Optimal Stopping in a Model of Speculative Attacks

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Abstract

When faced with a speculative attack, banks and governments often hesitate, attempting to withstand the attack but giving up after some time, suggesting they have some ex-ante uncertainty about the attack they will face. I model that uncertainty as arising from incomplete information about speculators’ payoffs and find conditions such that unsuccessful partial defences are possible equilibrium outcomes. There exist priors over the distribution of speculators’ payoffs that can justify any possible partial defence strategy. With Normal uncertainty, partial resistance is more likely when there is more aggregate uncertainty regarding agents’ payoffs and less heterogeneity among them.

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1 Introduction

In August and September of 1992, the Bank of England sold billions of dollars of foreign reserves in an attempt to maintain the pound’s exchange rate within the bands of the European ERM. On September 16 it finally gave up and abandoned the ERM. The cost of having attempted to defend the parity was later estimated at approximately £ 3.3 billion (UK Treasury 1997).

This pattern, of first attempting to defend the existing regime and giving up after some time, is a common feature of speculative attacks and explaining it presents a theoretical challenge. So-called first-generation models based on Krugman (1979), such as Flood and Garber (1984) and Broner (2007) account for it in a very simple way: by assuming the government follows and attempts to defend an unsustainable policy for exogenous reasons and abandons it only when forced to do so. However, these models leave unanswered the question of why a government would behave this way.

As formalized by Obstfeld (1996) and others, speculative attacks often have a self-fulfilling aspect: if enough agents believe the government will abandon a regime, they will act in ways that make it optimal for the government to indeed abandon it. The unsatisfying conclusion of models of self-fulfilling equilibria is that, at least within some range of parameters, the outcome is arbitrary or depends on ad-hoc unmodeled factors. Following Morris and Shin (1998), many authors have argued that modifying the common-knowledge assumptions of games that have self-fulfilling equilibria may help to resolve this indeterminacy and provide more definite predictions.

The basic building block of models in this literature is a game played by many small agents (“speculators”) who are incompletely informed about the relevant parameters of the economy, and one large agent (“the central bank”) who has complete information. Typically, the focus of the analysis is on the structure of actions and information of the game played by the speculators. In contrast, the central bank’s information and objectives are usually described in very simple terms so that its strategy can be summarized, or even replaced, by a simple rule such as “defend the existing regime unless a mass of speculators larger than $A^*$ attacks it”.

This paradigm (and for that matter the multiple-equilibria paradigm too) fails to account for why the central bank, acting rationally, would ever engage in an unsuccessful partial defence of the regime, as the Bank of England did in 1992. In these models, the central bank knows the “fundamentals” of the economy and can therefore perfectly predict (or in some versions observe) what the size of the speculative attack is going to be. As long as defending the regime is costly, it would never be the case that it attempts to defend it but surrenders after some time, since this failure would have been foreseen.

However, in some contexts the possibility of a temporary, unsuccessful defence of the status quo makes an important difference. For example, if we wish to apply these methods to the
study of bank runs, as Goldstein and Pauzner (2005) do, the only reason why depositors would run is if they believe that the bank will pay some of them before falling or deciding to suspend convertibility.

How can the theory account for the phenomenon of unsuccessful defences? One possibility, implicitly assumed by Morris and Shin (1998), is that defending the regime is not costly at the margin; conditional on regime change, the central bank does not have a preference for how far it held out. In many contexts this is not a reasonable assumption: the losses to the central bank’s balance sheet are greater the more reserves it has spent trying to defend a fixed exchange rate; the liquidation costs a bank incurs in are smaller the sooner it suspends convertibility; the retaliation against a dictator is likely to be harsher the longer he held on to power; the casualties to a surrendering garrison are greater the longer it attempted to withstand a siege.

If unsuccessful defences are costly, a theory that accounts for them must somehow allow the central bank to have uncertainty about the size of the attack it is going to face. With uncertainty and suitable timing assumptions, the central bank’s decision may be viewed as an optimal stopping problem: as the attack escalates, it must decide whether to surrender or to continue to defend the regime in the hope that the attack will be over soon, using its appropriately updated beliefs about how large the attack is likely to be.

There is more than one way to introduce uncertainty into the central bank’s problem. One approach, pursued by Ennis and Keister (2010) is to let speculators play a correlated (sunspot) equilibrium, and assume that the central bank does not observe the sunspot. A related possibility is to abandon pure-strategy Nash equilibrium as a solution concept. In any pure-strategy Nash equilibrium, the central bank knows the strategies of the speculators, and is thus able to predict the size of the speculative attack with no uncertainty. However, under appropriate conditions (although not in the Morris-Shin limit), both attacking a regime and not attacking it are rationalizable actions. If the requirement that the central bank know the speculators’ strategies is dropped, it is possible to simply endow it with beliefs about the joint distribution of (rationalizable) actions the speculators might take, and under these beliefs a policy of partial defence may indeed be optimal. The trouble with explaining the phenomenon along either of these lines is that these explanations rely on arbitrary assumptions about the central bank’s beliefs and/or speculators’ reactions to the sunspot.

This paper introduces uncertainty into the central bank’s decision problem in a different way. As in Diamond and Dybvig (1983), Wallace (1988), Green and Lin (2003) and Peck and Shell (2003), there is aggregate uncertainty about the distribution of (heterogeneous) preferences in the population of speculators. This distribution is governed by a single random parameter $\theta$.

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1This is also noted by Goldstein, Ozdenoren and Yuan (2008). In their model the central bank has uncertainty about the value of maintaining a fixed exchange rate and may learn about this by observing the speculative attack. Their model does not, however, allow the central bank to surrender to the attack midway.
and neither the central bank nor the speculators know its realization. Although the central bank knows the equilibrium strategies, the equilibrium size of the attack, conditional on the central bank’s information, is a random variable, so it faces a nontrivial optimal stopping problem.

For the central bank’s optimal stopping problem to have an interior solution, in which the central bank surrenders after some time despite having incurred sunk costs of defence, it must be that as the attack progresses the central bank becomes sufficiently more pessimistic about the magnitude of the attack that it will face. In particular, it must think that an attack that is not over by the stopping point is unlikely to be over soon after that, which is consistent with some ex-ante beliefs \( f(A) \) about the size of the attack \( A \) but not with others. However, I show that given any probability distribution \( f(A) \), it is possible to reverse-engineer a prior about \( \theta \) such that \( f(A) \) is indeed the endogenous probability distribution in an equilibrium of the game. Hence a basic finding is that the model is able to deliver the kinds of uncertainty that could justify partial defences.

This alone, however, provides no clear guidance as to what factors make unsuccessful defences likely to arise. To answer this question, I specialize the model to a simple example with linear payoffs and normal uncertainty. In this case, I show that partial defences are more likely to arise and last longer if they do arise if heterogeneity in preferences is small relative to aggregate uncertainty about average preferences. The model has multiple equilibria for some parameter values. The central bank’s uncertainty and the possibility of partial unsuccessful defences, however, are not due to this but to uncertainty about outcomes \textit{within} a given equilibrium.

The model introduces dynamics into the model in an extremely limited way. The speculators must still decide whether or not to attack at the beginning of the game and are not allowed to learn from one another’s actions. This abstracts from what is certainly a very important dimension in real-life speculative attacks. Furthermore, although I will informally describe the central bank’s actions as “waiting” there is no real temporal dimension to the problem: the attack is assumed to build up continuously and the central bank must choose a point in this continuum to stop defending the attack.

