Exercises on independent random variables and product measures

Solve Exercises [1.3.65], [1.4.15], [1.4.18] in Amir Dembo’s lecture notes.

Exercises on \( L_p \) spaces

Fix a probability space \((\Omega, \mathcal{F}, P)\). Throughout this exercise, we will say that two random variables \(X, Y\) are equivalent if \(P(\{\omega \mid X(\omega) = Y(\omega)\}) = 1\).

1. Show that the above indeed defines an equivalence relation.

Let, for \(p > 0\), \(L_p(\Omega, \mathcal{F}, P)\) denote the space of (equivalence classes of) random variables \(X\) such that \(E\{|X|^p\} < \infty\).

2. Show that, for \(p \geq 1\), \(\|X\|_p \equiv E\{|X|^p\}^{1/p}\) is a norm on this space.

For \(p = \infty\), \(L_\infty(\Omega, \mathcal{F}, P)\) is the space of equivalence classes of random variables \(X\) such that there exists \(M < \infty\) such that \(P(\{\omega \mid |X(\omega)| \leq M\}) = 1\).

3. Show that \(\|X\|_\infty \equiv \inf\{M : P(\{\omega \mid |X(\omega)| \leq M\}) = 1\}\) is a norm on \(L_\infty(\Omega, \mathcal{F}, P)\).

4. For \(X \in L_p(\Omega, \mathcal{F}, P)\), show that \(X \in L_p(\Omega, \mathcal{F}, P)\) for any \(p > 0\), and that \(\|X\|_p \to \|X\|_\infty\) as \(p \to \infty\).

5. For \(X\) a random variable, let \(S(X) \equiv P(\{\omega \mid X(\omega) \neq 0\})\). Show that, if \(X \in L_q(\Omega, \mathcal{F}, P)\) for some \(q > 0\), then \(\lim_{p \to 0} \|X\|_p^p = S(X)\).

6. Show that the space of simple functions \(SF\) is dense in \(L_p(\Omega, \mathcal{F}, P)\) for any \(0 < p \leq \infty\).

[Note that for \(0 < p < 1\) the space \(L_p(\Omega, \mathcal{F}, P)\) is not a normed space. The statement has to be interpreted in the following sense. For any random variable \(X\) with \(E\{|X|^p\} < \infty\), there exist a sequence of simple functions \(X_n\) such that \(E\{|X_n - X|^q\} \to 0\).]

Optional: For the enthusiasts

This will not be graded, but is an interesting and well known fact: \(L_p(\Omega, \mathcal{F}, P)\) is a Banach space for all \(1 \leq p \leq \infty\). This means (beyond the fact of being a normed space) the following. If \(\{X_n\}\) is a Cauchy sequence of (equivalence classes of) random variables in \(L_p(\Omega, \mathcal{F}, P)\), then \(\{X_n\}\) converges to a limit \(X_\infty \in L_p(\Omega, \mathcal{F}, P)\). Cauchy means that, for any \(\epsilon > 0\) there exists \(N(\epsilon) < \infty\) such that, if \(m, n \geq N(\epsilon)\), then \(\|X_m - X_n\|_p \leq \epsilon\).
At this point of the course, you know all that is needed to prove this (or to read a proof :-).

[Hint: Every Cauchy sequence converges if the following happens: for any sequence \( \{Z_n\} \) such that \( \sum_{k=1}^{\infty} \|Z_n\|_p < 1 \) the sums \( W_n = \sum_{k=1}^{n} Z_k \) converge.]