prices that are nearest to the Vickrey prices in a Euclidean metric.
Package Exchanges

- **Definition**: An exchange with multiple buyers, multiple sellers, and package bids (and possibly with some players who buy and sell different items).

- **Applications**:
  - Securities trading, with packages consisting of orders to buy and sell related securities.
  - Spectrum trading, in an attempt to shift the broadcast bands to higher value uses.
Conclusion

- Package auctions are finding increasing use for hard resource allocation problems.

- Vickrey auctions problematic because of low revenue, non-core outcomes.

- New designs trade off incentive and distributional properties.

- Package exchanges are fundamentally hard due to empty cores, but some interesting new ideas are being studied.