Section 2 introduces the model and defines equilibrium conditions. Section 3 explores, in the general case, how uncertainty about preferences translates into uncertainty about outcomes. Section 4 discusses the special case of normal uncertainty and linear payoffs. Section 5 discusses the results and draws some comparisons with the existing literature. Section 6 briefly concludes. The Appendix contains the proofs omitted in the text.
2 The Model

The backbone of the model is a variant of the simple binary action game of Morris and Shin (2003). There is a measure-one continuum of speculators, indexed by $i \in [0, 1]$ and a central bank. At the beginning of the game, each speculator chooses one of two actions: either attack the current policy regime, $a_i = 1$, or not attack, $a_i = 0$. Agents who decide to attack form a queue and inform the central bank of their decision one by one. The central bank cannot observe the length of the queue. After each one of them announces his decision the central bank has two options: either abandon the regime or to continue to support it. If the central bank abandons the regime the game ends in Defeat for the central bank. If it continues to support it, two things may happen. If the attacker was the last in the queue, the game ends in Survival for the central bank; otherwise, the game continues with the next attacker.

The assumption that the bank cannot observe the length of the queue is analogous to the sequential service constraint in the bank-run literature. In particular, it corresponds to the version formalized by Peck and Shell (2003), where the bank only observes depositors who choose to withdraw and not those who do not.

2.1 Speculators’ Payoffs

The payoff from not attacking is normalized to zero whereas attacking has an idiosyncratic cost $c_i$ and brings a benefit of 1 if the regime is defeated. Agent $i$’s payoff is therefore given by

$$u_i = a_i \left[ I\{\text{Defeat}\} - c_i \right]$$

Costs of attacking are distributed in the population according to the distribution $G(c_i|\theta)$ with density $g(c_i|\theta)$, where $\theta$ is a parameter. $G(c_i|\theta)$ is assumed to be decreasing in $\theta$, so $c_i$ is increasing in $\theta$ in a FOSD sense. $c_i$ is not constrained to lie in $[0, 1]$ so one of the actions may be dominant for some of the speculators. In the context of bank runs, where attacking means withdrawing a deposit, $G(0|\theta)$ can represent the fraction of depositors who receive a liquidity shock and need to withdraw, as in Diamond and Dybvig (1983), and $1 - G(1|\theta)$ may represent the fraction of depositors who have long-term deposits, or are out of town or are otherwise unable to participate in the run. In the context of currency attacks, $G(0|\theta)$ may represent the demand for foreign currency to pay for imports and $1 - G(1|\theta)$ may represent

\footnote{For an application to bank runs, a more complete model would require modeling the dependence of depositors’ payoff on the size of the run and not just the survival of the bank. For instance, some depositors may join the queue but never get paid because the bank surrenders before they reach the front of the queue. This introduces strategic substitutability as well as strategic complementarity in depositors’ actions since, beyond the point where the run topples the bank, the incentive to run diminishes with the size of the run. Goldstein and Pauzner (2005) show how to adapt global games techniques to account for this.}
the post-devaluation domestic money demand, as in Krugman (1979).

2.2 Central Bank’s payoff

Let $A = \int_0^1 a_i d_i$ denote the total mass of speculators who attack. The central bank’s payoff is given by a pair of functions: $S(A), D(\tau)$. $S(A)$ is the payoff the central bank obtains if the game ends in Survival after $A$ attackers. $D(\tau)$ is the payoff the central bank obtains if the game ends in Defeat after $\tau$ attackers, i.e. if the central bank abandons the regime after supporting it against $\tau$ attackers. Assume $S$ and $D$ are weakly decreasing, so defending the regime is costly at the margin, and $S(0) > D(0)$, so if no one attacks the central bank prefers to survive.

The four panels in Figure 1 illustrate different possible cases of payoffs the central bank might have. Panel (i) shows the payoffs assumed by Morris and Shin (1998), which are the special case where $S(A) = v - A$ and $D(\tau) = 0$. A constant $D$ function means that, conditional on eventually being defeated, the central bank places no value on ending its resistance soon. For instance, this could be the case if it perceived no value in having reserves once it has abandoned a fixed exchange rate. As mentioned in the introduction, for these payoffs the central bank’s problem is trivial since it would always defend the regime up to the point where $A = v$. Panel (ii) shows an interpretation where the central bank has finite reserves equal to $v$. $S(A)$ is constant up to $A = v$; at this point reserves are exhausted so $S(A)$ drops discontinuously to a point below the minimum level of $D(\tau)$. As proposed by Bagehot (1873), the central bank does not mind using reserves as long as it succeeds in maintaining the regime. However, the bank would prefer not to expend reserves if the defence is to ultimately prove unsuccessful, so $D(\tau)$ is downward-sloping. Panel (iii) shows a case where there is a fixed benefit $v$ of maintaining the regime and the cost of defending it (by giving up reserves) is linear in the size of the attack, both for successful and failed defences. Panel (iv) shows a similar example but with increasing marginal cost of using reserves.

2.3 Information

At the beginning of the game, nature draws the random variable $\theta$ from some prior distribution $P(\theta)$ with density $p(\theta)$. No one observes the realized $\theta$, but speculators observe their own realized $c_i$. A speculator’s individual cost is informative about the distribution of costs in the population summarized by $\theta$; applying Bayes’ rule, a speculator’s posterior is:

$$p(\theta|c_i) = \frac{g(c_i|\theta)p(\theta)}{\int_0^\theta g(c_i|\theta)p(\theta)d\theta}$$ (1)
Definition 1. \( g(c_i|\theta) \) satisfies the monotone inference property if, for any \( p(\theta) \), the posterior distribution \( P(\theta|c_i) \) is decreasing in \( c_i \).

If \( g(c_i|\theta) \) satisfies this property then a speculator who observes a higher cost \( c_i \) for himself will infer that the parameter \( \theta \) is likely to be higher, which implies that the costs of the other speculators are also likely to be higher.

2.4 Equilibrium

I focus on symmetric monotone equilibria, defined as perfect Bayesian equilibria such that a speculator attacks if and only if \( c_i \) is less than some threshold \( c^* \), which is identical for all speculators. In such an equilibrium, the aggregate size of the attack is \( A(\theta) = \Pr[c_i \leq c^*|\theta] = G(c^*|\theta) \). Since the central bank does not know \( \theta \), it does not know the realized value of \( A \). Instead, it has beliefs given by the density

\[
f(A) = p\left(G^{-1}(c^*; A)\right)\left|\frac{\partial G^{-1}(c^*; A)}{\partial A}\right|
\]

(2)

where \( G^{-1} \) is the inverse of \( G \) with respect to its second argument, i.e. \( A \equiv G(c|G^{-1}(c; A)) \).

While the attack is taking place, the central bank gradually learns about the realized value of \( A \).
but in a limited way: it only learns from the fact that the attack is not over yet. Hence its optimal stopping problem can be formulated simply as the following one-dimensional optimization problem:

\[ \max_{\tau \in [0,1]} V(\tau) = \int_0^\tau S(A) f(A) dA + [1 - F(\tau)] D(\tau) \] (3)

The first term of (3) is the value the central bank obtains from the possibility of surviving if the size of the attack turns out to be less than \( \tau \). The second term is the value it will obtain if it surrenders after \( \tau \) attackers. Program (3) may or may not have an interior optimum. In case it does, the first order necessary condition is

\[ V'(\tau) = [S(\tau) - D(\tau)] f(\tau) + [1 - F(\tau)] D'(\tau) = 0 \]

\[ \Rightarrow h(\tau) [S(\tau) - D(\tau)] = -D'(\tau) \] (4)

where \( h(A) \equiv \frac{f(A)}{1 - F(A)} \) is the hazard function of the size of the attack. The marginal benefit of waiting, given by the difference between the value of survival and defeat, \( S(\tau) - D(\tau) \) times the instantaneous probability that the attack will be over, \( h(\tau) \), must equal the marginal cost of waiting, which is \(-D'(\tau)\). Since the function \( V(\tau) \) is not necessarily concave, condition (4) could also denote a local minimum. The second order condition for a local maximum is:

\[ V''(\tau) = [S'(\tau) - 2D'(\tau)] f(\tau) + [S(\tau) - D(\tau)] f'(\tau) + [1 - F(\tau)] D''(\tau) \leq 0 \] (5)

Even if (4) and (5) hold, they may identify a local but not global maximum, so in general it is only possible to say that the central bank’s best response is given by \( \tau^* \in \arg\max_{\tau \in [0,1]} V(\tau) \).

