Non-linear Income Taxation

1 Setup

- readings: Mirrlees (REStud 1971), Diamond (AER 1998), Saez (REStud 2001), Werning (working paper 2007), Scheuer/Werning (working paper 2015)

- preferences
  \[ U(c, Y, \theta), \]
  e.g. \[ U(c, y/\theta) \] where \( l \equiv y/\theta \) is labor supply and \( \theta \) is productivity

- skill distribution with cdf \( F(\theta) \)

- technology
  \[ G + \int (c(\theta) - Y(\theta))dF(\theta) \leq e \] (1)
  equivalent to
  \[ G - e \leq \int T(Y(\theta))dF(\theta), \]
  government budget constraint

- non-linear income taxation: choose a tax schedule \( T(Y) \) such that the budget set becomes
  \[ B \equiv \{(c, Y)|c \leq Y - T(Y)\}, \]
  where we call \( R(Y) \equiv Y - T(Y) \) the retention function

- normative criterion:
  1. social welfare function (Mirrlees, Diamond, Saez)
  2. Pareto efficiency (Werning, Scheuer/Werning)

2 Feasibility and Incentive Compatibility

- agents solve
  \[ \max_{c, Y} U(c, Y, \theta) \text{ s.t. } c \leq R(Y) \] (2)
• **Definition:** An allocation $c(\theta), Y(\theta)$ and a tax function $T(Y)$ are feasible if (i) (1) holds and (ii) $c(\theta), Y(\theta)$ solves (2) given $T(Y)$.

• An allocation $c(\theta), Y(\theta)$ is feasible if there is a tax function $T(Y)$ such that $c(\theta), Y(\theta), T(Y)$ are feasible.

• whenever we have $c(\theta), Y(\theta)$, we have $R(Y)$ and hence $T(Y)$

- Observation: If $c(\theta), Y(\theta)$ is feasible, then
  \[ U(c(\theta), Y(\theta), \theta) \geq U(c(\theta'), Y(\theta'), \theta), \quad \forall \theta, \theta', \]
  incentive compatibility constraints

- the converse is also true (**Lemma**): If $c(\theta), Y(\theta)$ satisfies (1) and (3), then it is feasible.

- idea: can always find a retention function that implements an incentive-compatible allocation, e.g.
  \[ R(\tilde{Y}) \equiv \sup_{\tilde{\epsilon}} \{ \tilde{\epsilon} | U(c(\theta), Y(\theta), \theta) \geq U(\tilde{\epsilon}, \tilde{Y}, \theta), \forall \theta \} , \]
  lower envelope of indifference curves

- by construction,
  \[ U(c(\theta), Y(\theta), \theta) \geq U(R(\tilde{Y}), \tilde{Y}, \theta), \forall \tilde{Y}, \theta , \]
  so that agents faced with retention function $R(\tilde{Y})$ choose $\tilde{Y} = Y(\theta), R(\tilde{Y}) = c(\theta)$
• I.e. at points chosen by some type, retention function is just consumption in the allocation. At other points, it ‘fills the gaps.’

• Taxation principle or revelation principle (with private information). Here, we did not assume private information about $\theta$, but just that we are restricted to a budget set $B$ that has to be the same for all $\theta$.

• Marginal taxes (from (2)): If $T'(\bar{Y})$ exists and $\bar{Y} = Y(\theta)$, then

$$T'(\bar{Y}) = T'(Y(\theta)) = 1 - MRS(c(\theta), Y(\theta), \theta)$$

(4)

with

$$MRS(c, Y, \theta) \equiv -\frac{U_Y(c, Y, \theta)}{U_c(c, Y, \theta)}$$

• slope of 1 in $(Y, c)$-space means zero marginal tax rate

• adding structure: preferences satisfy single-crossing if

$$MRS(c, Y, \theta)$$ is decreasing in $\theta$

• Observation: If $c(\theta), Y(\theta)$ is incentive compatible and preferences satisfy single crossing, then $c(\theta)$ and $Y(\theta)$ must be increasing in $\theta$ (monotonicity)

• Define

$$v(\theta) \equiv U(c(\theta), Y(\theta), \theta) = \max_{\theta'} U(c(\theta'), Y(\theta'), \theta) \forall \theta$$
by the (global) incentive constraints (3)

- **FOC (local incentive constraints)**

\[
U_c(c(\theta), Y(\theta), \theta)c'(\theta) + U_Y(c(\theta), Y(\theta), \theta)y'(\theta) = 0,
\]

evaluated at truth-telling

- equivalent envelope condition:

\[
v'(\theta) = U_\theta(c(\theta), Y(\theta), \theta)
\]

or in integral form

\[
v(\theta) = \int_\theta^\theta U_\theta(c(\bar{\theta}), Y(\bar{\theta}), \bar{\theta})d\bar{\theta} + v(\theta)
\]

- As we have seen, incentive constraints (3) imply
  1. local incentive constraints (6) and
  2. monotonicity of \(Y(\theta)\) (and \(c(\theta)\)).

