Problem 5.1: Transformation of y-velocity

This is Spacetime Physics L-7

Similar setup as the previous problem but now start with an object that is moving in the \( y' \) direction, \( v'_y = \frac{\Delta y'}{\Delta t'} \). The rocket frame \( (S') \) is moving in the \( x \) direction with speed \( v_{\text{rel}} \) as in the previous problem. Show that the \( x \) and \( y \) components of the velocity, as observed in the \( S \) frame are,

\[
\begin{align*}
    v_x &= v_{\text{rel}} \\
    v_y &= v'_y \sqrt{1 - \frac{v_{\text{rel}}^2}{c^2}}
\end{align*}
\]

This is pretty straightforward. We need the inverse Lorentz transforms,

\[
\begin{align*}
    t &= \gamma [t' + \frac{v_{\text{rel}} x'}{c^2}] \\
    x &= \gamma [x' + v_{\text{rel}} t'] \\
    y &= y'
\end{align*}
\]

Or, in terms of differences,

\[
\begin{align*}
    \Delta t &= \gamma [\Delta t' + \frac{v_{\text{rel}} \Delta x'}{c^2}] \\
    \Delta x &= \gamma [\Delta x' + v_{\text{rel}} \Delta t'] \\
    \Delta y &= \Delta y'
\end{align*}
\]

First note that \( \Delta x' = 0 \), because in the primed frame the object does not move in the \( x' \) direction. Then we have,

\[
\begin{align*}
    \Delta t &= \gamma [\Delta t'] \\
    \Delta x &= \gamma [v_{\text{rel}} \Delta t'] \\
    \Delta y &= \Delta y'
\end{align*}
\]

Now we can calculate \( v_y \) with ease,

\[
v_y = \frac{\Delta y}{\Delta t} = \frac{\Delta y'}{\gamma [\Delta t']} = \frac{v'_y}{\gamma} = v'_y \sqrt{1 - \frac{v_{\text{rel}}^2}{c^2}}
\]

With \( v_x = \frac{\Delta x}{\Delta t} \) we get,

\[
v_x = \frac{\Delta x}{\Delta t} = \frac{\gamma [v_{\text{rel}} \Delta t']}{\gamma [\Delta t']} = v_{\text{rel}}
\]

Problem 5.2: Tilted meter stick

This is Spacetime Physics L-10

This problem use the result of L-7 (however note that we want the opposite result \( v'_y \) given \( v_y \) so result is flipped).

a) Consider the two events, which are simultaneous in the lab frame: Passage of the left end of the meter stick by the \( x \) axis and passage of the right end. In the rocket frame these two events will not be simultaneous. The rocket is traveling in the \(+x\) direction, at speed \( v \), as viewed in the lab frame. Also, meter stick is along the line of simultaneity in the lab frame, which appears tilted in the rocket frame.

b) Set up the problem so that at time \( t = 0 \) the stick is just passing the \( x \) axis in the lab frame. We can set \( t = 0 \) when the center of the meter stick passes the origin in the rocket frame. Thus,

<table>
<thead>
<tr>
<th>Lab frame at ( t = 0(x, y, t) )</th>
<th>Left end</th>
<th>Middle</th>
<th>Right end</th>
<th>Event 1</th>
<th>Event 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y_L = 0 )</td>
<td>( y_C = 0 )</td>
<td>( y_R = 0 )</td>
<td>( y_C = t = 0 )</td>
<td>( y_R = t = 0 )</td>
<td></td>
</tr>
<tr>
<td>( x_L = -0.5m )</td>
<td>( x_C = 0m )</td>
<td>( x_R = +0.5m )</td>
<td>( x_C = 0 )</td>
<td>( x_R = +0.5m )</td>
<td></td>
</tr>
</tbody>
</table>

| Rocket frame \( (x', y', t' = 0) \) | \( y'_L = -y' \) | \( y'_C = 0 \) | \( y'_R = y \) | \( y'_C = t' = 0 \) | \( y'_R = y', t' = 0 \) |
\[ x'_L = -x' \quad x'_{t'} = 0 \quad x'_R = x' \quad x'_{t'} = 0 \quad x'_R = x' \]

As should be evident from a spacetime diagram (for \( ct - x \)) \( E_2 \) will occur before \( E_1 \) in the \( S' \) (rocket) frame. What we want is to find where the ends of the meter stick is at time \( t' = 0 \). Thus we define a third event, \( E_3 \) as the event when the right end passes the \( x' \) axis.

We only need to calculate one side to find our answer. Our goal is to find the coordinates of event 3, which are \((y'_3 = y'_R(t' = 0), x'_3 = x'_R(t' = 0))\). The Lorentz transformation will map the origin of \( S \) into the origin of \( S' \), meaning that \( E_1: (t' = x' = y' = 0) \). Then from trigonometry we have,

\[
\tan \phi' = \frac{y'_R(t = 0)}{x'_R(t = 0)} = \frac{y'_3}{x'_3}
\]

So we use the (inverse) Lorentz transformation to map relate event 3 in the two frames. Again, in \( S' \) we have \( E_3: (t' = 0, x' = x'_3, y' = y'_3) \) where the last two variables are to be found.