The distinction between interior and corner solutions to program (3) is important. If \( \tau^* = 0 \) then the central bank does not attempt to resist at all and falls as soon as the attack begins. If \( \tau^* = 1 \) then the central bank withstands the attack no matter how large it is (although this does not rule out that it may ex-post regret having done so if the realized value of \( S(A(\theta)) \) is less than \( D(0) \)). Unsuccessful partial defences only occur when \( \tau^* \in (0,1) \) and the realized \( A(\theta) \) happens to be greater than \( \tau^* \).

The regime will survive iff \( A(\theta) = G(c^*|\theta) \leq \tau^* \). Higher values of \( \theta \) are associated with FOSD higher costs of attacking and therefore, for given \( c^* \), smaller attacks. If \( \lim_{\theta \to -\infty} G(c^*|\theta) > \tau^* \), then the smallest possible attack is sufficiently large for the central bank to fail; conversely if \( \lim_{\theta \to -\infty} G(c^*|\theta) \leq \tau^* \), then the largest possible attack is too small to make the central bank fail. Otherwise there exists a unique \( \theta^* \) such that the central bank fails whenever \( \theta < \theta^* \). This

3Denoting by \( W(a) \) the value function conditional on having resisted an attack of size \( a \), then the problem can be represented by the differential equation \( W'(a) = -h(a)[S(a) - W(a)] \), with value matching condition \( W(\tau) = D(\tau) \) and smooth pasting condition \( W'(\tau) = D'(\tau) \). These are equivalent to the first order conditions in the text.
critical value is defined by

\[ G(c^*|\theta^*) = \tau^* \] (6)

For an individual speculator, attacking is a best response if, given his information, the probability of the central bank failing is greater than the cost of attacking, i.e. if

\[ \text{Pr} [\theta \leq \theta^*|c_i] = P(\theta^*|c_i) > c_i \] (7)

If \( g(c_i|\theta) \) satisfies the monotone inference property then the LHS of (7) is decreasing in \( c_i \), so given \( \theta^* \) there exists a unique \( c^* \in [0, 1] \) such that the speculators attack iff \( c_i \leq c^* \). This shows that the best response to threshold strategies are threshold strategies. The speculator who has cost \( c_i = c^* \) must be indifferent between attacking and not attacking, which implies

\[ P(\theta^*|c^*) = c^* \] (8)

**Definition 2.** A monotone equilibrium consists of a threshold \( \theta^* \in \mathbb{R} \cup \{-\infty, +\infty\} \), strategies \( \tau^* \in [0, 1] \) and \( c^* \in [0, 1] \) and beliefs \( f(A) \) such that

1. Either (i) (6) holds, (ii) \( G(c^*|\theta) > \tau^*, \forall \theta \) and \( \theta^* = +\infty \) or (iii) \( G(c^*|\theta) < \tau^*, \forall \theta \) and \( \theta^* = -\infty \).
2. \( \tau^* \) solves program (3) using \( f(A) \), so the central bank is best-responding given its beliefs.
3. (8) holds, so the speculators are best-responding.
4. \( f(A) \) satisfies (2), so the central bank’s beliefs are consistent with the speculators’ strategies.

### 3 Beliefs about the size of the attack

Suppose the probability distribution of the sizes of attacks \( A \) were known to be given by some density function \( f(A) \). In principle, \( f(A) \) could be estimated empirically from the sizes of actual (successful or unsuccessful) speculative attacks. Could the model account for \( f(A) \) as arising from the equilibrium of the game described above? The answer is that it is always possible to reverse-engineer some prior \( p(\theta) \) such that the game has an equilibrium where the unconditional density of \( A \) is \( f(A) \).

**Proposition 1.** Let \( f(A) \) be an arbitrary continuous pdf on \([0, 1]\) and let \( g(c_i|\theta) \) be a continuous pdf on \( \mathbb{R} \) such that

1. \( g(c_i|\theta) \) satisfies the monotone inference property.
2. \( \lim_{\theta \to \infty} G(c_i|\theta) = 0, \forall c_i \) and \( \lim_{\theta \to -\infty} G(c_i|\theta) = 1, \forall c_i \)

Then for any payoff functions \( D \) and \( S \) there exists a prior \( p(\theta) \) such that under primitives \( g(c_i|\theta) \), \( p(\theta) \) there is an equilibrium where the unconditional distribution of \( A \) is \( f(A) \).

The way to construct such an equilibrium is as follows. Beliefs \( f(A) \) immediately imply a best response \( \tau^* \) for the central bank. Given any strategy \( c^* \) for the speculators, it is mechanically possible to find a prior about the value of \( \theta \), \( p_{c^*}(\theta) \), such that the posterior belief about the size of the attack is indeed \( f(A) \). Under prior \( p_{c^*}(\theta) \), the speculators’ best response to a stopping point \( \tau^* \) and a cutoff strategy \( c^* \) will be some other cutoff strategy \( T(c^*,p_{c^*}) \). An equilibrium consists of a fixed point such that \( T(c^*,p_{c^*}) = c^* \) and continuity implies that such a fixed point exists.

Table 1 shows examples of what priors could justify different \( f(A) \) functions. In all cases \( f(A) \) is some Beta distribution, the central bank’s payoffs are \( S(A) = 0.6 - A \), \( D(\tau) = -\tau \) and \( g(c_i|\theta) = \phi(c_i - \theta) \), where \( \phi \) is the density for a standard normal, which satisfies the conditions of Proposition 1, as shown in Lemma 1 below.

**Lemma 1.** \( g(c_i|\theta) = \sqrt{\alpha_\theta} \phi(\sqrt{\alpha_\theta} (c_i - \theta)) \) satisfies the monotone inference property

Proposition 1 also implies that making precise predictions requires imposing more structure on the problem, since under the general specification virtually anything could happen. The following section explores a simple special case of the model.

## 4 Normal uncertainty and linear payoffs

Consider the following special case of the model. The central bank’s payoffs are \( S(A) = v - A \) and \( D(\tau) = -\tau \), so there is a fixed value of survival \( v \in (0,1) \) and a constant marginal cost of defending the regime, as in panel (iii) of Figure 1. The costs of attacking are normally distributed in the population, with mean \( \theta \) and variance \( \frac{1}{\alpha_\theta} \), and the prior on \( \theta \) is also normal, with mean \( \mu \) and variance \( \frac{1}{\alpha_\mu} \), i.e. \( g(c_i|\theta) = \sqrt{\alpha_\theta} \phi\left(\sqrt{\alpha_\theta} (c_i - \theta)\right) \) and \( p(\theta) = \sqrt{\alpha_\mu} \phi\left(\sqrt{\alpha_\mu} (\theta - \mu)\right) \).

\( \frac{1}{\alpha_\theta} \) and \( \frac{1}{\alpha_\mu} \) are measures of heterogeneity and aggregate uncertainty respectively.