These are also sufficient.

- **Lemma**: If preferences satisfy single-crossing, then the allocation \(c(\theta), Y(\theta)\) is incentive compatible if and only if (6) is satisfied and \(Y(\theta)\) is non-decreasing for all \(\theta\).
3 Pareto Improvements and Laffer Effects

- start from a tax schedule $T_0(Y)$, which induces incomes $Y_0(\theta)$
- is this Pareto efficient?
- by resource constraint, we must have

$$G - e \leq \int T_0(Y_0(\theta))dF(\theta)$$

(7)

- Lemma: If the inequality in (7) is strict, then there exists another tax schedule $T_1 > T_0$ (in the Pareto sense)
- Idea: Assume $T^* \equiv \max_{\theta} T_0(Y_0(\theta)) < \infty$ for simplicity (e.g. bounded type space). Define $T_\varepsilon(Y) = \min\{T_0(Y), T^* - \varepsilon\}$ for some small $\varepsilon > 0$. The lost tax revenue moves continuously with $\varepsilon$ (it may even increase if types with lower $Y$ move up to higher incomes and pay $T^* - \varepsilon$ now). This way, one can close the gap in (7) and clearly $T_\varepsilon > T_0$.

- Suppose $T_1 > T_0$. Then it must be that

$$G - e = \int T_0(Y_0(\theta))dF(\theta) \leq \int T_1(Y_1(\theta))dF(\theta)$$

(8)

for feasibility, where $Y_1(\theta)$ is induced by $T_1(Y)$.

- Lemma: For a Pareto improvement, it must be that

$$T_1(Y_1(\theta)) \leq T_0(Y_1(\theta)) \forall \theta.$$ 

(9)
To see this, note that

\[
U(Y_1(\theta) - T_1(Y_1(\theta)), Y_1(\theta), \theta) \geq U(Y_0(\theta) - T_0(Y_0(\theta)), Y_0(\theta), \theta)
\]

\[
\geq U(Y_1(\theta) - T_0(Y_1(\theta)), Y_1(\theta), \theta),
\]

where the first inequality follows from the Pareto improvement and the second from the assumption that \(T_0(Y)\) induces agents to choose incomes \(Y_0(\theta)\).

- Hence, (8) and (9) imply that Pareto-improving tax reforms must take the form of tax reductions that do not reduce tax revenue: (sophisticated) Laffer effects

### 4 Pareto Efficient Taxation

- change in variables from \(c, Y\) to \(v, Y\) with

\[
v(\theta) = U(c(\theta), Y(\theta), \theta)
\]

\[
c(\theta) = e(v(\theta), Y(\theta), \theta).
\]

- use dual approach to Pareto problem: maximize resources subject to delivering \(\bar{v}(\theta)\) or more to all \(\theta\):

\[
\max_{v(\theta), Y(\theta)} \int (Y(\theta) - e(v(\theta), Y(\theta), \theta)) \, dF(\theta)
\]

s.t.

\[
v'(\theta) = U_\theta(e(v(\theta), Y(\theta), \theta), Y(\theta), \theta) \forall \theta
\]

\[
v(\theta) \geq \bar{v}(\theta) \forall \theta
\]

and the monotonicity constraint that \(Y(\theta)\) must be increasing.

- Ignore monotonocity constraint and check later whether the solution satisfies it. If not, need to consider bunching/ironing.

