Dynamic Optimization - Part 1

1 Setup

- We’ll look at problems of the form:

\[
\begin{align*}
\sup_{\{x_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \\
\text{s.t.} \\
x_{t+1} \in \Gamma(x_t) \quad \forall t \\
x_0 \text{ given}
\end{align*}
\]  

(1)

- Recall that for the Neoclassical Growth Model
  
  \begin{itemize}
  \item $x_t \rightarrow k_t$
  \item $F(x_t, x_{t+1}) \rightarrow u(f(k_t) + (1-\delta)k_t - k_{t+1})$
  \item $\Gamma(x_t) = [0, f(k_t) + (1-\delta)k_t]$
  \end{itemize}

- Use the notation $\tilde{x} = \{x_t\}_{t=0}^{\infty}$ to refer to complete sequences (SLP calls these plans).

- Let

\[
\Pi(x_0) = \{\tilde{x} : x_{t+1} \in \Gamma(x_t) \forall t\}
\]

be the set of feasible plans

- Remarks:
  
  1. We write sup because there is no presumption that the maximum exists
      - (But usually this is not an issue)
  2. The infinite sum

\[
\sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})
\]
means
\[ \lim_{T \to \infty} \sum_{t=0}^{T} \beta^t F(x_t, x_{t+1}) \]

- SLP assume that the limit exists for any \( \tilde{x} \in \Pi(x_0) \) (assumption 4.2) (although it could be \(+\infty\) or \(-\infty\)

3. \( x_t \) is a “state variable”. It belongs to a set \( X \).

4. \( \Gamma \) is a correspondence: it maps \( x_t \) into a set \( \Gamma(x_t) \subseteq X \)

2 A Variational Approach

- The goal is to find conditions such that FOCs are sufficient for an optimum

2.1 Assumptions

Assumption 1. \( F \) is increasing in the first argument.

This is just a normalization; otherwise redefine \( x \).

Assumption 2. \( X \subseteq \mathbb{R}_+^n \).

E.g. the capital stock is nonnegative. We could easily generalize to any nonzero lower bound on \( x \).

Assumption 3. \( F \) is differentiable.

So we can take FOCs

Assumption 4. \( F \) is concave.

Strict concavity will give uniqueness

2.2 Euler equation

- Suppose \( \bar{x}^* \) is the optimum, and it is interior, i.e.

\[ x_{t+1}^* \in \text{int}(\Gamma(x_t^*)) \]

- Then

\[ x_{t+1}^* \in \arg \max_{x_{t+1}} F(x_t^*, x_{t+1}) + \beta F(x_{t+1}, x_{t+2}^*) \]
• This implies Euler equation:

\[ F_y(x_t^*, x_{t+1}^*) + \beta F_x(x_{t+1}^*, x_{t+2}^*) = 0 \]  

(2)

• Notation: \( F_x \) is the derivative w.r.t. the first argument and \( F_y \) is the derivative w.r.t. the second argument.

• Remark: \( x \) could be a vector, so the Euler equation (2) should be read as a system of equations, one for each dimension of \( x \).

• Example:

\[
\begin{align*}
F' (x_t^*, x_{t+1}^*) &= u (f(k_t) + (1 - \delta) k_t - k_{t+1}) \\
F_y (x_t^*, x_{t+1}^*) &= -u' (f(k_t) + (1 - \delta) k_t - k_{t+1}) \\
F_x (x_{t+1}^*, x_{t+2}^*) &= u' (f(k_{t+1}) + (1 - \delta) k_{t+1} - k_{t+2}) [f' (k_t) + 1 - \delta] \\
\Rightarrow u' (f(k_t) + (1 - \delta) k_t - k_{t+1}) &= \beta u' (f(k_{t+1}) + (1 - \delta) k_{t+1} - k_{t+2}) [f' (k_t) + 1 - \delta] \\
\Rightarrow u' (c_t) &= \beta u' (c_{t+1}) [f' (k_t) + 1 - \delta]
\end{align*}
\]

2.3 Transversality condition

• Suppose we had a finite horizon, with last period \( T \)

• Then in any interior solution, it would have to be that

\[ F_y (x_T^*, x_{T+1}^*) = 0 \]

• Generalization to infinite horizon case:

\[ \lim_{T \to \infty} \beta^T F_y (x_T^*, x_{T+1}^*) x_{T+1}^* = 0 \]

or, using the Euler equation

\[ \lim_{T \to \infty} \beta^T F_x (x_T^*, x_{T+1}^*) x_T^* = 0 \]  

(3)

• Equation (3) is known as a transversality condition.

