Linear Capital Taxation and Tax Smoothing

1 Finite Horizon

1.1 Setup

- 2 periods $t = 0, 1$
- preferences $U_i(c_0, c_1, l_0)$
- sequential budget constraints in $t = 0, 1$

$$c_0^i + pb_1^i + k_1^i \leq w_0(1 - \tau^i)l_0^i + R_0 k_0^i$$

$$c_1^i \leq R_1 k_1^i + b_1^i$$

for all $i$, where

$$R_t \equiv 1 + (r_t - \delta)(1 - \tau^k_t)$$

- idea: firms just rent capital at rental rate $r_t$. Consumers own capital, get paid the rental rate but incur depreciation costs. But they can deduct this from rental income when computing capital tax liability.

- can combine to period-0 present value budget constraint

$$c_0^i + pc_1^i \leq w_0(1 - \tau^i)l_0^i + R_0 k_0^i + (pR_1 - 1)k_1^i$$

- sequential aggregate resource constraints

$$k_1 + c_0 + g_0 \leq F^0(k_0, l_0) + (1 - \delta)k_0$$

$$c_1 + g_1 \leq F^1(k_1) + (1 - \delta)k_1$$

- firms:

$$\max_{k_0, l_0} F^0(k_0, l_0) - r_0 k_0 - w_0 l_0$$
with FOCs

\[ r_0 = F_k^0(k_0, l_0) \]
\[ w_0 = F_l^0(k_0, l_0) \]

- no arbitrage requires

\[ R_1 = \frac{1}{p'} \]

i.e. the return on bonds and the return on capital must be the same

- consumers’ budget constraint becomes

\[ c_0^i + p c_1^i \leq w_0(1 - \tau^l)l_0^i + R_0 k_0^i, \]

where \( k_1^i \) has vanished, so from the perspective of the consumers, we are left with 3 goods \( c_0, c_1, l_0 \)

### 1.2 Uniform Taxation

- for instance, consider linear technology with

\[ F^0(k_0, l_0) = \bar{r}k_0 + \bar{w}l_0 \]
\[ F^1(k_1) = \bar{r}k_1 \]

so that the aggregate resource constraints can be combined to

\[ c_0 + g_0 + \bar{R}^{-1}(c_1 + g_1) \leq \bar{R}k_0 + \bar{w}l_0 \]

with \( \bar{R} \equiv 1 + \bar{r} - \delta \)

- again, we’re back to a 3 goods economy with \( c_0, c_1, l_0 \)

- since we’ve normalized the tax on \( c_0 \) to zero, the question whether \( c_0 \) and \( c_1 \) should be taxed uniformly comes down to asking under what conditions it’s optimal to set

\[ \frac{1}{p'} = R_1 = \bar{R} \]

and thus equivalently under what conditions

\[ \tau_{1k} = 0 \]
• tax on capital income in period 1 is nothing but a distortion on the price between consumption in periods 0 and 1

• hence we can apply the uniform commodity taxation result from the last note, which applies if

\[ U^i(c_0, c_1, l_0) = U^i(G(c_0, c_1), l_0) \]

is separable and \( G(.) \) is homothetic

• then \( R_1 = \overline{R} \) and thus \( \tau^k_1 = 0 \) in any Pareto optimum

• for instance, this would be satisfied if preferences are of the form

\[ U^i(c_0, c_1, l_0) = \frac{c_0^{1-\sigma}}{1-\sigma} + \beta \frac{c_1^{1-\sigma}}{1-\sigma} - v^i(l_0) \]

and e.g. \( v^i(l_0) = v(l_0/\theta^i) \)

• Atkinson-Stiglitz (1976) theorem: if we could tax labor income non-linearly, then homogeneity of \( G(.) \) would not be required, only that

\[ U^i(c_0, c_1, l_0) = U^i(G(c_0, c_1), l_0) \]

for uniform taxation to be Pareto optimal

• note: the sub-utility function \( G(c_1, c_2) \) must be the same for all individuals. See Saez (2002) and Diamond and Spinnewijn (2011) for deviations from this (heterogeneous discount factors \( \beta^i \)). Then capital taxation is generally welfare improving.

