Mixed Taxation and Production Efficiency

1 Overview

1. Uniform commodity taxation under non-linear income taxation
   → Atkinson-Stiglitz (JPubE 1976) Theorem
   → Application to capital taxation

2. Uniform commodity taxation under linear income taxation
   → Primal approach to commodity taxation
   → Useful for thinking about capital taxation
   → Application: production efficiency theorem (Diamond-Mirrlees AER 1971)

2 Mixed Taxation

- Atkinson-Stiglitz (1976)
- consumption goods \( x \in \mathbb{R}^m \), labor \( Y \in \mathbb{R} \)
- preferences \( U^i(x, Y) \), arbitrary heterogeneity
- \( B \) — budget set: \( (x, Y) \in B \Rightarrow (x, Y) \) is affordable
  → may be non-linear set
- If we had \( B^i \) → First-best achievable
- Here: \( B \) anonymous (\( B \) is the government’s choice variable)
- Assume \( U^i[G(x), Y] \). I.e. weak separability and no heterogeneity in \( G \)
  → Result: uniform commodity taxation
- Proof: Given \( B \), consumers choose

\[
(x^i, Y^i) \in \arg \max_{(x,Y)\in B} U^i(x, Y)
\]  

(1)
(not necessarily unique max; if they are indifferent, just pick one)

- Technology: linear (can be relaxed, see below)
  \[ \sum_i \sum_j p_j x^i_j \leq \sum_i Y^i \] (2)

- \{x^i, Y^i\}_{i \in I}, B feasible iff (1) and (2) hold

- Start from \(B_0 \Rightarrow \{x^{i_0}, Y^{i_0}\}\) feasible. \(B_0\) can be arbitrary, e.g. with commodity taxes.

- Look for reform (\(B\) and feasible \(\{x^i, Y^i\}\)) that leaves utilities unchanged and saves resources.

- If \(U^i = U^i(G(x), Y)\), have 2-stage optimization:
  1. for a given \(G\), decide on optimal \(x\)
  2. then choose between \(G\) and \(Y\) optimally

- (1) \(\Rightarrow \) \(G(x^{i_0}), Y^{i_0}\) \(\in \arg \max_{(g,Y) \in b_0} U^i(g,Y)\) with
  \(b_0 = \{(g,Y) | \exists g = G(x) \land (x,Y) \in B_0\}\)

  (so \(b_0\) is fully pinned down by \(B_0\))

- Look for reform for \(B\) that has the same implied budget set \(b_0\)

  → Individuals choose the same \(g, Y\) and hence get same utility
  → But now let them choose the \(x\)’s efficiently

- Cost function \(e^G(g, p) = \min_x p \cdot x \text{ s.t. } g = G(x)\)

- Then \(B_{AS} = \{(x, Y) | p \cdot x \leq e^G(g, p) \text{ and } (g, Y) \in b_0\}\)

- For example
  \(\hat{b} = \{(y,Y) | y = e^G(g, p) \land (g, Y) \in b_0\}\),

  with \(Y\) denoting before-tax income and \(y\) after-tax income. Hence, \(\hat{b}\) corresponds to a nonlinear income tax schedule.

  \(\Rightarrow B_{AS} = \{(x, Y) | p \cdot x \leq y \text{ and } (y, Y) \in \hat{b}\}\)
→ individuals choose same \( g^i, Y^i \) as before

- \( x^i \)'s potentially changed, but by definition of \( e \), it minimized \( p \cdot x \), so LHS of resource constraint is decreased! (RHS unchanged)

\[ \Rightarrow (2) \text{ is now satisfied with inequality (strict if } x^{i0} \neq x^{iAS}) \]

- More general technology: 
  \[ F \left( \sum_i x^i \pi^i, \sum_i Y^i \pi^i \right) \leq 0 \]

→ set \( p_j = \frac{\partial F}{\partial x_j} \text{ at the new allocation} \)

- Application to capital taxation: see next class

3 Primal Approach to Commodity Taxation

- Return to linear tax framework from the beginning of the quarter

- Introduce an alternative approach (called the “primal” approach) that will be useful to characterize optimal capital taxation next class