- Once we’ve found the solution to this problem, we can find \(c(\theta) = e(v(\theta), Y(\theta), \theta)\) and with \(c(\theta), Y(\theta)\) find the retention function \(R(Y)\) and the optimal tax schedule \(T(Y)\)

- Put some multiplier \(\zeta(\theta) = \lambda(\theta)\frac{f(\theta)}{\eta}\) on the last constraints. \(\bar{v}(\theta)\) and \(\zeta(\theta)\) will be related at the optimum.
• solve
\[
\max_{v(\theta), Y(\theta)} \int (Y(\theta) - e(v(\theta), Y(\theta), \theta)) dF(\theta) + \frac{1}{\eta} \int \lambda(\theta)v(\theta)f(\theta)d\theta
\]

s.t.
\[
v'(\theta) = U_\theta(e(v(\theta), Y(\theta), \theta), Y(\theta), \theta) \forall \theta
\]
or equivalently
\[
\frac{1}{\eta} \left\{ \max_{v(\theta), Y(\theta)} \eta \int (Y(\theta) - e(v(\theta), Y(\theta), \theta)) dF(\theta) + \int \lambda(\theta)v(\theta)f(\theta)d\theta \right\},
\]
which is the Lagrangian that comes out of the dual problem
\[
\max_{v(\theta), Y(\theta)} \int \lambda(\theta)v(\theta)f(\theta)d\theta
\]

s.t.
\[
v'(\theta) = U_\theta(e(v(\theta), Y(\theta), \theta), Y(\theta), \theta) \forall \theta
\]
\[
\int (Y(\theta) - e(v(\theta), Y(\theta), \theta))dF(\theta) \geq G - e
\]
for some $G - e$ (which will be related to $\eta$ at the optimum)

• Utilitarian social welfare function is then captured by $\lambda(\theta) = 1 \forall \theta$, the Rawlsian case by $\lambda(\theta) = 0$ for all $\theta$ except $\bar{\theta}$

• Solve using
  - optimal control with $v$ as state and $Y$ as control variable
  - Lagrangian and integration by parts

• form Lagrangian
\[
L = \int (Y(\theta) - e(v(\theta), Y(\theta), \theta)) dF(\theta) + \frac{1}{\eta} \int \lambda(\theta)v(\theta)f(\theta)d\theta
\]
\[
+ \int \mu(\theta)v'(\theta)d\theta - \int \mu(\theta)U_\theta(e(v(\theta), Y(\theta), \theta), Y(\theta), \theta)d\theta
\]
and note that (after integration by parts)
\[
\int \mu(\theta)v'(\theta)d\theta = \mu(\bar{\theta})v(\bar{\theta}) - \mu(\bar{\theta})v(\bar{\theta}) - \int \mu'(\theta)v(\theta)d\theta
\]
• let’s take pointwise FOCs for $Y(\theta)$:

$$(1 - e_Y(\theta))f(\theta) - \mu(\theta) [U_{\theta c}(\theta)e_Y(\theta) + U_{\theta Y}(\theta)] = 0 \forall \theta,$$ (10)

where I simplified notation

• note that

$e_Y(\theta) = \frac{dc(\theta)}{dY(\theta)} = - \frac{U_Y(\theta)}{U_c(\theta)} = MRS(\theta)$

and hence

$$1 - e_Y(\theta) = 1 - MRS(\theta) = T'(Y(\theta)) \equiv \tau(\theta).$$

• also,

$$-U_{\theta c}(\theta)\frac{U_Y(\theta)}{U_c(\theta)} + U_{\theta Y}(\theta) = U_c(\theta)\frac{U_{\theta Y}(\theta)U_c(\theta) - U_{\theta c}(\theta)U_Y(\theta)}{U_c(\theta)^2}$$

$$= -U_c(\theta)\frac{\partial}{\partial \theta} \left[-\frac{U_Y(\theta)}{U_c(\theta)}\right] = -U_c(\theta)\frac{\partial MRS(\theta)}{\partial \theta}.$$ (11)

• with this, (10) becomes

$$\tau(\theta)f(\theta) = -\mu(\theta)U_c(\theta)\frac{\partial MRS(\theta)}{\partial \theta}$$

or, since $MRS(\theta) = 1 - \tau(\theta)$,

$$\frac{\tau(\theta)}{1 - \tau(\theta)}f(\theta) = -\mu(\theta)U_c(\theta)\frac{\partial \log MRS(\theta)}{\partial \theta}.$$ (12)

• the FOC for $v(\theta)$ (for interior $\theta$) can be written as

$$-e_v(\theta)f(\theta) - \mu'(\theta) - \mu(\theta)U_{\theta c}(\theta)e_v(\theta) + \lambda(\theta)f(\theta)/\eta = 0,$$ (13)

or, noting that $e_v(\theta) = 1/U_c(\theta)$ and $\lambda(\theta)f(\theta)/\eta \geq 0$,

$$-U_c(\theta)\mu'(\theta) - \mu(\theta)U_{\theta c}(\theta) \leq f(\theta).$$ (14)

• I.e. any Pareto efficient allocation has to satisfy this inequality, since otherwise we cannot find non-negative multipliers $\xi(\theta)$ on the Pareto constraints $v(\theta) \geq v(\theta)$. In particular, the Rawlsian allocation would have to satisfy (14) with equality for all (interior) $\theta$. 