In order to highlight the role of the central bank’s uncertainty and speculators’ incomplete information, I first analyze variants of the game with common knowledge and where the central bank is informed.
Table 1: Examples of priors that would lead to beliefs about $A$

<table>
<thead>
<tr>
<th>Beliefs about attack $f(A)$</th>
<th>Prior $p(\theta)$ that results in $f(A)$</th>
<th>Equilibrium</th>
</tr>
</thead>
<tbody>
<tr>
<td>\textit{Beta} (1, 1) (uniform)</td>
<td><img src="image" alt="Graph" /></td>
<td><img src="image" alt="Graph" /></td>
</tr>
<tr>
<td>$f(A)$</td>
<td>$p(\theta)$</td>
<td>0</td>
</tr>
<tr>
<td>$Beta$ (.25, .25)</td>
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</tr>
<tr>
<td>$Beta$ (2, 1)</td>
<td><img src="image" alt="Graph" /></td>
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</table>
4.1 Common knowledge benchmark

Assume first that $\theta$ is common knowledge for both the central bank and the speculators. The central bank’s decision becomes simpler because, in any Nash equilibrium, it knows what size of attack it will face. If the central bank decides that it will not outlast the attack then it finds it optimal to choose $\tau^* = 0$. As in Morris and Shin (1998), the central bank will abandon the regime iff $A(\theta) > v$.

A mass $\Phi(-\sqrt{\alpha_c}\theta)$ of speculators have $c_i < 0$ so attacking is dominant for them. Conversely, a mass $1 - \Phi(\sqrt{\alpha_c}(1-\theta))$ have $c_i > 1$ so not attacking is dominant for them. If $\theta$ is such that $v - \Phi(-\sqrt{\alpha_c}\theta) > 0 > v - \Phi(\sqrt{\alpha_c}(1-\theta))$, i.e. if $\theta \in \left[ -\frac{\Phi^{-1}(v)}{\sqrt{\alpha_c}}, 1 - \frac{\Phi^{-1}(v)}{\sqrt{\alpha_c}} \right]$ then the game will have multiple equilibria. If the central bank expects all the speculators who have $c_i \in [0, 1]$ to attack, then $\tau = 0$ is a best response, which in turn justifies their decision to attack. Conversely, if the central bank expects all the speculators who have $c_i \in [0, 1]$ not to attack, then any $\tau^* > \Phi(-\sqrt{\alpha_c}\theta)$ is a best response, which justifies the speculators’ decision.

4.2 Informed central bank benchmark

Now assume instead that the central bank knows the realized value of $\theta$ but the speculators do not. As with common knowledge, in any Nash equilibrium the central bank will always choose either $\tau^* = 0$ or $\tau^* = 1$ because it will know the size of the attack. I will look for a monotone equilibrium such that the central bank chooses $\tau^* = 1$ iff $\theta \geq \theta^*$.

From equation (1), a speculator’s Bayesian posterior about $\theta$ is a normal distribution, with mean $\frac{\alpha_\mu \mu + \alpha_c c^*}{\alpha_\mu + \alpha_c}$ and variance $\frac{1}{\alpha_\mu + \alpha_c}$. Given a cutoff $\theta^*$ for the central bank’s strategy, speculators’ best response cutoff $c^*$ is given by:

$$
\Phi\left(\sqrt{\alpha_\mu + \alpha_c} \left(\theta^* - \frac{\alpha_\mu \mu + \alpha_c c^*}{\alpha_\mu + \alpha_c} \right)\right) = c^* \tag{9}
$$

which is just a special case of the indifference condition (8).

The size of the attack will be given by

$$
A(\theta) = \Phi\left(\sqrt{\alpha_c} (c^* - \theta)\right)
$$

which is decreasing in $\theta$. This implies that, as long as $v \in (0, 1)$, there will be a unique cutoff $\theta^*$ such that the central bank prefers $\tau = 1$ to $\tau = 0$ iff $\theta \geq \theta^*$. The cutoff is given by the indifference condition

$$
\Phi\left(\sqrt{\alpha_c} (c^* - \theta^*)\right) = v \tag{10}
$$
Solving (9) and (10), the speculators’ cutoff must satisfy
\[
\Phi\left(\sqrt{\alpha}\left(\frac{\alpha_{\mu}(c^{*} - \mu)}{\alpha_{\mu} + \alpha_{c}} - \frac{\Phi^{-1}(c^{*})}{\sqrt{\alpha_{\mu} + \alpha_{c}}}\right)\right) = v
\]  
(11)

The derivative of the left hand side of (11) is
\[
\phi(\cdot)\sqrt{\alpha}\left(\frac{\alpha_{\mu}}{\alpha_{\mu} + \alpha_{c}} - \frac{1}{\phi(\Phi^{-1}(c^{*}))}\right) \leq \phi(\cdot)\sqrt{\alpha}\left(\frac{\alpha_{\mu}}{\alpha_{\mu} + \alpha_{c}} - \frac{\sqrt{2\pi}}{\sqrt{\alpha_{\mu} + \alpha_{c}}}\right)
\]

Therefore if
\[
\frac{\alpha_{\mu}}{\sqrt{\alpha_{\mu} + \alpha_{c}}} \leq \sqrt{2\pi}
\]  
(12)

the left hand side is decreasing and (11) has a unique solution. Furthermore, if condition (12) does not hold, there exist values of \(\mu\) and \(v\) such that (11) has multiple solutions. Condition (12) says that in order to guarantee uniqueness in the game where the central bank has complete information, heterogeneity \(\frac{1}{\alpha_{c}}\) must be small relative to aggregate uncertainty \(\frac{1}{\alpha_{\mu}}\). When heterogeneity is small, the idiosyncratic cost \(c_{i}\) is a very good signal about \(\theta\), which rules out multiple self-fulfilling equilibria. Morris and Shin (2000) derive a condition for uniqueness similar to (12), though not identical because here a speculator’s cost \(c_{i}\) plays a direct payoff role in addition to an informational role.

The following proposition summarizes the above benchmark results.

Proposition 2.

1. Under common knowledge, there are multiple equilibria if \(\theta \in \left[-\frac{\Phi^{-1}(v)}{\sqrt{\alpha_{c}}}, 1 - \frac{\Phi^{-1}(v)}{\sqrt{\alpha_{c}}}\right]\).

2. If the central bank knows the realized \(\theta\) but the speculators do not, there is a unique equilibrium for every \(\mu, v\) iff \(\frac{\alpha_{\mu}}{\sqrt{\alpha_{\mu} + \alpha_{c}}} \leq \sqrt{2\pi}\).

3. In both cases \(\tau^{*} \in \{0, 1\}\), so there are never unsuccessful defences.

4.3 Uninformed central bank

Now consider the game as described in section 2, where the central bank does not observe \(\theta\). Given a threshold \(\theta^{*}\) for the central bank’s survival, speculators’ best response cutoff is still given by (9). The threshold \(\theta^{*}\) is related to the central bank’s strategy by equation (6), which reduces to
\[
\theta^{*} = c^{*} - \frac{\Phi^{-1}(\tau^{*})}{\sqrt{\alpha_{c}}}
\]  
(13)

if \(\tau \in (0, 1)\), \(\theta^{*} = +\infty\) if \(\tau^{*} = 0\) and \(\theta^{*} = -\infty\) if \(\tau^{*} = 1\).
With linear $S(\cdot)$ and $D(\cdot)$ payoff functions, the FOC and SOC for an interior solution to the central bank’s optimal stopping problem (3) simplify to:

$$vh(\tau) = 1 \quad (14)$$
$$h'(\tau) \leq 0 \quad (15)$$

The FOC (14) equates the marginal benefit of instantaneous resistance, which is equal to the constant benefit of survival $v$ times the instantaneous probability that the attack ends $h(\tau)$, to the marginal cost, which is constant. The SOC (15) says that for the central bank to find it optimal to abandon its defence at some interior point $\tau$, $f(A)$ must be such that it is decreasingly likely that the attack will be over soon.

Conditions (14) and (15) only identify a local optimum. For a global optimum, the central bank must compare the value it obtains from an interior $\tau$ to the value of offering no resistance $V(0) = 0$ and the value of resisting any possible attack $V(1) = v - E(A)$ (as well as comparing it to other local optima if there are any).