3.1 Setup

- representative consumer, no heterogeneity

- linear taxation, no lump-sum tax to finance exogenous government expenditure (the “Ramsey problem”)

- numeraire good labor \( l \), untaxed

- \( n \) consumption goods \( c_1, ..., c_n \), prices \( p_i \), taxed at linear rate \( t_i \)

- consumers:

\[
\max_{c_1, ..., c_n, l} U(c_1, ..., c_n, l) \quad \text{s.t.} \quad \sum_i p_i(1 + t_i)c_i \leq l. \quad (3)
\]

- firms: CRS technology

\[ F(x_1, ..., x_n, l) \leq 0, \]

\[ \text{e.g.} \]

\[ \sum_i \bar{p}_i x_i - l \leq 0, \]

where \( 1/\bar{p}_i \) is the productivity of labor in good \( i \)
• profit maximization

\[
\max_{x_1,\ldots,x_n,l} \sum_i p_i x_i - l \quad \text{s.t.} \quad F(x_1,\ldots,x_n,l) \leq 0
\] (4)

• government: exogenous government expenditures \{g_i\}, budget constraint

\[
\sum_i p_i g_i \leq \sum_i p_i t_i c_i \quad (5)
\]

• note: since \{g_i\} is fixed, we could easily allow for preferences

\[U(c_1,\ldots,c_n,l;g_1,\ldots,g_n)\]
without changing results

3.2 Competitive Equilibrium

A competitive equilibrium with taxes \{t_i\} and government expenditures \{g_i\} is an allocation \{c_i, x_i, l\} and prices \{p_i\} such that

1. \{c_i, l\} solves the consumers’ problem (3) given prices \{p_i\} and taxes \{t_i\}

2. \{x_i, l\} solves (4) given \{p_i\} and firms make zero profits

3. \{c_i, g_i\} and \{p_i, t_i\} satisfy the government budget constraint (5)

4. all markets clear, i.e.

\[c_i + g_i = x_i \quad \forall i = 1,\ldots,n\] (6)

Lemma 1. \{c_i, l\} and \{p_i\} is part of a competitive equilibrium with \{t_i\} and \{g_i\} if and only if

\[F(c_1 + g_1,\ldots,c_n + g_n,l) = 0,\] (7)

\[p_i = \frac{F_i(c_1 + g_1,\ldots,c_n + g_n,l)}{F(c_1 + g_1,\ldots,c_n + g_n,l)}\]

and \{c_i, l\} solves (3) given \{p_i, t_i\}.

Proof. • only if: clear

• set \[x_i = c_i + g_i\], so 4. is satisfied
• necessary conditions for (4) are

\[ p_i = \gamma F_i \]

for some \( \gamma \) and

\[ -1 = \gamma F_l \]

• if \( p_i = -F_i / F_l \), then these conditions are satisfied and profits are

\[ \sum_i p_i x_i - l = -\sum_i \frac{F_i}{F_l} x_i - l = -\frac{1}{F_l} \left( \sum_i F_i x_i + F_l l \right) = 0 \]

by CRS and Euler’s theorem, so 2. is satisfied

• as for 3., note

\[ \sum_i p_i g_i = \sum_i t_i p_i c_i \iff \sum_i p_i g_i = \left( \sum_i p_i (1 + t_i) c_i - l \right) - \left( \sum_i p_i c_i - l \right) = -\left( \sum_i p_i c_i - l \right) \]

by the consumers’ budget constraint,

\[ \iff \sum_i p_i (g_i + c_i) - l = \sum_i p_i x_i - l = 0, \]

since profits are zero as shown above. Thus, 3. is satisfied.