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• let’s change variables from $\mu(\theta)$ to $\hat{\mu}(\theta) \equiv U_c(\theta)\mu(\theta)$, so that (recall that $U_c(\theta) = U_c(c(\theta), Y(\theta), \theta)$)

$$\hat{\mu}'(\theta) = U_c(\theta)\mu'(\theta) + \mu(\theta) [U_{c\theta}(\theta) + U_{cY}(\theta)c'(\theta) + U_{cY}(\theta)Y'(\theta)]$$

• substituting in (14) yields

$$-\hat{\mu}'(\theta) + \hat{\mu}(\theta) \frac{U_{cc}(\theta)c'(\theta) + U_{cY}(\theta)Y'(\theta)}{U_c(\theta)} \leq f(\theta)$$

• note that

$$\frac{U_{cc}(\theta)c'(\theta) + U_{cY}(\theta)Y'(\theta)}{U_c(\theta)} = \frac{U_{cc}(\theta)c'(\theta)}{U_c(\theta)} + \frac{U_{cY}(\theta)Y'(\theta)}{U_c(\theta)} = -\frac{\partial}{\partial c} \left[ \frac{U_Y(\theta)}{U_c(\theta)} \right] Y'(\theta) = -\frac{\partial MRS(\theta)}{\partial c} Y'(\theta)$$

since $c'(\theta)/Y'(\theta) = -U_Y(\theta)/U_c(\theta)$ by the local incentive constraints (5)

• substituting all this, the two conditions for Pareto efficiency (12) and (14) become

$$\frac{\tau(\theta)}{1 - \tau(\theta)} f(\theta) = -\hat{\mu}(\theta) \frac{\partial \log MRS(c(\theta), Y(\theta), \theta)}{\partial \theta}$$  \hspace{1cm} (15)$$

$$-\hat{\mu}'(\theta) - \hat{\mu}(\theta) \frac{\partial MRS(c(\theta), Y(\theta), \theta)}{\partial c} Y'(\theta) \leq f(\theta)$$  \hspace{1cm} (16)$$

• for any given preferences $U(c, Y, \theta)$, (differentiable) tax schedule $T(Y)$ and skill distribution $F(\theta)$ (and thus the resulting allocation $c(\theta), Y(\theta)$), (15) gives $\hat{\mu}(\theta)$ uniquely

• then (16) is the test for Pareto efficiency

5 An Example and Interpretation

• consider preferences with no income effect and a constant labor supply elasticity (Diamond 1998)

• in this case, $\partial MRS(c(\theta), Y(\theta), \theta)/\partial c = 0$

• e.g. functional form

$$U(c, Y, \theta) = c - \frac{1}{\alpha} \left( \frac{Y}{\theta} \right)^a$$
implying a (constant) wage elasticity of labor supply $\varepsilon \equiv \frac{1}{\alpha - 1}$

- under what conditions would a “flat” income tax (Hall/Rabushka 1995) with a constant marginal tax rate $\tau$ be Pareto efficient?

- with quasilinear preferences, the agent’s problem

$$\max_Y (1 - \tau)Y - \frac{1}{\alpha} \left( \frac{Y}{\theta} \right)^\alpha$$

implies

$$1 - \tau = Y(\theta)^{\alpha-1} \theta^{-\alpha} = MRS(Y(\theta), \theta)$$

and hence

$$\frac{\partial \log MRS(c(\theta), Y(\theta), \theta)}{\partial \theta} = -\frac{\alpha}{\theta} = -\frac{1 + \varepsilon}{\varepsilon}$$

- substituting in (15) yields

$$-\hat{\mu}(\theta) = -\frac{\tau}{1 - \tau} \frac{\varepsilon}{1 + \varepsilon} \theta f(\theta) \Rightarrow -\hat{\mu}'(\theta) = -\frac{\tau}{1 - \tau} \frac{\varepsilon}{1 + \varepsilon} \left[ f(\theta) + \theta f'(\theta) \right]$$