Finally, by equation (2), the central bank’s beliefs about the attack it will face are given by:

$$f(A) = \frac{\sqrt{\frac{\alpha_{c}}{\alpha_{\mu}}} \phi \left( \sqrt{\frac{\alpha_{\mu}}{\alpha_{c}}} \Phi^{-1}(A) - \sqrt{\alpha_{\mu} \left(c^{*} - \mu\right)} \right)}{\phi \left( \Phi^{-1}(A) \right)} \quad (16)$$

In what follows I make use of the following properties of $f(A)$

**Lemma 2.**

1. $E(A) = \Phi \left( \frac{c^{*} - \mu}{\sqrt{\frac{\alpha_{\mu}}{\alpha_{c}}}} \right)$

2. If $\alpha_{\mu} < \alpha_{c}$, (or $\alpha_{\mu} = \alpha_{c}$ and $c^{*} < \mu$), then $\lim_{A \to 0} f(A) = \infty$

3. If $\alpha_{\mu} > \alpha_{c}$ and $c^{*} > \mu$, then $f(A)$ has increasing hazard

Part 1 of Lemma 2 simply computes the expected size of the attack from the point of view of the central bank. If the cutoff $c^{*}$ for not attacking is high compared to $\mu$, the prior mean of $c_{i}$, then the central bank expects that many speculators will be below the cutoff and therefore attack. Hence the unconditional expectation of $A$ will be high. The expected attack will be more sensitive to the difference between $c^{*}$ and $\mu$ when the central bank has less overall (aggregate plus idiosyncratic) uncertainty about any speculator’s $c_{i}$. Part 2 of Lemma 2 states that, if aggregate uncertainty is large relative to heterogeneity, then the probability that attacks will be very small is high. The reason is that, for a given $c^{*}$, small heterogeneity means there are relatively few speculators for whom the idiosyncratic component of $c_{i}$ is so low that a high
realization of $\theta$ will not push them above the cutoff into the non-attacking region; moreover, large aggregate uncertainty makes extreme realizations of $\theta$ (and in particular, high relocations) more likely. Part 3 of Lemma 2 says that if instead aggregate uncertainty is small relative to heterogeneity and in addition the cutoff $c^*$ is higher than the prior mean of $\theta$, then $f(A)$ has increasing hazard. The reason is that relatively small aggregate uncertainty shifts $f(A)$ towards the center rather than the extremes, while $c^* > \mu$ shifts it to the right, which suffices for the hazard function to be increasing.

### 4.4 Characterization of equilibria

I distinguish between three different kinds of equilibria: “no resistance” equilibria, with $\tau^* = 0$; “full resistance” equilibria, with $\tau^* = 1$ and “waiting” equilibria, with $\tau \in (0, 1)$. The following proposition, which follows from Lemma 2, summarizes the necessary and sufficient conditions under which each may exist.

**Proposition 3.** The following are necessary and sufficient conditions for no resistance, full resistance and waiting equilibria respectively

<table>
<thead>
<tr>
<th>Condition</th>
<th>Necessary conditions</th>
<th>Sufficient conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>No resistance</td>
<td>$\Phi \left( \frac{1-\mu}{\sqrt{\frac{1}{\alpha \mu} + \frac{1}{\pi_c}}} \right) \geq v$</td>
<td>$\Phi \left( \frac{1-\mu}{\sqrt{\frac{1}{\alpha \mu} + \frac{1}{\pi_c}}} \right) \geq v$</td>
</tr>
<tr>
<td></td>
<td>$\alpha_\mu &gt; \alpha_c$, (or $\alpha_\mu = \alpha_c$ and $\mu \leq 1$)</td>
<td>$\alpha_\mu &gt; \alpha_c$</td>
</tr>
<tr>
<td></td>
<td>$\mu &lt; 1$</td>
<td>$\mu &lt; 1$</td>
</tr>
<tr>
<td>Full resistance</td>
<td>$\Phi \left( \frac{-\mu}{\sqrt{\frac{1}{\alpha \mu} + \frac{1}{\pi_c}}} \right) \leq v$</td>
<td>$\Phi \left( \frac{-\mu}{\sqrt{\frac{1}{\alpha \mu} + \frac{1}{\pi_c}}} \right) \leq v$</td>
</tr>
<tr>
<td></td>
<td>$\alpha_\mu &gt; \alpha_c$</td>
<td>$\alpha_\mu &gt; \alpha_c$</td>
</tr>
<tr>
<td></td>
<td>$\mu &lt; 0$</td>
<td>$\mu &lt; 0$</td>
</tr>
<tr>
<td>Waiting</td>
<td>$\alpha_\mu \leq \alpha_c$ (or $\mu \geq 0$)</td>
<td>$\alpha_\mu &lt; \alpha_c$</td>
</tr>
<tr>
<td></td>
<td>$\Phi \left( \frac{-\mu}{\sqrt{\frac{1}{\alpha \mu} + \frac{1}{\pi_c}}} \right) &gt; v$</td>
<td>$\Phi \left( \frac{-\mu}{\sqrt{\frac{1}{\alpha \mu} + \frac{1}{\pi_c}}} \right) &gt; v$</td>
</tr>
</tbody>
</table>

In a no resistance equilibrium, the central bank settles for obtaining $V(0) = 0$ by giving up immediately. Since the regime always falls, i.e. $\theta^* = \infty$, then any speculator for whom not attacking is not dominant will attack, i.e. $c^* = 1$.

A simple necessary condition for no resistance to be optimal is that it must be better that resisting for a very short period, i.e. it must be that $V'(0) = vf(0) - 1 \leq 0$. As long as the density at 0 is greater than $\frac{1}{v}$, i.e. as long as there is sufficient chance that the attack will be very small relative to the central bank’s desire to survive, at least a little resistance will be preferable to no resistance at all. Hence a necessary condition for a no resistance equilibrium is that the conditions of Lemma 2.2 not hold, i.e. $\alpha_\mu > \alpha_c$, (or $\alpha_\mu = \alpha_c$ and $\mu \leq 1$).

---

4Strictly speaking, the density at 0 may not be well defined. Let $f(0)$ mean $\lim_{A \to 0} f(A)$. 

Under the conditions of Lemma 2.3, there can never be an interior solution to the central bank’s problem since an increasing hazard would contradict (15). The central bank will not value the option to wait and will simply choose between \( \tau = 0 \) and \( \tau = 1 \). By Lemma 2.1, the former is preferred if \( \Phi \left( \frac{(1-\mu)}{\sqrt{\frac{1}{\alpha \mu} + \frac{1}{\alpha c}}} \right) > v \). Hence the conditions \( \alpha \mu > \alpha c \), \( 1 > \mu \) and \( \Phi \left( \frac{(1-\mu)}{\sqrt{\frac{1}{\alpha \mu} + \frac{1}{\alpha c}}} \right) > v \) are sufficient to ensure that there is a no resistance equilibrium.

Overall, the central bank will choose no-resistance when it believes that speculators’ costs of attacking are low on average (low \( \mu \)), when it’s quite certain about this (high \( \alpha \mu \)) and when it thinks that there is sufficient heterogeneity (low \( \alpha c \)) that many speculators are likely to attack regardless of \( \theta \) so that it is not worthwhile to offer some resistance in the hope that the realization of \( \theta \) was unexpectedly high.

In a full resistance equilibrium, the central bank obtains a value \( V(1) = v - E(A) \). The regime never falls, so \( \theta^* = -\infty \) and \( c^* = 0 \). Since the central bank could always obtain zero by choosing \( \tau = 0 \), a necessary condition for this kind of equilibrium to exist is \( v \geq E(A) = \Phi \left( -\mu/\sqrt{\frac{1}{\alpha \mu} + \frac{1}{\alpha c}} \right) \). Furthermore, under the conditions for Lemma 2.3, there can never be an interior solution for the central bank’s problem, so conditions \( \alpha \mu > \alpha c \), \( \mu < 0 \) and \( v \geq \Phi \left( -\mu/\sqrt{\frac{1}{\alpha \mu} + \frac{1}{\alpha c}} \right) \) are sufficient to ensure that there is a full resistance equilibrium.