3.3 Ramsey Problem

•

\[
\max_{c_1, \ldots, c_n, l} U(c_1, \ldots, c_n, l)
\]

s.t.

\[ F(c_1 + g_1, \ldots, c_n + g_n, l) = 0 \]

and

\[ \{c_1, \ldots, c_n, l\} \in \arg \max_{c_1, \ldots, c_n, l} U(c_1, \ldots, c_n, l) \text{ s.t. } \sum_i c_i (1 + t_i) p_i = l \]

• we optimize over quantities \( \{c_i, l\} \) and prices/taxes \( \{p_i, t_i\} \), but the two are related through the last condition
• two approaches:
  1. dual: solve quantities as a function of prices and optimize over prices (as we did at the beginning of the quarter)
  2. primal: solve prices as a function of quantities and optimize over quantities
• we pursue the second approach now as it will be very useful for dynamic taxation later

3.4 Primal Approach
• by convexity of the consumers’ problem, FOCs are necessary and sufficient:

$$U_i = \lambda (1 + t_i)p_i$$
$$U_l = -\lambda$$

• solve for prices

$$(1 + t_i)p_i = -\frac{U_i}{U_l}$$

• substitute in budget constraint

$$\sum_i U_i c_i + U_l l = 0 \quad (8)$$

• “implementability constraint,” no prices left

**Proposition 1.** Consider any allocation \( \{c_i^*, l^*\} \) that satisfies the implementability constraint (8) and the feasibility constraint (7). Then there exist prices and taxes \( \{p_i^*, t_i^*\} \) such that \( \{c_i^*, l^*\} \) and \( \{t_i^*, p_i^*\} \) is part of a competitive equilibrium with taxes.

• note: many solutions since 2 constraints, but \( n + 1 \) variables

**Proof.** set

$$p_i^* = -\frac{F_i(c_1^* + g_1, ..., c_n^* + g_n, l^*)}{F_i(c_1^* + g_1, ..., c_n^* + g_n, l^*)}$$

so 2. is satisfied

• given this, FOC for consumers

$$p_i^*(1 + t_i^*) = -\frac{U_i(c_1^*, ..., c_n^*, l^*)}{U_l(c_1^*, ..., c_n^*, l^*)},$$
thus set
\[ 1 + t_i^* = \frac{U_i^* F_i^*}{U_i^* F_i^*} \]  
(9)

• moreover, consumers’ budget constraint is satisfied since
\[ \sum_i p_i^*(1 + t_i^*)c_i^* = l^* \]
is equivalent to (substituting \( \{p_i^*, t_i^*\} \) from above)
\[ -\sum_i U_i^* c_i^* = U_i^* l^*, \]
which is the imposed implementability constraint (8). Hence, 1. is satisfied.

• market clearing was guaranteed when deriving \( p_i^* \)

• government budget constraint is satisfied by Walras’ law: consumers’ budget constraint
\[ \sum_i p_i^*(1 + t_i^*)c_i^* = l^* \]
zero profits
\[ \sum_i p_i^*(c_i^* + g_i^*) = l^* \]
and subtracting
\[ \sum_i p_i^* t_i^* c_i^* = \sum_i p_i^* g_i \]

3.5 Optimal Tax Rules

• Ramsey problem
\[ \max_{c_1, \ldots, c_n, l} U(c_1, \ldots, c_n, l) \]
s.t. (7) and (8)

• Lagrangian
\[ \mathcal{L} = U(c_1, \ldots, c_n, l) + \mu \left( \sum_i U_i c_i + U_i l \right) - \gamma F(c_1 + g_1, \ldots, c_n + g_n, l) \]
• FOCs for good $c_j$ and $l$

\[
(1 + \mu)U_j + \mu \left( \sum_i U_{ij}c_i + U_{ij}l \right) = \gamma F_j
\]

\[
(1 + \mu)U_l + \mu \left( \sum_i U_{il}c_i + U_{il}l \right) = \gamma F_l
\]

or

\[
\frac{U_j}{U_l} \frac{1 + \mu + \mu \sum_i U_{ij}c_i + U_{ij}l}{1 + \mu + \mu \sum_i U_{il}c_i + U_{il}l} = \frac{F_j}{F_l}
\]

• we know from (9) that

\[
1 + t_j = \frac{U_j}{U_l} \frac{F_j}{F_l} = \frac{1 + \mu + \mu \sum_i U_{ij}c_i + U_{ij}l}{1 + \mu + \mu \sum_i U_{il}c_i + U_{il}l} \equiv \frac{1 + \mu - \mu H_l}{1 + \mu - \mu H_j}
\]

