18.100C Writing Assignment 6

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1 Rudin Ch.6, Problem 10

Let \( p \) and \( q \) be positive real numbers such that

\[
\frac{1}{p} + \frac{1}{q} = 1. \tag{1}
\]

Prove the following statements.

1.1 If \( u \geq 0 \) and \( v \geq 0 \), then \( uv \leq \frac{u^p}{p} + \frac{v^q}{q} \). Equality holds if and only if \( u^p = v^q \).

If \( u = 0 \), \( v = 0 \) or both, the inequality is trivial since the LHS is zero, while the RHS is strictly nonnegative. So, we now treat the case where \( u > 0 \) and \( v > 0 \).

We wish to come up with a more convenient form of the inequality to work with. So, divide Eq (1) by the quantity \( v^q \) to obtain:

\[
\frac{u}{v^{q-1}} \leq \frac{1}{p} \cdot \frac{u^p}{v^q} + \frac{1}{q} \tag{2}
\]

Define \( z = \frac{u^p}{v^q} \). Given the bounds on \( u, v, p, q \), it is easy to see that \( z \) is a real variable taking on the values in \([0, \infty)\). Furthermore, recognizing

\[
\frac{u}{v^{q-1}} = \frac{u}{v^{q/p}} = \left( \frac{u^p}{v^q} \right)^{\frac{1}{p}} = z^{\frac{1}{p}} \tag{3}
\]

we can write Eq (2) as:

\[
z^{\frac{1}{p}} \leq \frac{1}{p} \cdot z + \frac{1}{q} \tag{4}
\]

\[
0 \leq \frac{1}{p} \cdot z - z^{\frac{1}{p}} + \frac{1}{q} \tag{5}
\]

So our task is now to show that the function \( f(z) = \frac{1}{p} \cdot z - z^{\frac{1}{p}} + \frac{1}{q} \) is nonnegative over \([0, \infty)\). At the boundaries, we find that \( f \) is positive. This is obvious for the case \( f(z = 0) = \frac{1}{q} \). The boundary as \( z \to \infty \)
is clear if we write \( f \) as:

\[
\begin{align*}
    f(z) &= \frac{1}{p} \cdot z - z^{\frac{1}{p} + \frac{1}{q}} \\
    &= z^{\frac{1}{p}} \cdot \left( \frac{1}{p} \cdot z^{\frac{1}{q}} - 1 \right) + \frac{1}{q},
\end{align*}
\]

since it is readily apparent that both the terms in the product are positive in the limit \( z \to \infty \). It remains to be shown that \( f \) is nonzero in the interval \((0, \infty)\).

Because we have shown that \( f \) at its two endpoints is positive, if \( f \) is negative in between, it must possess a local minimum at some point \( z_0 \) characterized by \( f'(z_0) = 0 \) and \( f(z_0) < 0 \). Computing the derivative of \( f \) gives:

\[
\begin{align*}
f'(z) &= \frac{1}{p} - \frac{1}{z^{\frac{1}{p}-1}} \\
    &= \frac{1}{p} \cdot \left( 1 - z^{-\frac{1}{q}} \right)
\end{align*}
\]

Since the only real solution to \( 1 - z^{-\frac{1}{q}} = 0 \) is \( z_0 = 1 \), there is only one point to investigate. Evaluating \( f \) at \( z_0 \) gives:

\[
\begin{align*}
f(z_0 = 1) &= \frac{1}{p} \cdot 1 - \frac{1}{\frac{1}{p} + \frac{1}{q}} \\
    &= \frac{1}{p} + \frac{1}{q} - 1 = 0
\end{align*}
\]

We see that the only local minimum (and hence the absolute minimum, since we also investigated the boundaries) of \( f \) over \([0, \infty)\) is 0, so we conclude that \( f(z) \geq 0 \), establishing the inequality we wanted to prove.

We now wish to show that the equality holds if and only if \( u^p = v^q \).

Suppose \( u^p = v^q \). In our language of \( f \) and \( z \), this corresponds to the case when \( z = 1 \). Returning to Eq (4) we find:

\[
\begin{align*}
z^{\frac{1}{p}} &\leq \frac{1}{p} \cdot z + \frac{1}{q} \\
1 &\leq \frac{1}{p} + \frac{1}{q} = 1
\end{align*}
\]

so the equality follows.

Now suppose we have the equality, i.e. \( z^{\frac{1}{p}} = \frac{1}{p} \cdot z + \frac{1}{q} \). We want to prove that \( z_0 = 1 \) (i.e. \( u^p = v^q \)) is the only possible solution.

Put \( g(z) = z^{\frac{1}{p}} \) and \( h(x) = \frac{1}{p} \cdot z + \frac{1}{q} \). Suppose, to get a contradiction, that there is a \( z_1 \neq z_0 \) such that \( g(z_1) = h(z_1) \).