- substituting in (16) yields

- **Proposition**: Given quasilinear preferences with constant wage elasticity $\varepsilon$ and a skill density $f(\theta)$, a flat tax with constant marginal tax rate $\tau$ is Pareto efficient if and only if

$$\frac{\tau}{1 - \tau} \frac{1}{1 + 1/\varepsilon} \left[ -1 - \frac{f'(\theta)\theta}{f(\theta)} \right] \leq 1 \ \forall \theta$$

(18)
• Interpretation:

1. Note the role of the skill distribution: For any $\tau$ and $\varepsilon$, there exists a set of skill densities $f(\theta)$ such that $\tau$ is Pareto efficient and a set of skill densities such that it is Pareto inefficient.

2. Also: For any $\varepsilon$ and $f(\theta)$, there exists a set of flat tax schedules $\tau$ that are Pareto efficient and a set of $\tau$’s that are Pareto inefficient. Hence, without guidance on $f(\theta)$ (more on that later), “anything goes.”

3. Rawlsian flat tax would satisfy (18) with equality at $\max_\theta \left\{-1 - f'(\theta)\theta / f(\theta)\right\}$

4. $\varepsilon$ and $d \log f(\theta) / d \log \theta$ enter intuitively. Notably, the latter captures the (local) Laffer effects that can lead to Pareto inefficiency.

5. Problem Set: check condition for different skill distributions

- More generally, suppose preferences are still quasilinear with a constant wage elasticity, but we aim at testing the Pareto efficiency of any non-linear (differentiable) tax schedule

- (15) implies

\[
-\hat{\mu}(\theta) = -\frac{\varepsilon}{1 + \varepsilon \frac{\tau(\theta)}{1 - \tau(\theta)}} \theta f(\theta) \quad \Rightarrow \quad -\hat{\mu}'(\theta) = -\frac{\varepsilon}{1 + \varepsilon \frac{\tau(\theta)}{1 - \tau(\theta)}} \frac{d}{d\theta} \left[ \frac{\tau(\theta)}{1 - \tau(\theta)} \theta f(\theta) \right]
\]
substituting in (16) yields

\[-\varepsilon \frac{d}{1 + \varepsilon d\theta} \left[ \frac{\tau(\theta)}{1 - \tau(\theta)} \theta f(\theta) \right] \leq f(\theta) \ \forall \theta\]

and hence, after integrating,

• **Proposition:** Given quasilinear preferences with constant wage elasticity \(\varepsilon\) and a skill distribution \(F(\theta)\), a differentiable tax schedule with marginal tax rates \(\tau(\theta) = T'(Y(\theta))\) is Pareto efficient if and only if

\[\frac{1}{1 + 1/\varepsilon} \frac{\tau(\theta)}{1 - \tau(\theta)} f(\theta)\theta + F(\theta)\]

is weakly increasing in \(\theta\).

• “integral form” of (18)

• easy to see that flat tax with \(\tau(\theta) = \tau\) is special case

• skill distribution again plays crucial role

6 **Identifying the Skill Distribution**

• “anything goes” with Pareto efficiency unless we have restrictions on the skill distribution

• but we don’t observe the skill distribution

• Saez’s (2001) identification step: Suppose we observe an income distribution with cdf \(H(Y)\) induced by a tax schedule \(T(Y)\). Then we can back out the skill distribution from the relationship

\[H(Y(\theta)) = F(\theta) \iff h(Y(\theta))Y'(\theta) = f(\theta) \ \forall \theta.\]

• Moreover, given the tax schedule \(T(Y)\) and preferences \(U(c, Y, \theta)\), the agents’ utility maximization problem gives \(Y(\theta)\) from

\[Y(\theta) = \arg \max_Y U(Y - T(Y), Y, \theta),\]
where $Y(\theta)$ is implicitly defined by the FOC

$$U_c(Y(\theta) - T(Y(\theta)), Y(\theta), \theta)(1 - T'(Y(\theta))) + U_Y(Y(\theta) - T(Y(\theta)), Y(\theta), \theta) = 0. \quad (19)$$