• Interpretation:

\( - \beta^T F_x (x_T^*, x_{T+1}^*) \) is the net present marginal value of \( x \) at time \( T \): how much does the value of the plan (measured as of time 0) increase if we increase \( x_T \).
- $x_T$ is the level in period $T$
- Equation (3) is saying that the present value of $x$ goes to zero as $T \to \infty$

2.4 Sufficiency result

**Proposition 1.** Suppose Assumptions 1-4 hold and $\bar{x}^*$ is interior, satisfies (2) and (3) and $\sum_{t=0}^{\infty} \beta^t F (x_t^*, x_{t+1}^*) < \infty$. Then $\bar{x}$ is a solution to problem (1). If Assumption 4 holds strictly, then $\bar{x}$ is the unique solution.

**Proof.** Let $\tilde{x}$ be any plan other than $\bar{x}^*$, i.e. such that $x_{t+1}^* \neq x_{t+1}^*$ for some $t$. Let

$$\Delta = \sum_{t=0}^{\infty} \beta^t \left[ F (x_t^*, x_{t+1}^*) - F (x_t, x_{t+1}) \right]$$

Then

$$\Delta = \lim_{T \to \infty} \sum_{t=0}^{T} \beta^t \left[ F (x_t^*, x_{t+1}^*) - F (x_t, x_{t+1}) \right]$$

$$\geq \lim_{T \to \infty} \sum_{t=0}^{T} \beta^t \left[ F_x (x_t^*, x_{t+1}^*) [x_t^* - x_t] + F_y (x_t^*, x_{t+1}^*) [x_{t+1}^* - x_{t+1}] \right]$$

$$= F_x (x_0^*, x_1^*) [x_0^* - x_0] + \lim_{T \to \infty} \sum_{t=0}^{T} \beta^{t+1} F_x (x_{t+1}, x_{t+2}^*) [x_{t+1}^* - x_{t+1}] + \sum_{t=0}^{T} \beta^t F_y (x_t^*, x_{t+1}^*) [x_{t+1}^* - x_{t+1}]$$

$$= F_x (x_0^*, x_1^*) [x_0^* - x_0] + \lim_{T \to \infty} \sum_{t=0}^{T} \beta^{t} (\beta F_x (x_{t+1}, x_{t+2}^*) + F_y (x_t^*, x_{t+1}^*)) [x_{t+1}^* - x_{t+1}]$$

$$- \lim_{T \to \infty} \beta^{T+1} F_x (x_{t+1}, x_{t+2}^*) [x_{t+1}^* - x_{t+1}]$$

$$= - \lim_{T \to \infty} \beta^T F_x (x_t^*, x_{t+1}^*) [x_t^* - x_t]$$

$$\geq 0$$

- The first step follows from concavity (Assumption 4) and differentiability (Assumption 3) and is strict inequality if we have strict concavity.
- The next two steps are rearranging.
- The following step uses that $x_0^* - x_0 = 0$ because both plans start from the same point and that the Euler equation holds for plan $\bar{x}^*$.
- The final step follows from the fact that $F_x \geq 0$ (Assumption 1), the transversality condition holds for plan $\bar{x}^*$ and $x_t \geq 0$ (Assumption 1).
• Remarks
  – SLP prove a weaker version of this (Theorem 4.15).
    * We didn’t require bounded $F$ (useful because often unbounded), just finite value of plan $\bar{x}$.
  – The exact form of the transversality condition depends on the problem
    * E.g. it’s different if you don’t assume 1
  – We have proved (2) and (3) are sufficient, not that they are necessary.
    * Easy to see Euler is necessary
    * Necessity of transversality condition requires a bit more work
  – The result is valid even for nonstationary problems, i.e. when $F$ and $\Gamma$ also have $t$ as an argument

3 Examples

3.1 Neoclassical Growth Model

• We had
  \[ F(x_t, x_{t+1}) \rightarrow u(f(k_t) + (1 - \delta)k_t - k_{t+1}) \]

• Euler equation is
  \[ -u'(f(k_t) + (1 - \delta)k_t - k_{t+1}) + \beta u'(f(k_{t+1}) + (1 - \delta)k_{t+1} - k_{t+2}) [f'(k_{t+1}) + (1 - \delta)] = 0 \]
  which is a second-order difference equation in $k_t$