• what about taxing initial capital \( k^0_i \) using \( \tau^k_0 \neq 0 \)? Imitates a lump sum tax. If there is only one representative agent, would want to set \( \tau^k_0 \) as high as possible (not necessarily so with heterogeneity). Time inconsistency problem.

2 Infinite Horizon

2.1 Overview

Judd (JPubE 1985), Chamley (ECTA 1986)

• Famous results: zero capital taxation in steady state

• Significant policy impact
• Straub-Werning (2015): overturn these results to a large degree

2.2 Judd (1985)

• Two types of agents
  – Capitalists: save (do not work), income from returns to capital
  – Workers: supply 1 unit of labor inelastically, live hand-to-mouth

• Government taxes return to capital and pays transfers to workers

• Common discount factor $\beta$

• Capitalists $U(C_t)$

• Workers $u(c_t)$, labor endowment $n_t = 1$

• Technology $F(k_t, n_t)$, CRS, $\delta$ depreciation

• In equilibrium, $n_t = 1$, so will be convenient to work with $f(k) \equiv F(k, 1)$

• constant government expenditure $g \geq 0$
  – Resource constraint
    \[ c_t + C_t + g + k_{t+1} \leq f(k_t) + (1 - \delta)k_t \]  
    \[ (RC) \]
    given $k_0$

• Perfectly competitive markets $\rightarrow w^*_t = F_L(k_t, n_t) = f(k_t) - k_tf'(k_t)$
  – Before-tax return to capital
    \[ R^*_t = f'(k_t) + 1 - \delta \]

  – After-tax return to capital
    \[ R_t = 1 + (1 - \tau_t)(R^*_t - 1) \]

    We will say that capital is taxed if $R^* > R$, subsidized if $R^* < R$ and not taxed if $R^* = R$. 

• Taking $R_t$ as given, capitalists solve

$$\max_{C_t, a_{t+1}} \sum_{t=0}^{\infty} \beta^t U(C_t)$$

s.t.

$$C_t + a_{t+1} = R_t a_t,$$

$a_{t+1} \geq 0, a_0$ given.

• Euler equation: $U'(C_t) = \beta R_{t+1} U'(C_t)$, necessary and sufficient condition for optimum (together with transversality condition, $\beta^t U'(C_t) a_{t+1} \to 0$ as $t \to \infty$).

• Workers’ consumption equals disposable income

$$c_t = w_t^* + T_t = f(k_t) - f'(k_t) k_t + T_t$$

• Judd assumes balanced government budget in each period (no debt, this is one difference to Chamley)

• Total wealth equals capital stock $a_t = k_t$

$$g_t + T_t = (R_t^* - R_t) k_t \quad (BC)$$

• To set up the planning problem, we again follow the primal approach

  – Which allocations are consistent with the capitalists’ optimal savings decision for a given linear capital tax?
  – Must satisfy Euler equation
  – Use to solve out for prices ($R_t$) as function of quantities

$$\Rightarrow R_t = \frac{U'(C_{t-1})}{\beta U'(C_t)} \quad (*)$$

  – Substitute in sequential budget constraint to obtain sequential implementability constraint

$$C_t + k_{t+1} = R_t k_t = \frac{U'(C_{t-1})}{\beta U'(C_t)} k_t$$

$$\Rightarrow \beta U'(C_t)(C_t + K_{t+1}) = U'(C_{t-1}) k_t \quad (ImC)$$
• Planning Problem

\[
\max_{\{c_t, C_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta_t^t (u(c_t) + \gamma U(C_t))
\]

s.t. (RC) and (ImC), where \(\gamma\) is the Pareto weight on capitalists

• From last lecture, we know that, for any allocation that solves (RC) and (ImC), we can back out prices and taxes (from (*)) that ensures that this allocation is part of a Competitive Equilibrium with these prices and taxes.

• Set up Lagrangian

\[
L = \sum_{t=0}^{\infty} \beta_t^t (u(c_t) + \gamma U(C_t)) + \sum_{t=0}^{\infty} \beta_t \lambda_t (f(k_t) + (1 - \delta)k_t - c_t - C_t - g - k_{t+1}) + \sum_{t=0}^{\infty} \beta_t \mu_t (\beta U'(C_t)(C_t + k_{t+1}) - U'(C_{t-1})k_t)
\]

with first \(\mu_0 = 0\) since there is no implementability constraint in period 0.