- That way, we can identify the skill distribution from the distribution of observed incomes and the observed tax schedule, assuming some preferences.
- see Problem Set for an example
- this allows us to formulate a test for Pareto efficiency directly in terms of the observed income distribution
- in particular, (19) implies (simplifying notation)

$$Y'(\theta) = - \frac{U_c \theta (1 - T') + U_Y \theta}{U_{cc} (1 - T')^2 + 2 U_{cY} (1 - T') - U_c T'' + U_{YY}} \quad (20)$$

and note that the numerator is

$$U_c \theta (1 - T') + U_Y \theta = -U_c \theta \frac{U_Y}{U_c} + U_Y \theta = -U_c \frac{\partial MRS(c(\theta), Y(\theta), \theta)}{\partial \theta} = -U_c (1 - T') \frac{\partial \log MRS(c(\theta), Y(\theta), \theta)}{\partial \theta}$$

from (11) and $(1 - T') = MRS$.

- We can thus rewrite (20) as follows

$$Y'(\theta) = - \left( \frac{- \partial \log MRS}{\partial \theta} \right) \frac{- \left( \frac{U_c}{U_c (1 - T')} + 2 \frac{U_{cY}}{U_c (1 - T')} + \frac{U_{YY}}{U_c (1 - T')} \right) + \frac{T''}{1 - T'}}{1 - T'}.$$  

$$(21)$$

- (19) also implies that

$$\frac{dY}{d(1 - T')} = - \frac{U_c}{U_{cc} (1 - T')^2 + 2 U_{cY} (1 - T') + U_{YY}}$$

and hence the (after-tax) wage elasticity of income is

$$\varepsilon_w^* = \frac{dY}{d(1 - T')} \frac{1 - T'}{Y} = - \frac{U_c / Y}{U_{cc} (1 - T') + 2 U_{cY} + U_{YY} / (1 - T')}.$$
Substituting in the denominator of (21) finally gives

\[ Y'(\theta) = -\frac{\partial \log \text{MRS}(c, Y, \theta)}{\partial \theta} \frac{1}{\epsilon^*_w Y + \frac{T''}{1 - T'}}. \] \hspace{1cm} (22)

Defining \( \hat{\mu}(Y) \equiv \hat{\mu}(Y^{-1}(Y)) \), the two conditions for Pareto efficiency (15) and (16) can thus be rewritten exclusively in terms of \( Y \) and \( h(Y) \) as follows

\[ \hat{\mu}(Y) \equiv \frac{T'(Y)}{1 - T'(Y)} \frac{h(Y)}{1 + Y \epsilon^*_w(Y) \frac{T''(Y)}{1 - T'(Y)}} \] \hspace{1cm} (23)

\[-\hat{\mu}'(Y) - \hat{\mu}(Y) \frac{\partial \text{MRS}}{\partial c} \leq h(Y). \] \hspace{1cm} (24)

Saez (2001) defines the “virtual density” as\(^1\)

\[ h^*(Y) = \frac{h(Y)}{1 + Y \epsilon^*_w(Y) \frac{T''(Y)}{1 - T'(Y)}} = \frac{h(Y)}{\Phi(Y)} \]

then (23) becomes

\[ \hat{\mu}(Y) = \frac{T'(Y)}{1 - T'(Y)} h^*(Y) \epsilon^*_w(Y) Y \] \hspace{1cm} (25)

in logs

\[ \log \hat{\mu}(Y) = \log \left( \frac{T'(Y)}{1 - T'(Y)} \right) + \log h^*(Y) + \log \epsilon^*_w(Y) + \log Y \]

and differentiating w.r.t. \( \log Y \)

\[ \frac{\hat{\mu}'(Y)Y}{\hat{\mu}(Y)} \frac{d \log \hat{\mu}(Y)}{d \log Y} = \frac{\log \left( \frac{T'(Y)}{1 - T'(Y)} \right)}{d \log Y} + \frac{d \log h^*(Y)}{d \log Y} + \frac{d \log \epsilon^*_w(Y)}{d \log Y} + 1 \] \hspace{1cm} (26)

returning to (24), let’s substitute the definition of the virtual density \( h^*(Y) \) to obtain

\[ -\hat{\mu}'(Y) - \hat{\mu}(Y) \frac{\partial \text{MRS}}{\partial c} \leq h^*(Y) \Phi(Y) \]