(10)

3.5.1 Uniform Taxation Rule

• suppose $U$ is separable such that

\[
U(c_1, ..., c_n, l) \equiv U(G(c_1, ..., c_n), l),
\]

where $U_G = 0$ and $G(.)$ is homogeneous of degree one

• if double total spending on all consumptions goods, then double demand for each individual good, and demand for consumption goods does not depend on $l$

• then

\[
U_j = U_G G_j
\]

and

\[
U_{ij} = U_{GG} G_i G_j + U_{Gij}
\]

and

\[
U_{lj} = U_{lG} G_j = 0
\]
• thus
\[ \frac{\sum_i U_{ij}c_i + U_{lj}l}{U_l} = \frac{\sum_i (U_{GG}G_iG_jc_i + U_{G}G_{ij}c_i)}{U_{G}G_j} \]

• now use
\[ \sum_i U_{GG}G_iG_jc_i = U_{GG} \sum_i G_i c_i = U_{GG}G_j \]
and
\[ \sum_i U_{G}G_{ij}c_i = U_{G} \sum_i G_{ij}c_i = 0 \]
by Euler’s theorem and since \( G \) is homogeneous of degree one and \( G_j \) is homogeneous of degree zero

• hence
\[ \frac{\sum_i U_{ij}c_i + U_{lj}l}{U_l} = \frac{U_{GG}G}{U_{G}} \]

• therefore
\[ 1 + t_j = \frac{1 + \mu + \mu \frac{U_{lj}l}{U_l}}{1 + \mu + \mu \frac{U_{GG}G}{U_{G}}} \]
independent of \( j \)

• exercise: show that the same result more generally goes through when \( G(.) \) is homothetic, i.e.
\[ G(c_1, ..., c_n) \equiv k(K(c_1, ..., c_n)), \]
where \( k'(K) \neq 0 \) and \( K(c_1, ..., c_n) \) is homogeneous of degree \( \rho \).

• can also be generalized to heterogeneous agents with preferences \( U^k(G(c_1, ..., c_n), l) \), where \( k \) is the household index. Then the uniform commodity taxation rule applies in any Pareto optimum if (i) \( U^k(.) \) is separable for all \( k \), (ii) the sub-utility \( G(.) \) is the same for all \( k \) and (iii) \( G(.) \) is homothetic.

• Note the difference to the Atkinson-Stiglitz (1976) theorem: Non-linear taxation of labor, no homotheticity of \( G(.) \) (nor \( U_{Gl} = 0 \)) required.

3.5.2 Inverse Income Elasticities Rule

• rearrange (10) to
\[ \frac{t_j}{1 + t_j} = \frac{\mu}{1 + \mu - \mu H_l} \frac{H_j - H_l}{H_l} \]
• thus \( t_j > t_i \) if \( H_j > H_i \)

• suppose now that \( U \) is separable across all arguments, so that

\[
H_j = -\frac{U_{jj} c_j}{U_j} \quad \text{and} \quad H_i = -\frac{U_{ii} l}{U_l}
\]

• to consider income effects, suppose consumers are endowed with some exogenous non-labor income \( I \), so that the FOC for their utility maximization problem is

\[
U_i(c_i(q, I)) = \lambda(q, I) q_i,
\]

where

\[
q = [p_1(1 + t_1), \ldots, p_n(1 + t_n)]
\]

• differentiate w.r.t. \( I \)

\[
U_{ii} \frac{\partial c_i}{\partial I} = q_i \frac{\partial \lambda}{\partial I} = \frac{U_i \partial \lambda}{\lambda \partial I}
\]

so that \( H_i \) becomes

\[
H_i = -\frac{U_{ii} c_i}{U_i} = -\frac{c_i \partial \lambda / \partial I}{\lambda \partial c_i / \partial I}
\]

• define the income elasticity

\[
\eta_i \equiv \frac{\partial c_i}{\partial I} \frac{I}{c_i}
\]

to get

\[
H_i = -\frac{\partial \lambda / \partial I \cdot \lambda}{\eta_i}
\]

• numerator is positive, so \( H_i \) is negatively related to income elasticity \( \eta_i \)