Overall, the central bank will choose full resistance when it is optimistic that costs of attacking are high (high \( \mu \)), when it is quite confident in this assessment (high \( \alpha \mu \)) and when there is high heterogeneity (low \( \alpha c \)) so that no matter how long an attack continues the central bank rationally attributes it to speculators with a low idiosyncratic component of \( c_i \) (and thus likely to be over soon) and not to a low realization of \( \theta \).

In a waiting equilibrium, the central bank chooses some intermediate \( \tau^* \in (0, 1) \). By (15), this requires that the hazard function be decreasing at some point. By Lemma 2.3, this means that either \( \alpha \mu \leq \alpha c \) or \( \mu \geq 0 \) must hold. Furthermore, if \( V'(0) > 0 \) and \( V(0) > V(1) \) then any optimum must necessarily be interior. Therefore, if the conditions for Lemma 2.2 hold and \( v < \Phi \left( -\mu/\sqrt{\frac{1}{\alpha \mu} + \frac{1}{\alpha c}} \right) \) we know that any equilibrium must be a waiting equilibrium.

Overall, the bank will choose to wait when it is uncertain about average costs of attacking (low \( \alpha \mu \)), so that it is worth offering some resistance in case they turn out to be low, but there is low heterogeneity in the population (high \( \alpha c \)) so that, if the attack continues beyond a certain point, the central bank is fairly confident that it is not just facing speculators with a low idiosyncratic component of \( c_i \) but rather a low realization of \( \theta \). This means that the attack is unlikely to be over soon and persuades the central bank to stop defending the regime.

Notice that Proposition 3 does not rule out the possibility of multiple equilibria. Indeed, if

\[
v \in \left[ \Phi \left( \frac{-\mu}{\sqrt{\frac{1}{\alpha \mu} + \frac{1}{\alpha c}}} \right), \Phi \left( \frac{1 - \mu}{\sqrt{\frac{1}{\alpha \mu} + \frac{1}{\alpha c}}} \right) \right]
\]  

(17)
then the necessary conditions for both full resistance equilibria and no resistance equilibria hold. This interval always exists, although it becomes smaller as either $\alpha_\mu$ or $\alpha_c$ become large. If, in addition, $\mu < 0$, the sufficient conditions for both full resistance and no resistance equilibria hold as well and we are certain to have multiple equilibria.

The logic behind multiple self-fulfilling equilibria is not exactly the same as in the case where the central bank knows $\theta$. In that case, multiplicity arises when information is sufficiently common across speculators to allow for coordination on different equilibria. By condition (12), this requires small aggregate uncertainty, so that speculators’ common prior is highly informative and large heterogeneity so that speculators’ idiosyncratic cost is not very informative about the aggregate state. In the game where the central bank is uninformed, multiplicity arises if the expected mass of speculators who do not have a dominant strategy is sufficiently large to justify each of the central bank’s possible decision rules. This requires that the central bank’s desire to survive $v$ take intermediate values and that the unconditional distribution of $c_i$ be sufficiently concentrated, which requires small heterogeneity and small aggregate uncertainty.

When there are multiple equilibria, the central bank (ex-ante) always prefers the one where its stopping point is latest. Consider two equilibria with stopping points $\tau_0$ and $\tau_1$ respectively, with $\tau_0 < \tau_1$. Since each is an equilibrium, the stopping points are best responses to the distribution of $A$ in each case. But these distributions are different because by equations (6) and (8), $c^*$ is decreasing in $\tau$ so the equilibrium with $\tau_1$ has smaller attacks. The central bank prefers this because $S(\cdot)$ and $D(\cdot)$ are decreasing.

Proposition 3 does not fully characterize the possible equilibria that will arise for each combination of parameters, but it is possible to compute the equilibria numerically in order to find sharper boundaries for the regions in the parameter space where each equilibrium occurs. Figure 2 shows the regions of $\alpha_\mu$, $\alpha_c$ where each of the types of equilibrium occurs, fixing $\mu = 0.3$ and setting $v = 0.4$ or $v = 0.6$ in each case.

In the case where $v = 0.4$, Proposition 3 implies that no resistance equilibria will exist iff $\alpha_\mu > \alpha_c$, i.e. in the northwest half of the graph. When this condition does not hold, $V'(0) > 0$ so some resistance is preferable to no-resistance and there exists a waiting equilibrium. Moreover, when the unconditional distribution of $c_i$ is concentrated (the northeast of the graph), there also exists a full-resistance equilibrium. In it, speculators’ knowledge that $\tau^* = 1$ shifts enough mass towards a no-attack strategy that full resistance is desirable.

In the case where $v = 0.6$, the central bank’s increased valuation of the regime’s survival makes the northeast region of full-resistance equilibria expand. Furthermore, in the west of the graph where heterogeneity is very high, then most speculators will have a dominant action and $E(A)$ approaches $\frac{1}{2}$. Since $v > \frac{1}{2}$, then full resistance is better than no resistance. Furthermore, the large heterogeneity means that the central bank learns little about the aggregate state as the attack progresses and rationally believes that the hazard of the attack ending is increasing,
so the only equilibrium is full resistance.

Within the waiting-equilibrium region in the right panel of Figure 2, Figure 3 shows how the equilibrium stopping point $\tau^*$ is affected by various parameters.

Panels (i) and (ii) show the effects of the central bank’s uncertainty about speculators’ cost of attacking. Panel (i) shows the effects of varying $Var(c_i) = \frac{1}{\alpha_c} + \frac{1}{\alpha_c}$, while holding $\frac{\alpha_c}{\alpha_c}$ constant, i.e. making the central bank more uncertain about a given speculator’s costs while holding constant the fraction of uncertainty that arises from idiosyncratic as opposed to aggregate factors. The figure shows that higher overall uncertainty for the central bank leads to more waiting. This is because in these examples it happens that $c^* > \mu$, so by (16), this makes small attacks more likely, which justifies waiting more. Panel (ii) shows what happens if the idiosyncratic component of the variance increases relative to the aggregate, while holding $Var(c_i)$ constant. High ratios of heterogeneity to aggregate uncertainty lead to lower resistance in waiting equilibria (as well as ruling out waiting equilibria entirely if they are sufficiently high). Higher heterogeneity means more speculators with very high or very low costs of attacking and leads to a $f(A)$ with less mass at the extremes. This implies that $h(\tau)$ will be decreasing only for very low values of $\tau$. Panel (iii) shows that the more the central bank values survival the longer it is willing to wait. This is due to two reinforcing effects: firstly, given a function $f(A)$, (14) implies that higher $v$ requires a lower hazard for the central bank not to wish to continue defending; since the hazard must be decreasing at an optimum, higher $v$ implies waiting more; secondly, a higher $\tau^*$ leads to a lower equilibrium $c^*$ for the speculators, which makes a small attack more likely and justifies waiting more. Finally, panel (iv) shows that the higher the central bank’s prior belief about speculators’ costs of attacking, the more it will be willing to wait, simply because this will lead it to believe that attacks are likely to be small.
Figure 3: Equilibrium $\tau^*$ for different parameters
5 Discussion

The question of how to respond to a gradually unfolding speculative attack has been examined from a slightly different angle in the banking literature. One class of papers (Wallace 1988, Green and Lin 2003, Peck and Shell 2003), based on the Diamond and Dybvig (1983) framework, pose the problem in the following way: If there is uncertainty about the number of impatient consumers, what is the optimal allocation subject to a sequential service constraint? If a bank is set up so that the optimal allocation results from a Nash equilibrium, is there another equilibrium in which depositors run?