\(^1\)Saez (2001) shows that this is the density of incomes that would occur at \( Y \) if the non-linear tax schedule \( T(Y) \) were replaced by the linear tax schedule that is tangent to \( T(Y) \) at income level \( Y \), i.e. a tax schedule with a constant marginal tax rate \( \tau = T'(Y) \) and intercept \( T = Y - T(Y) - Y(1 - \tau) \).
and multiplying through by $Y / \hat{\mu}(Y)$

$$-\frac{\hat{\mu}'(Y)}{\hat{\mu}(Y)} Y - \frac{\partial \text{MRS}}{\partial c} Y \leq h^*(Y) \Phi(Y) \frac{Y}{\hat{\mu}(Y)}.$$

- substitute (25) on the RHS and (26) on the LHS to get

$$-\frac{d \log \left( \frac{T'(Y)}{1-T'(Y)} \right)}{d \log Y} - \frac{d \log h^*(Y)}{d \log Y} - \frac{d \log \epsilon^*_w(Y)}{d \log Y} - 1 - \frac{\partial \text{MRS}}{\partial c} Y \leq \frac{Y h^*(Y) \Phi(Y)}{1-T'(Y) h^*(Y) \epsilon^*_w(Y) Y},$$

which simplifies and leads to the following result

- **Proposition:** A tax schedule $T(Y)$ is Pareto efficient if and only if

$$\frac{T'(Y)}{1-T'(Y)} \frac{\epsilon^*_w(Y)}{\Phi(Y)} \left[ -\frac{d \log \left( \frac{T'(Y)}{1-T'(Y)} \right)}{d \log Y} - \frac{d \log h^*(Y)}{d \log Y} - \frac{d \log \epsilon^*_w(Y)}{d \log Y} - 1 - \frac{\partial \text{MRS}}{\partial c} Y \right] \leq 1.$$

(27)

- **Interpretation:**

1. Many terms cancel if $T(Y)$ is flat tax, there are no income effects and the wage elasticity is constant: $T'(Y) = \tau, \Phi(Y) = 1, h^*(Y) = h(Y)$,

$$\frac{d \log \left( \frac{T'(Y)}{1-T'(Y)} \right)}{d \log Y} = 0, \frac{d \log \epsilon^*_w(Y)}{d \log Y} = 0, \frac{\partial \text{MRS}}{\partial c} = 0,$$

so that we get back to

$$\frac{\tau}{1-\tau} \epsilon^*_w \left( -1 - \frac{d \log h(Y)}{d \log Y} \right) \leq 1,$$

but now the test is in terms of the observable income distribution induced by the flat tax.

2. Otherwise,

- income effects $\partial \text{MRS} / \partial c > 0$,
- an increasing wage elasticity $d \log \epsilon^*_w(Y) / d \log Y > 0$, and
- progressivity $d \log \left( \frac{T'(Y)}{1-T'(Y)} \right) / d \log Y > 0$

help with justifying higher marginal tax rates. Intuition: again local Laffer effects.
7 Relation to Utilitarian Social Welfare Function

- How does the test (27) relate to the optimal tax formulas in Mirrlees (1971) and Saez (2001)?

- by the duality of the problem that we pointed out above, the utilitarian social welfare function approach is equivalent to setting $\zeta(\theta) = \lambda(\theta)f(\theta)/\eta = f(\theta)/\eta$ in (13)

- (15) and (16) turn into the optimality conditions

\[
\frac{\tau(\theta)}{1 - \tau(\theta)} f(\theta) = -\mu(\theta) \frac{\partial \log MRS(c(\theta), Y(\theta), \theta)}{\partial \theta}
\]

\[
-\mu'(\theta) - \mu(\theta) \frac{\partial MRS(c(\theta), Y(\theta), \theta)}{\partial c} Y'(\theta) = f(\theta) \left( 1 - \frac{U_c(c(\theta), Y(\theta), \theta)}{\eta} \right)
\]

with the transversality conditions $\hat{\mu}(\theta) = \hat{\mu}(-\theta) = 0$ (if the skill distribution is bounded)

- do transformation to income as above, and use virtual density $h^*(Y) = h(Y)/\Phi(Y)$:

\[
\hat{\mu}(Y) = \frac{T'(Y)}{1 - T'(Y)} \epsilon_w^*(Y) Y h^*(Y)
\]

\[
-\frac{\partial MRS(c(\theta), Y(\theta), \theta)}{\partial c} \hat{\mu}(Y) = h(Y) \left( 1 - \frac{U_c(c(\theta), Y(\theta), \theta)}{\eta} \right) + \hat{\mu}'(Y),
\]