(In the following the terms \( v \) and \( \gamma \) relate to the \( x \) direction movement, \( v_y \) will relate to the speed of the meter stick in \( S \), and \( x_3 = \frac{1}{2} m \).

\[
t'_3 = 0 = \gamma(t_3 - \frac{vx_3}{c^2}) \quad \rightarrow \quad t_3 = \frac{vx_3}{c^2}
\]

\[
x'_3 = \gamma [x_3 - v_y t_3] = \gamma [x_3 - v (\frac{vx_3}{c^2})] = \gamma x_3 \left( 1 - \frac{v^2}{c^2} \right)
\]

\[
y'_3 = y_3 = v_y t_3 = v_y \frac{vx_3}{c^2}
\]

**Problem 5.3: The Rising Manhole**

This is *Spacetime Physics* L-11

This uses the tilted meter stick problem to answer. In the frame of the meter stick the manhole will appear tilted and the ends do not pass through at the same time.

**Problem 5.4: Superluminal expansion of quasar 3C273**

This is *Spacetime Physics* 3-16

To reinstating the factors of \( c \), the following statements should be changed.

a): "\( \Delta t(1 - v^2 \cos \theta) \)" should be replaced with "\( \Delta t(1 - \frac{v^2}{c^2} \cos \theta) \)"
Problem 5.5: Contraction or rotation?

This is *Spacetime Physics* 3-17

A cube at rest in the rocket frame has an edge of length 1 meter in that frame. In the laboratory frame the cube is Lorentz contracted in the direction of motion. Determine this Lorentz contraction, for example, from locations of four clocks at rest and synchronized in the laboratory frame lattice with which the four corners of the cube $E, F, G < H$ coincide when all four clocks read the same time. This latticework measurement eliminates time lags in the travel of light from different corners of the cube.

Now for a different observing procedure! Stand in the lab frame and look at the cube with one eye as the cube passes overhead. What one sees at any time is light that enters the eye at that time, even if it left the different corners of the cube at different times. Hence, what ones sees visually may not be the same as what one observes with the latticework of clocks. If the cube is viewed from the bottom then the distance $GO$ is equal to the distance $HO$, so the light that leaves $G$ and $H$ simultaneously will arrive at $O$ simultaneously. Hence, when one sees the cube to be overhead one will see the Lorentz contraction of the bottom edge.

![Figure](attachment:image.png)

a) Light from $E$ that arrives at $O$ simultaneously with light from $G$ will have to leave $E$ earlier than light from $G$ left $G$. How much earlier? How far has the cube moved in this time? What is the value of the distance $x$ in the right top figure?

Since we are assuming the cube is far away from the observer, the light has to leave $E$ approximately $\frac{1}{3 \times 10^{-9}}$ seconds (the cube has a side length of one meter) before the light leaves $G$. Thus,

$$\Delta t = 3 \times 10^{-9} \text{s}$$

The cube is traveling at speed $v$, so during this time it traverses a distance $v \Delta t$:

$$x = v \times 3 \times 10^{-9} \text{s} = \frac{v}{c} \times (1 \text{m})$$

b) Suppose the eye interprets the projection in the figures as a rotation of a cube that is not Lorentz contracted. Find an expression for the angle of apparent rotation $\phi$ of this uncontracted cube. Interpret this expression for the limiting two cases of cube speed in the lab frame: $v \rightarrow 0$ and $v \rightarrow c$.

The angle $\phi$ can be found using the definition of sine and the expression for $x$ that we just found:

$$\sin \phi = \frac{x}{1} = \frac{v}{c}$$

$$\phi = \sin^{-1} \frac{v}{c}$$
c) **Discussion questions**: Is the word “really” an appropriate word in the following quotations?

1. An observer using the rocket latticework of clocks says, “The stationary cube is really neither rotated nor contracted.”
   
   *This can be “really” said to be true.*

2. Someone riding in the rocket who looks at the stationary cube agrees, “The cube is really neither rotated nor contracted.”

   *This can be “really” said to be true. Since all objects of interest are at rest, no problem.*

3. An observer using the laboratory latticework of clocks says, “The passing cube is really Lorentz contracted but not rotated.”

   *This can be “really” said to be true. This is a measurement of the passing cube.*

4. Someone standing in the lab frame looking at the passing cube says, “The cube is really rotated but not Lorentz contracted.”

   *This situation is an observational effect and not “really” what is going on. I.e. it is not a measurement of the passing cube, it is an observation, open to distortion.*

What can one rightfully say – in a sentence or two – to make each observer think it reasonable that the other observers should come to different conclusions?