• FOCs

\[
u'(c_t) = \lambda_t \tag{1}
\]

\[
\frac{\lambda_{t+1}}{\lambda_t} (f''(k_{t+1}) + 1 - \delta) = \frac{1}{\beta} + \frac{U'(C_t)}{\lambda_t}(\mu_{t+1} - \mu_t) \tag{2}
\]

\[
\mu_{t+1} = \frac{\mu_t}{k_{t+1}} \left( C_t + k_{t+1} + \frac{U'(C_t)}{U''(C_t)} \right) + \frac{1}{\beta k_{t+1}} \left( \gamma \frac{U'(C_t)}{U''(C_t)} - \frac{\lambda_t}{U''(C_t)} \right) \tag{3}
\]

Theorem 1 (Judd (1985)). Suppose multipliers and quantities converge to an interior steady state, i.e. \(c_t \to c, C_t \to C, k_t \to k\) (\(c, C, k > 0\)) and \(\mu_t \to \mu\). Then the tax on capital is zero in the steady state.

Proof. (1) and (2), steady state ⇒
Moreover, from Euler equation (*) in steady state: \( R = \frac{1}{\beta} \Rightarrow R = R^* \).

- Extremely powerful result
  - independent of preferences (in contrast to Atkinson-Stieglitz)
  - independent of \( \gamma \), i.e. don’t want to tax capitalists even if only care about workers

- Issues:
  - Really only used one FOC (for \( k \))
  - Strong assumptions on endogenous outcomes (convergence of allocation and multipliers) \( \rightarrow \) have we shown anything, or just assumed the result?

- Seems somewhat natural that optimum converges to interior steady state, and that multipliers converge in such a steady state. But: both are crucial for the result
  1. If \( c_t \rightarrow 0 \), we cannot guarantee that \( \frac{u(c_{t+1})}{u'(c_t)} \rightarrow 1 \) in (2)
  2. If \( \mu_t \) does not converge, \( \mu_{t+1} - \mu_t \) may not vanish in (2)

- To investigate, rewrite FOCs using \( \alpha_t \equiv \frac{k_t}{c_{t-1}} \), \( v_t \equiv \frac{U'(C_t)}{u'(c_t)} \), \( \sigma_t = -\frac{U''(C_t)C_t}{U'(C_t)} \) (capitalists coefficient of relative risk aversion, or the inverse of their elasticity of intertemporal substitution):

\[
(2) \Rightarrow \frac{u'(c_{t+1})}{u'(c_t)} \left( f'(k_{t+1}) + 1 - \delta \right) = \frac{1}{\beta} + v_t(\mu_{t+1} - \mu_t) \\
(3) \Rightarrow \mu_{t+1} = \mu_t \left( \frac{1 - 1/\sigma_t}{k_{t+1}} + 1 \right) + \frac{1}{\beta k_t \sigma_t v_t} (1 - \gamma v_t)
\]

- Consider the case with log-preference (\( \sigma = 1 \)) and suppose allocation converges to
interior steady state ($c_t, C_t, k_t$ all converge to strictly positive values). Then

$$(2') \Rightarrow \mu_{t+1} - \mu_t = \frac{R^* - \frac{1}{\beta}}{v}$$

$$(3') \Rightarrow \mu_{t+1} - \mu_t = \frac{1 - \gamma v}{\beta k v}$$

$\Leftrightarrow R^* - \frac{1}{\beta} = \frac{1 - \gamma v}{\beta k}$$

- Moreover, again from Euler equation: $R = \frac{1}{\beta}$. Hence

$$R^* - R = \frac{1 - \gamma v}{\beta k} > 0$$

if $\gamma$ sufficiently small.

$\rightarrow$ Positive long-run tax on capital if do not care too much about capitalist.