- Note that the ODE (31) has an analogy with the valuation of an asset that pays a flow profit $\pi(t)$ in continuous time and is discounted at interest rate $r(t)$, leading to the no arbitrage condition for the value $V(t)$

\[
r(t)V(t) = \pi(t) + V'(t),
\]

which is the similar to (31). This yields

\[
V(s) = \int_s^\infty e^{-\int_t^s r(z) dz} \pi(t) dt.
\]

- analogously, integrating (31) in the same way and using (30) yields

\[
\frac{T'(Y)}{1 - T'(Y)} = \frac{1}{\epsilon_w^*(Y) h^*(Y) Y} \int_y^\infty \left( 1 - \frac{U_c}{\eta} \right) \exp \left( \int_y^x \frac{\partial MRS}{\partial c} (z) dz \right) \frac{h(x)}{1 - H(Y)} dx,
\]
which is the optimal income tax formula given in Saez (2001)

- The marginal tax rate is negatively related to the elasticity and virtual density at $Y$ and positively to income effects, since

\[
\exp \left( \int_y^x \frac{\partial MRS}{\partial c}(z) dz \right) > 1 \quad \forall x > y
\]

if $\partial MRS/\partial c > 0$

8 An Example and Numerical Computation

- How to compute the optimal income tax schedule (given a utilitarian SWF) numerically?

- Consider again the simpler case where preferences exhibit no income effect and the wage elasticity is constant. In particular, assume

\[
U(c, Y, \theta) = c - \frac{1}{2} \left( \frac{Y}{\theta} \right)^2
\]

and a social welfare function $W(v) = \log(v)$ as in Saez (2001)

- In this case, (28) can be used to solve for $Y(\theta)$ because $\partial \log MRS/\partial \theta = -2/\theta$ as seen above, $MRS = Y\theta^{-2}$ and hence

\[
\frac{\tau(\theta)}{1 - \tau(\theta)} f(\theta) = -\tilde{\mu}(\theta) \frac{\partial \log MRS(c(\theta), Y(\theta), \theta)}{\partial \theta} \iff \frac{1 - Y(\theta)\theta^{-2}}{Y(\theta)\theta^{-2}} f(\theta) = 2\tilde{\mu}(\theta)/\theta
\]

\[
\iff Y(\theta) = \frac{f(\theta)\theta^3}{2\tilde{\mu}(\theta) + f(\theta)\theta}
\]

(33)

- Moreover, (29) simplifies to

\[
-\tilde{\mu}'(\theta) = f(\theta) \left( 1 - \frac{1}{\eta v(\theta)} \right)
\]

(34)

and the local incentive constraint to

\[
v'(\theta) = U_\theta(c(\theta), Y(\theta), \theta) = \frac{Y(\theta)^2}{\theta^3} = \left( \frac{f(\theta)}{2\tilde{\mu}(\theta) + f(\theta)\theta} \right)^2 \theta^3.
\]

(35)
• for any given density $f(\theta)$, the optimum can therefore be computed as follows

1. fix some multiplier $\eta$
2. solve the system of ODE (34) and (35) for $v(\theta)$ and $\hat{\mu}(\theta)$ using the boundary conditions $\hat{\mu}(\theta) = \hat{\mu}(\overline{\theta}) = 0$. In particular
   - start with $\hat{\mu}(\theta) = 0$ and some (guessed) $v(\theta)$ as initial conditions
   - solve the ODE system to obtain $\hat{\mu}(\overline{\theta})$
   - repeat this while adjusting $v(\theta)$ until $\hat{\mu}(\overline{\theta}) = 0$
3. from the solved $\hat{\mu}(\theta)$ and $v(\theta)$ schedules, compute $Y(\theta)$ and $c(\theta)$ using (33) and
   \[ c(\theta) = v(\theta) + \frac{1}{2} \left( \frac{Y(\theta)}{\theta} \right)^2 \]
4. repeat this while adjusting $\eta$ until the resource constraint
   \[ \int (Y(\theta) - c(\theta)) dF(\theta) = 0 \]
   is satisfied
5. finally, check whether $Y(\theta)$ is increasing in $\theta$, so that the monotonicity constraint is satisfied. Otherwise, an additional iterative procedure is required to determine optimal bunching.

9 Untouched Issues

• extensive margin, occupational choice
• dynamics, life-cycle
• uncertainty, tax system as part of social insurance system
• richer heterogeneity

→ Scheuer/Werning (2016)