Suppose \( z_1 > z_0 \). Since \( g \) is continuous and differentiable on \((0, \infty)\), by the MVT there is a \( c \in (z_0, z_1) \) such that:

\[
g'(c) = \frac{g(z_1) - g(z_0)}{z_1 - z_0}
\]
\[
\frac{h(z_1) - h(z_0)}{z_1 - z_0} = \frac{1}{p}
\]

but this is impossible since \(g'(z) = \frac{1}{p} \cdot \frac{1}{z^{\frac{1}{p}}}\) and hence \(g'(z) < \frac{1}{p}\) for all \(z \in (z_0, \infty)\).

If \(z_1 < z_0\), we similarly obtain a contradiction, since we can then bound \(g'\) below by \(\frac{1}{p}\).

So the only solution to \(g(z) = h(z)\) on \([0, \infty)\) is \(z_0 = 1\). In conclusion, we see that equality implies \(z = 1\) and hence \(u^p = v^q\).

**1.2** If \(f \in R(\alpha)\) and \(g \in R(\alpha)\), \(f \geq 0, g \geq 0\) and \(\int_a^b f^p d\alpha = \int_a^b g^q d\alpha = 1\), then \(\int_a^b fg d\alpha \leq 1\)

It is clear that \(f\) and \(g\) pointwise satisfies the inequality:

\[
fg \leq \frac{f^p}{p} + \frac{g^q}{q}
\]

where \(p\) and \(q\) are chosen as in the first part of this assignment. By Rudin 6.12(b) then:

\[
\int_a^b f^p d\alpha \leq \int_a^b \left( \frac{f^p}{p} + \frac{g^q}{q} \right) d\alpha
\]

\[
\leq \frac{1}{p} \int_a^b f^p d\alpha + \frac{1}{q} \int_a^b g^q d\alpha
\]

\[
\leq \frac{1}{p} + \frac{1}{q} = 1
\]

where we have used the linearity of the integral with respect to the integrand (Rudin 6.12a) and the properties of \(f^p, g^q\) as well our earlier definition of \(p\) and \(q\). The result is proved.

**1.3** If \(f\) and \(g\) are complex functions in \(R(a)\), then show the following:

\[
\left| \int_a^b fg d\alpha \right| \leq \left( \int_a^b |f|^p d\alpha \right)^{\frac{1}{p}} \cdot \left( \int_a^b |g|^q d\alpha \right)^{\frac{1}{q}}
\]

(7)

First, from Rudin 6.13b, we have:

\[
\left| \int_a^b fg d\alpha \right| \leq \int_a^b |f||g| d\alpha
\]

(8)

We wish to apply the results of the previous section to \(|f|\) and \(|g|\), which are by definition nonnegative. Define two constants \(F, G\) by:

\[
F = \left( \int_a^b |f|^p d\alpha \right)^{\frac{1}{p}}
\]

\[
G = \left( \int_a^b |g|^q d\alpha \right)^{\frac{1}{q}}
\]
First we assume that $F, G$ are nonzero. Then define $u = |f|/F$ and $v = |g|/G$. Trivially, we have the following:

$$\int_a^b u^p d\alpha = \int_a^b \frac{|f|^p}{F^p} d\alpha = \frac{1}{F^p} \int_a^b |f|^p d\alpha = \frac{1}{F^p} \cdot \int_a^b |f|^p d\alpha = 1$$

Similarly, we obtain, $\int_a^b v^q = 1$.

Then $u$ and $v$ satisfies the premises for the previous results, and we have:

$$\int_a^b uv d\alpha \leq 1$$

$$\frac{1}{FG} \int_a^b |f| |g| d\alpha \leq 1$$

$$\int_a^b |f| |g| d\alpha \leq FG \left( \int_a^b |f|^p d\alpha \right)^{\frac{1}{p}} \left( \int_a^b |g|^q d\alpha \right)^{\frac{1}{q}}$$

and we have the result, provided that $F, G$ are nonzero.

Now we return to the case where $F, G$ or both are zero. We assume that $f$ and $g$ are continuous functions, so that $|f|, |g|$ are also continuous (Rudin 4.7: Continuity of compositions). We wish to show that under these conditions, the inequality is trivial.

Suppose $F = 0$. In the recent problem set, we showed (Rudin Ch.6, Problem 2) that if $h(x) \geq 0$, $h$ is continuous on $[a, b]$, and $\int_a^b h(x) dx = 0$, then $h(x) = 0$ for all $x \in [a, b]$. It’s clear that for the case of integration with respect to $\alpha$, the generalization of this theorem is that $h(x) = 0$ for those points for which $\frac{d\alpha}{dx} > 0$. Put more intuitively, $h$ is zero when it contributes to the integral.

So, $F = 0$ implies that $|f|^p = 0$ for all $x \in [a, b]$ for which $\frac{d\alpha}{dx} > 0$. In turn, this implies $|f| = 0$ and $f = 0$ for those points. Therefore $\int_a^b fg d\alpha = 0$ since $f = 0$ for those points for which $\frac{d\alpha}{dx} > 0$. A similar conclusion is obtained if we assume $G = 0$. So in the case where one of the integrals vanish, Holder’s inequality assumes the trivial form: $0 \leq 0$.

We have covered all possibilities.

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1We consider only $\frac{d\alpha}{dx} > 0$ since $\alpha$ is monotonically increasing.