- In addition $\mu_t$ does not converge at the optimum (it explodes when the tax is positive!). Indeed, this observation has been made in the literature (Lansing, JPubE 1999), but it was thought to be a knife-edge problem only with log-utility

- Straub-Werning (2015): not knife-edge, holds for any CRRA with $\sigma > 1$ (and some more general results)

- For simplicitly, $\gamma = 0$. Towards a contradiction, suppose allocation were to converge to interior steady state $\rightarrow k_t, v_t$ converge to positive values $k, v$

$$(3') \Rightarrow v(\mu_{t+1} - \mu_t) = \mu_t \frac{\sigma - 1}{\sigma k} v + \frac{1}{\beta k \sigma}$$

$$(2') \Rightarrow f'(k) + 1 - \delta - \frac{1}{\beta} = v(\mu_{t+1} - \mu_t)$$

$\Rightarrow \mu_t$ must converge (recall $\sigma > 1$)

$\Rightarrow$ in the limit, $\mu = -\frac{1}{(\sigma-1)v\beta} < 0$

- Now check whether this can be possible. Recall $\mu_0 = 0$. From (3'), we know that whenever $\mu_t \geq 0$, then $\mu_{t+1} \geq 0$. Hence, $\mu_t \geq 0 \ \forall t$, and we have the desired contradiction. This proves

**Theorem 2.** If $\sigma > 1$ and $\gamma = 0$, then for any initial $k_0$ the solution to the planning problem does not converge to the zero-tax steady state, or any other interior steady state.

8
Straub-Werning (2015) shows that, in fact, solution converges to \( c_t \to 0, k_t \to k, C_t \to C, \tau_t \to \tau > 0 \). Moreover, \( \mu_t \to \infty \).

- More precisely, since \( c_t \to 0 \) and \( C = \frac{1-\beta}{\beta} k \) by (ImC), \( k \) is the smaller solution to

\[
\frac{1}{\beta} k + g = f(k) + (1 - \delta)k.
\]

- Hence, if \( g \to 0 \), the entire economy shrinks to zero in the long run: \( c_t, C_t, k_t \to 0 \). The capital tax rate converges to a strictly positive constant, but the tax base vanishes.

- If \( g > 0 \), the economy converges to the smallest capital stock consistent with those expenditures. Since \( c_t \to 0 \) but \( w_t^* = f(k_t) - f'(k_t)k_t \to w^* > 0 \), this means that the transfer to workers converges to a strictly negative constant. In other words, the capital tax is not high enough to finance \( g \), but workers also need to contribute.

- Show numerically that optimum involves very high capital tax

- For \( \sigma < 1 \), solution converges to zero-tax steady state, but very slowly so (especially for \( \sigma \) close to 1). E.g. takes 300 years even for \( \sigma = 0.75 \) to reach something close to 0.

- Intuition: suppose announce higher future tax on capital

How do capitalist react today? \( \to \) Substitution and income effects

- \( \sigma > 1 \) (i.e. elasticity of inter temporal substitution < 1): substitution effect < income effect
  * Capitalists lower consumption today to match drop in future consumption
  * Savings increase in short-run (even if fall in long-run)
- \( \sigma < 1 \) capitalist increase consumption today
- \( \sigma = 1 \) current consumption and savings unchanged

- Increasing capital is desirable for workers, as it increases tax base and wages

- \( \sigma > 1 \) \( \to \) achieves this by promising higher taxes in the future
  \( \to \) increasing path for taxes
- \( \sigma < 1 \) \( \to \) decreasing path for taxes (drive to 0)
- \( \sigma = 1 \) \( \to \) constant path for taxes
• Then results about long-run levels follow from the desire slopes of the path for capital taxes, rather than the other way around. For example, if $\sigma > 1$ and $g = 0$, the entire economy is starved in the long-run, but this is the negative side of a tradeoff that is worthwhile earlier on for workers.

• Results for large $\gamma \rightarrow$ negative capital taxes to redistribute towards capitalists

2.3 Chamley (1986)

Chamley considered a deterministic economy, but I will generalize here to allow for aggregate uncertainty. This will be useful to consider fiscal policy later on.

2.3.1 Setup

• CRS technology

$$F \left( K(s^{t-1}), L(s^t), s^t, t \right),$$

where $s^t = (s_0, s_1, \ldots, s_i)$ captures the history of aggregate uncertainty up to $t$

• the capital stock present in period $t$ is chosen in period $t - 1$ and therefore depends on $s^{t-1}$ only

• representative agent (no heterogeneity) with preferences

$$\sum_{s^t} \beta^t \Pr(s^t) u \left( c(s^t), L(s^t) \right)$$

• Aggregate resource constraint

$$c(s^t) + g(s^t) + K(s^t) \leq F \left( K(s^{t-1}), L(s^t), s^t, t \right) + (1 - \delta)K(s^{t-1}) \forall s^t \tag{4}$$