• hence goods with a higher income elasticity are taxed at a lower rate: tax necessities at a higher rate than luxury goods

4 Production Efficiency

• How should goods that consumers do not consume directly be taxed, such as intermediate goods? A general result (Diamond and Mirrlees, 1971) is that the economy should always be on the production possibility frontier with optimal taxes. This implies that intermediate goods should not be taxed.
• We will use the primal approach from the preceding section to demonstrate a simple version of this result. Consider an economy with two sectors. The final goods sector has technology

\[ f(x, z, l_1) = 0, \]

where \( x \) is the final good, \( z \) is an intermediate good and \( l_1 \) is labor used in the final goods sector.

• The intermediate goods sector has technology

\[ h(z, l_2) = 0, \]

where \( l_2 \) is labor used in the intermediate goods sector.

• Let us first characterize a competitive equilibrium in this economy. Consumers maximize their utility subject to their budget constraint taking prices as given.

\[
\max_{c, l_1, l_2} U(c, l_1 + l_2) \\
s.t. \quad p(1 + \tau)c \leq w(l_1 + l_2),
\]

where \( \tau \) is the consumption tax and we normalized the tax on labor to be zero. \( p \) is the price of consumption and \( w \) is the wage.

• The FOCs are

\[ U_c = p(1 + \tau)\lambda \quad \text{and} \quad -U_l = w\lambda. \]

Substituting \( p(1 + \tau) \) and \( w \) from this in the budget constraint, we obtain the implementability constraint

\[ U_c c + U_l (l_1 + l_2) = 0. \]

• The final goods sector maximizes profits subject to the feasibility constraint, taking prices as given

\[
\max_{x, z, l_1} px - wl_1 - q(1 + \tau_z)z \\
s.t. \quad f(x, z, l_1) = 0,
\]

where \( \tau_z \) is the tax on the intermediate good and \( q \) is the price of the intermediate good.

• The FOCs are

\[ -w = \gamma f_{l_1} \]
\[-q(1 + \tau_z) = \gamma f_z\]

so that
\[
\frac{f_t}{f_z} = \frac{w}{q(1 + \tau_z)}. \tag{11}
\]

- The intermediate goods sector also maximizes profits subject to the feasibility constraint and taking prices as given

\[
\max_{qz, l_2} qz - wl_2
\]

s.t. \( h(z, l_2) = 0 \)

with the FOCs
\[
q = \gamma h_z
\]
\[-w = \gamma h_l
\]

so that
\[
\frac{h_l}{h_z} = -\frac{w}{q}
\]

and hence using (11)
\[
\frac{h_l}{h_z} = -(1 + \tau_z)\frac{f_l}{f_z}. \tag{12}
\]

- Finally, the government’s budget constraint is

\[
\tau pc + \tau z qz = pg
\]

and market clearing requires
\[
c + g = x.
\]

- The social planner’s problem can be written as

\[
\max_{c, l_1, l_2, z} U(c, l_1 + l_2)
\]

s.t. \( U(c) + U(l_1 + l_2) = 0 \)
\[
f(c + g, z, l_1) = 0
\]
\[
h(z, l_2) = 0,
\]

where we used the primal approach by substituting out prices from the consumers’
FOCs and budget constraint as before. 

• The FOC of the planning problem w.r.t \(z\) is

\[ f_z \gamma_f + h_z \gamma_h = 0 \]

or

\[ \frac{f_z}{h_z} = -\frac{\gamma_h}{\gamma_f}. \]

• The FOC w.r.t. \(l_1\) is

\[ U_l = f_l \gamma_f \]

and w.r.t. \(l_2\)

\[ U_l = h_l \gamma_h, \]

which implies that

\[ \frac{f_l}{h_l} = \frac{\gamma_h}{\gamma_f} \]

or

\[ \frac{f_l}{f_z} = -\frac{h_l}{h_z}. \]

• This shows that when taxes are set optimally, the marginal rate of transformation is undistorted across goods. Comparing with the condition for a competitive equilibrium in (12), we see that in the optimum \(\tau_z = 0\). Hence, no tax on intermediate goods should be imposed.