In this problem, since the bank does not know how many depositors are actually impatient and must satisfy sequential service, the optimal allocation is to pay successive impatient depositors a decreasing amount. The resulting allocation will generically not coincide with that which would obtain if there was no sequential service constraint. When it turns out there were few impatient depositors, the bank will ex-post wish it had paid each of them more; when there are many, it will ex-post wish it had paid less to the first few that showed up. The pattern of paying the first depositors more than the later ones has some similarity with what I call an unsuccessful partial defence which (in the currency run application) involves honouring the fixed exchange rate for the first \( \tau^* \) speculators and not for the rest.

There are two main differences between the outcomes in these models and in the current one. The first, less essential one, is that the banking models allow for various intermediate actions by the bank beyond “continue to defend” and “abandon”, so the optimal allocation does not clearly distinguish between defending the regime and not defending it. The reason this is not so essential is that if one were to constrain the bank to only offer two levels of withdrawals for the early depositors (“normal” and “suspension of convertibility”), the issue of how many “normal” withdrawals to allow before suspending convertibility would still arise, even assuming that patient depositors never run.

The second, more essential difference is that in the banking models the bank’s decisions never respond to the possibility that it may be facing a run: it’s only problem is how to manage demand uncertainty subject to sequential service.\(^5\) Even when a run equilibrium is actually played, the bank will doggedly follow the prescribed payments for the no-run optimum allocation. Suppose one maps the model into the bank-run environment by identifying speculators with \( c < 0 \) as impatient depositors (since they always attack / withdraw) and speculators with \( c > 0 \) as patient depositors (since they only attack if they think it’s sufficiently likely that the central bank will be defeated). The banking papers ask what is the optimal stopping point assuming that \( A = G(0|\theta) \), i.e. that only impatient depositors attack, and only then

\(^5\)Peck and Shell (2003) do analyze a banking arrangement that explicitly contemplates the possibility of a sunspot-based run, but only characterize a two-depositor example where the question of how far to resist a run cannot be posed.
ask whether under this stopping point there exists another equilibrium where patient depositors run. In my model, the central bank’s uncertainty includes uncertainty about how patient (c > 0) speculators will act, i.e. uncertainty about whether it will face an actual speculative attack and if so of what size. Indeed, it is possible for the central bank to be defeated even if S( maxθ G(0|θ)) > D(0) which, translated to the banking context is a sufficient condition for the bank (assuming no runs from patient depositors) to always pay depositors in full.

Ennis and Keister (2010) study a related problem. Their model has no uncertainty about the fraction of impatient depositors, and the only uncertainty the bank faces has to do with what equilibrium will be played, i.e. about how the patient depositors will act. Like Green and Lin (2003) (but unlike Peck and Shell (2003)), Ennis and Keister (2010), assume that depositors know their order in the queue before they decide whether or not to withdraw their deposit. Given the way the banking technology is assumed to work, this means that patient depositors who are sufficiently behind in the queue do not want to withdraw early, in order to avoid inefficient liquidation of the bank’s remaining assets. This is what makes partial runs possible and justifies partial defences: there are equilibria where patient depositors withdraw if the are ahead in the queue but not if they are behind. Whether the assumption that speculators know their place in the queue (and in particular how many speculators are behind them) before making their decisions is reasonable probably depends on the application. My analysis shows how one can obtain partial runs (the possibility of which justifies partial defences) even assuming speculators don’t know their place in line.

Moreover, the mechanism in Ennis and Keister (2010) is based on multiple equilibria. This makes it hard to use the model for comparative statics, since the key is that many different things can happen for given parameters. The “global games” approach does not completely rule out multiple equilibria, but it does limit the set of equilibria to the extent that some comparative statics questions can be studied. Under what conditions will the central bank be willing to withstand a speculative attack? How does the willingness to resist an attack change with the central bank’s preferences, the distribution of speculators’ preferences or the degree and type of uncertainty?

One point in common with the analysis by Ennis and Keister (2010) is that the timing of the central bank’s decision matters for the types of equilibria that may arise. If the central bank could decide on τ∗ before the game begins and commit to this, it would want to commit to full resistance in order to steer the equilibrium of the speculators’ subgame to c∗ = 0, where only the speculators for whom attacking is dominant attack and speculative attacks are small.6 If instead, as in Morris and Shin (1998), the central bank chose τ∗ after observing A, it would sometimes choose to fail; anticipating this, speculative attacks would be larger than under

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6Unless there is a sufficiently high probability that the mass of speculators for whom attacking is dominant is large. In this case the central bank may prefer partial or no resistance even with commitment.
commitment. Continuous decision-making without commitment, which I assume in common with Ennis and Keister (2010), is an intermediate case. For some combinations of parameters, there are full-resistance equilibria, where the central bank behaves just as under commitment. In these equilibria, during the course of play the central bank will always (correctly) believe that an ongoing attack is likely to be over soon. This justifies its persistence in defending the regime even when ex-post it would prefer not to have done so. If instead parameters are such that a waiting equilibrium with stopping point $\tau^*$ is played, the central bank would always gain by committing to a stopping point $\tau^* + \epsilon$ for small enough $\epsilon$, even if it could not commit to full resistance. Due to the first order condition (4), holding the distribution of $A$ constant, the central bank is indifferent to a small change in $\tau^*$ but (6) and (8) imply that a small increase in $\tau^*$ shifts the distribution of $A$ to the left, which results in a first-order benefit for the central bank.

Interestingly, the same forces that lead to unique equilibria in the informed-central-bank case lead to equilibria with waiting when the central bank is uninformed. When there is little heterogeneity and large aggregate uncertainty, private sources of information (i.e. the idiosyncratic cost $c_i$), which the central bank does not have access to, are relatively more informative. This force eliminates multiplicity in the informed-central-bank case, as in Morris and Shin (2000), but heightens the informational disadvantage of the central bank in the uninformed-central-bank case. For the uninformed central bank, an interior $\tau$ is justified when the attack is likely to be very small or very large: it is willing to wait up to $\tau$ because of the chance that the attack might be very small but if after $\tau$ attackers the attack is not over then it realizes that the attack will not be over soon and it abandons its defence of the regime. As discussed above, probability distributions of this kind, which have most of the mass in the extremes, arise when there is little heterogeneity relative to the amount of aggregate uncertainty.

One further prediction of the model is that one should expect to see failed partial defences against speculative attacks in the same kinds of environments when one also observes successful defences against small attacks. For instance, in a fixed-exchange regime where money demand is highly variable, the central bank will often experience what the model describes as small speculative attacks, simply from shifts in $G(0|\theta)$. It will therefore be more willing to engage in a partial defence than a central bank in a country where money demand is very stable and a speculative attack is not easily mistaken for day-to-day variation in money demand.

6 Final Remarks

If central banks and governments undertake costly defence measures when faced with speculative attacks and after some time decide to abandon them, then (assuming they are acting rationally) it must be that in the meantime they learned something about the environment or
the actions of the speculators that they did not know at first. This paper provides one possible explanation of where their original uncertainty may stem from: uncertainty about (some aspect of) the distribution of speculators’ payoffs from attacking or not attacking the regime.

Under fairly general conditions, virtually any beliefs about the attack could be consistent with the equilibrium of a simple coordination game. In a special case with normal uncertainty and linear payoffs, beliefs that would justify some, but not complete, defence of the status quo arise when aggregate uncertainty is great compared to the degree of heterogeneity in the population, so that very small or very large attacks are likely.

Of course, this is not the only possible source of uncertainty that central banks or governments may face in these episodes. They could be unsure, as in Goldstein, Ozdenoren and Yuan (2008), about the costs of regime change (\(v\) in this model) or about what information the public has. Part of the analysis of the present model is likely to extend to these settings, such as the nature of the optimal stopping problem and the key role of the (endogenous) hazard function. Other aspects, such as the role of heterogeneity, are more special to the exact way uncertainty is introduced in the model.