• hence aggregate uncertainty can result from technology shocks or government spending shocks

• Linear taxes on labour income

$$\tau^l(s^t)$$

• Linear taxes on capital income

$$\tau^k(s^t)$$

• (Taxes on consumption are redundant)
• The government has an initial debt equal to $B_0$

• Complete markets where the price of an Arrow-Debreu security is $p(s^t)$

• Note two key differences to Judd (1985): First, a representative agent, i.e. only motive for taxes is to raise revenue, no redistribution. Labor is now elastically supplied, and we have both linear capital and labor taxes available. The question is what is the optimal tax mix between those two in the long run to finance government spending. Second, the government no longer needs to run a balanced budget period by period. There is (fully state-contingent) government debt.

• Competitive markets with wages $w(s^t)$ and rental rate $r(s^t)$

2.3.2 Households, government and firms

• Government budget constraint

$$\sum_{t,s^t} p(s^t) \left[ g(s^t) - \tau^l(s^t)w(s^t)L(s^t) - \tau^k(s^t)(r(s^t) - \delta)K(s^t-1) \right] \leq -B_0$$

• Household budget constraint

$$\sum_{t,s^t} p(s^t) \left[ c(s^t) + K(s^t) - w(s^t) \left( 1 - \tau^l(s^t) \right) L(s^t) - R(s^t)K(s^t-1) \right] \leq B_0$$

where

$$R(s^t) = 1 + (1 - \tau^k(s^t))(r(s^t) - \delta)$$

is the gross after-tax return on capital

• Firm profits:

$$\pi(K, L, s^t, t) = F(K, L, s^t, t) - w(s^t)L - r(s^t)K$$

2.3.3 Equilibrium conditions

**Definition 1.** A competitive equilibrium is a policy $\{g(s^t), \tau^k(s^t), \tau^l(s^t)\}$, an allocation $\{c(s^t), K(s^t), L(s^t)\}$ and prices $\{w(s^t), r(s^t), p(s^t)\}$, such that households maximize utility s.t. budget constraint, firms maximize profits, the government budget constraint holds and markets clear.
• Firm FOC:

\[ r(s^t) = F_K \left( K(s^{t-1}), L(s^t), s^t, t \right) \]  
\[ w(s^t) = F_L \left( K(s^{t-1}), L(s^t), s^t, t \right) \]  

• Consumer FOC

\[
\beta^t \Pr(s^t) u_c \left( c(s^t), L(s^t) \right) - \lambda p(s^t) = 0 \\
\beta^t \Pr(s^t) u_L \left( c(s^t), L(s^t) \right) + \lambda p(s^t) \left( 1 - \tau^t(s^t) \right) w(s^t) = 0 \\
-\lambda p(s^t) + \lambda \sum_{s^{t+1}} p(s^{t+1}) R(s^{t+1}) = 0 
\]

• No arbitrage:

\[ p(s^t) = \sum_{s^{t+1}} p(s^{t+1}) R(s^{t+1}) \]  

• Intratemporal:

\[
\frac{\beta^t \Pr(s^t) u_c \left( c(s^t), L(s^t) \right)}{p(s^t)} = -\frac{\beta^t \Pr(s^t) u_L \left( c(s^t), L(s^t) \right)}{p(s^t) \left( 1 - \tau^t(s^t) \right) w(s^t)} \\
u_c \left( c(s^t), L(s^t) \right) = -\frac{u_L \left( c(s^t), L(s^t) \right)}{w(s^t) \left( 1 - \tau^t(s^t) \right)} \\
w(s^t) \left( 1 - \tau^t(s^t) \right) = -\frac{u_L \left( c(s^t), L(s^t) \right)}{u_c \left( c(s^t), L(s^t) \right)} 
\]