• Note that using the resource constraint
  \[ k_{t+1} = f(k_t) + (1 - \delta)k_t - c_t \]
  we get back to our usual equation in terms of consumption:
  \[ -u'(c_t) + \beta [f'(k_{t+1}) + (1 - \delta)] u'(c_{t+1}) = 0 \]

• If we find a path for $k_t$ such that
  \[ \lim_{t \to \infty} k_t = k_{ss} \]
  then the transversality condition is satisfied and we know that this is the unique optimal path.
3.2 A consumption problem with borrowing constraints and CRRA utility

- Problem:

\[
\max_{t=0} \infty \sum \beta^t u(c_t)
\]

s.t.

\[
a_{t+1} = y - c_t + (1 + r) a_t
\]

\[
a_{t+1} \geq -B
\]

\[
a_0 \text{ given}
\]

where

\[
u(c) = \frac{c^{1-\sigma}}{1-\sigma}
\]

- Here:

  - \(x_t \to a_t\)
  - \(F(x_t, x_{t+1}) \to u((1 + r) a_t + y - a_{t+1})\)
  - \(\Gamma(x_t) = [-B, y + (1 + r) a_t]\)

- This doesn’t meet assumption 2 but it’s easy to transform the problem such that is does:

  - \(z_t \equiv a_t + B\)
  - Budget constraint becomes

\[
z_{t+1} - B = y - c_t + (1 + r) (z_t - B)
\]

\[
z_{t+1} = y - rB - c_t + (1 + r) z_t
\]

  - \(F(z_t, z_{t+1}) = u((1 + r) z_t + y - z_{t+1} - rB)\)
  - \(\Gamma(z_t) = [0, y - rB + (1 + r) z_t]\)

- Euler:

\[
F_y \left(x_t^*, x_{t+1}^*\right) + \beta F_x \left(x_{t+1}^*, x_{t+2}^*\right) = 0
\]

\[
-u'((1 + r) z_t + y - z_{t+1} - rB) + \beta u'((1 + r) z_{t+1} + y - z_{t+2} - rB) (1 + r) = 0
\]

which, using the budget constraint, becomes

\[
u'(c_t) = \beta (1 + r) u'(c_{t+1})
\]
• Using CRRA:
\[
\frac{c_{t+1}}{c_t} = \left[ \beta (1 + r) \right]^{\frac{1}{\sigma}}
\]

• Exercise: transversality condition

4 A Recursive Approach

• Define
\[
V^* (x_0) \equiv \sup_{\{x_t\}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})
\]
\[
s.t.
\]
\[
x_{t+1} \in \Gamma(x_t) \quad \forall t
\]
\[
x_0 \text{ given}
\]

• Break up the infinite dimensional problem into 2-period problems

• Informal logic:
  - Suppose we have reached period 1, having chosen the optimal \( x_1 \). The value from then on would be \( V^* (x_1) \)
  - The overall value would then be
\[
F(x_0, x_1) + \beta V^* (x_1)
\]
  - Then we should be able to find the optimal plan by solving
\[
\sup_{x_0, x_1} F(x_0, x_1) + \beta V^* (x_1)
\]
\[
s.t.
\]
\[
x_1 \in \Gamma(x_0)
\]
  - (But for this we would need to know the function \( V^* \))
  - But the original value should then satisfy
\[
V^* (x_0) = \sup_{x_0, x_1} F(x_0, x_1) + \beta V^* (x_1)
\]
\[
s.t.
\]
\[
x_1 \in \Gamma(x_0)
\]
• Functional equation:

\[ V(x) = \sup_{x,y} F(x, y) + \beta V(y) \]

subject to

\[ y \in \Gamma(x) \]  

(4)

• Questions:

1. Does (4) have a solution?
2. Is it unique?
3. Is it the case that \( V^* \) (defined from the sequence problem) satisfies (4)?
4. (When) is it the case that a solution to (4) is \( V^* \)?
5. If \( \tilde{x}^* \) solves (1), is it the case that it maximizes the RHS of (4), i.e.

\[ V^*(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta V^*(x_{t+1}^*) \]  

?  

6. If \( \tilde{x}^* \) maximizes the RHS of (4), i.e.

\[ V^*(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta V^*(x_{t+1}^*) \]

then is it the case that it attains the supremum of problem (1)?