One important feature of real speculative attacks that the model does not include is the possibility that speculators may learn as the attack progresses. Gu (2011) and He and Manela (2012) study models along these lines.

Appendix

Proof of Proposition 1. Given \(f(A)\), let \(\tau^*\) solve program (3) and define the operator \(T(c^*)\) by the following series of steps:

1. Given \(c^*\), let \(\theta^*\) satisfy (6) if a solution exists, \(\theta = \infty\) if \(G(c^*|\theta) > \tau^*, \forall \theta\) and \(\theta = -\infty\) if \(G(c^*|\theta) < \tau^*, \forall \theta\).

2. Using (2), let

\[
p(\theta) = \frac{f(A)}{\left| \frac{\partial G^{-1}(c^*; A)}{\partial A} \right| \bigg|_{A=G(c^*|\theta)}}
\]

Property 2 in the statement of the proposition ensures that \(G^{-1}(c^*; A)\) exists so \(p(\theta)\) is well defined.

3. Let \(T(c^*)\) be the solution to (8) where \(\theta^*\) is the value obtained in step 1 and the function \(p(\theta)\) derived in step 2 is used in (1) to compute \(P(\theta^*|c)\). Since \(g(c_i|\theta)\) satisfies the monotone inference property, this equation always has a unique solution.
Since \( f (A) \) and \( g(c_i|\theta) \) are continuous, then the operator \( T (c^*) : \mathbb{R} \to [0, 1] \) is a continuous function, so it must have a fixed point in \([0, 1]\). If \( c^* \) is such a fixed point, then under prior \( p(\theta) \) given by (18), \( \{\tau^*, c^*, \theta^*, f(A)\} \) is an equilibrium of the game.

\[ \square \]

**Proof of Lemma 1.** Using (1)

\[
p(\hat{\theta}|c) = \frac{\sqrt{\alpha} \phi(\sqrt{\alpha} (c - \theta)) p(\theta)}{\int_{\theta} \sqrt{\alpha} \phi(\sqrt{\alpha} (c - \theta)) p(\theta) d\theta} \\
P(\hat{\theta}|c) = \frac{\int_{\theta \leq \theta} \Phi(\sqrt{\alpha} (c - \theta)) p(\theta) d\theta}{\int_{\theta} \Phi(\sqrt{\alpha} (c - \theta)) p(\theta) d\theta}
\]

Taking derivatives and rearranging:

\[
\frac{\partial P(\hat{\theta}|c)}{\partial c} = \frac{\alpha}{[\int_{\theta} \phi(\sqrt{\alpha} (c - \theta)) p(\theta) d\theta]^2} \left[ (\int_{\theta \leq \theta} \hat{\phi}(\sqrt{\alpha} (c - \theta)) p(\theta) d\theta) \left( \int_{\theta \geq \theta} \phi(\sqrt{\alpha} (c - \theta)) p(\theta) d\theta \right) - \left( \int_{\theta \geq \theta} \hat{\phi}(\sqrt{\alpha} (c - \theta)) p(\theta) d\theta \right) \left( \int_{\theta \leq \theta} \phi(\sqrt{\alpha} (c - \theta)) p(\theta) d\theta \right) \right]
\]

\[
\leq \frac{\alpha}{[\int_{\theta} \phi(\sqrt{\alpha} (c - \theta)) p(\theta) d\theta]^2} \left[ \hat{\phi} \left( \int_{\theta \geq \theta} \phi(\sqrt{\alpha} (c - \theta)) p(\theta) d\theta \right) \left( \int_{\theta \leq \theta} \phi(\sqrt{\alpha} (c - \theta)) p(\theta) d\theta \right) - \hat{\phi} \left( \int_{\theta \leq \theta} \phi(\sqrt{\alpha} (c - \theta)) p(\theta) d\theta \right) \left( \int_{\theta \geq \theta} \phi(\sqrt{\alpha} (c - \theta)) p(\theta) d\theta \right) \right]
\]

\[
= 0
\]

\[ \square \]

**Proof of Lemma 2.**

1. By direct computation

2.

\[
f(A) = \frac{\sqrt{\frac{\alpha \mu}{\alpha c}} \phi \left( \sqrt{\frac{\alpha \mu}{\alpha c}} \Phi^{-1}(A) - \sqrt{\frac{\alpha \mu}{\alpha c}} (c^* - \mu) \right)}{\phi(\Phi^{-1}(A))}
\]

\[
= \sqrt{\frac{\alpha \mu}{\alpha c}} \exp \left[ \frac{1}{2} \left( 1 - \frac{\alpha \mu}{\alpha c} \right) [\Phi^{-1}(A)]^2 + 2 \frac{\alpha \mu}{\sqrt{\alpha c}} (c^* - \mu) \Phi^{-1}(A) - \alpha \mu (c^* - \mu)^2 \right]
\]

Taking the limit as \( A \to 0 \) gives the result.

3. Let \( \gamma = \sqrt{\frac{\alpha \mu}{\alpha c}} \) and \( C = \sqrt{\alpha \mu} (c^* - \mu) \)

\[
h(A) = \frac{\phi(\gamma \Phi^{-1}(A) + C)}{\phi(\Phi^{-1}(A))} \int_A^1 \frac{\phi(\gamma \Phi^{-1}(a) + C)}{\phi(\Phi^{-1}(a))} da
\]

24
Define

\[ u(A) \equiv \phi \left( \gamma \Phi^{-1}(A) + C \right) \]
\[ v(A) \equiv \phi \left( \Phi^{-1}(A) \right) \]
\[ w(A) \equiv \int_{A}^{1} \frac{\phi \left( \gamma \Phi^{-1}(a) + C \right)}{\phi \left( \Phi^{-1}(a) \right)} \, da \]

Taking derivatives,

\[ h'(A) = \frac{u'(A) v(A) w(A) - u(A) v'(A) w(A) - u(A) v(A) w'(A)}{[w(A)]^2} \]

\[ = \frac{u(A)}{[w(A)]^2} \left\{ (1 - \gamma^2) \Phi^{-1}(A) - \gamma C \right\} \int_{\gamma \Phi^{-1}(A) + C}^{1} \frac{\phi \left( \gamma \Phi^{-1}(a) + C \right)}{\phi \left( \Phi^{-1}(a) \right)} \, da + \phi \left( \gamma \Phi^{-1}(A) + C \right) \}

\[ = \frac{u(A)}{[w(A)]^2} \left\{ (1 - \gamma^2) \Phi^{-1}(A) - \gamma C \right\} \left\{ \frac{1}{\gamma} \int_{\gamma \Phi^{-1}(A) + C}^{1} \phi(x) \, dx + \phi \left( \gamma \Phi^{-1}(A) + C \right) \right\} \}

\[ = \frac{u(A)}{[w(A)]^2} \left\{ 1 - \Phi \left( \gamma \Phi^{-1}(A) + C \right) \right\} \left\{ \frac{(1 - \gamma^2) \Phi^{-1}(A) - \gamma C}{\gamma} + H \left( \gamma \Phi^{-1}(A) + C \right) \right\} \}

where I have used the change of variable \( x = \gamma \Phi^{-1}(a) + C \) and \( H \) is the hazard function of the standard normal distribution. The assumptions of the Lemma can be restated as \( \gamma > 1, C < 0 \). For \( A < \frac{1}{2} \), they imply that all the terms in brackets are positive, so the hazard must be increasing. For \( A \geq \frac{1}{2} \), use the fact that \( H(y) > y \) so

\[ \frac{(1 - \gamma^2) \Phi^{-1}(A) - \gamma C}{\gamma} + H \left( \gamma \Phi^{-1}(A) + C \right) \geq -\gamma \Phi^{-1}(A) - C + H \left( \gamma \Phi^{-1}(A) + C \right) > 0 \]

\[ \square \]

References