• Intertemporal:

\[
\frac{\beta^t \Pr(s^t) u_c \left( c(s^t), L(s^t) \right)}{p(s^t)} = u_c \left( c_0, L_0 \right) \\
p(s^t) = \frac{\beta^t \Pr(s^t) u_c \left( c(s^t), L(s^t) \right)}{u_c \left( c_0, L_0 \right)} 
\]
• Back to household budget:¹

\[ \sum_{t,s^t} p(s^t) \left[ c(s^t) + K(s^t) - w(s^t) \left( 1 - \tau^l(s^t) \right) L(s^t) - R(s^t)K(s^{t-1}) \right] \leq B_0 \]

\[ \sum_{t,s^t} p(s^t) \left[ c(s^t) - w(s^t) \left( 1 - \tau^l(s^t) \right) L(s^t) \right] \leq B_0 + R_0K_0 \]

\[ \sum_{t,s^t} \beta^t \Pr(s^t)u_c(c(s^t), L(s^t)) \left[ c(s^t) + \frac{u_L(c(s^t), L(s^t))}{u_c(c(s^t), L(s^t))} L(s^t) \right] \leq B_0 + R_0K_0 \]

\[ \sum_{t,s^t} \beta^t \Pr(s^t) \left[ u_c(c(s^t), L(s^t)) c(s^t) + u_L(c(s^t), L(s^t)) L(s^t) \right] \leq u_c(c_0, L_0) [B_0 + R_0K_0] \]

where we used the no arbitrage and the inter- and intratemporal conditions

• This is again the implementability constraint familiar from the last note. Prices and taxes have disappeared (primal approach), except for the initial period.

**Lemma 1.** An allocation \( \{ c(s^t), K(s^t), L(s^t) \} \) can be part of a competitive equilibrium iff (4) and (11) hold with equality.

**Proof.**

• Only if: shown above

• If:

1. Construct prices and taxes
   (a) Find \( r(s^t) \) and \( w(s^t) \) from (6) and (7)
   (b) Find \( p(s^t) \) from (10)
   (c) Find \( \tau^l(s^t) \) from (9)
   (d) Find \( \tau^k(s^t) \) from (8) and (5). Note: many solutions. Many patterns of state-contingent capital-income tax/government debt can implement same allocation.

2. Check for equilibrium
   (a) Factor prices imply firm optimization and zero profits by Euler’s theorem

¹Note that I use \( K_0 \) to denote initial capital, although to be 100% consistent I should denote this \( K_{-1} \).
Suppose there is an upper bound to capital taxes (lump-sum tax). Then in the initial period where we would like to tax the existing capital stock as much as possible to replicate a household optimization

The government’s problem is

\[ \sum_{t,s} p(s^t) \left[ c(s^t) + K(s^t) - w(s^t) \left( 1 - \tau^l(s^t) \right) L(s^t) - R(s^t)K(s^{t-1}) \right] = B_0 \]

\[ \sum_{t,s} p(s^t) \left[ c(s^t) + K(s^t) - w(s^t)L(s^t) + w(s^t)\tau^l(s^t)L(s^t) \right] = B_0 

\[ \sum_{t,s} p(s^t) \left[ c(s^t) + K(s^t) - w(s^t)L(s^t) + w(s^t)\tau^l(s^t)L(s^t) - (1 - \delta)K(s^{t-1}) \right] = B_0 

\[ \sum_{t,s} p(s^t) \left[ c(s^t) + K(s^t) - F (K(s^{t-1}), L(s^t), s^t, t) - (1 - \delta)K(s^{t-1}) \right] = B_0 

\[ \sum_{t,s} p(s^t) \left[ -g(s^t) + w(s^t)\tau^l(s^t)L(s^t) + \tau^k(s^t)(r(s^t) - \delta)K(s^{t-1}) \right] = B_0 \]

so the government budget constraint holds. This is really just Walras’ Law.

\[ \square \]

2.3.4 Optimal Taxes

The government’s problem is

\[ \max_{c(s^t), L(s^t), K(s^t), \tau^l_0} \sum_{s,t} \beta^t \Pr(s^t)u \left( c(s^t), L(s^t) \right) \]

s.t. \[ c(s^t) + g(s_t) + K(s^t) = F \left( K(s^{t-1}), L(s^t), s^t, t \right) + (1 - \delta)K(s^{t-1}) \]

\[ \sum_{t,s} \beta^t \Pr(s^t) \left[ u_c \left( c(s^t), L(s^t) \right) c(s^t) + u_L \left( c(s^t), L(s^t) \right) L(s^t) \right] = u_c \left( c_0, L_0 \right) [B_0 + R_0K_0] \]

Suppose there is an upper bound to capital taxes \( \bar{\tau} \) (recall the initial period problem, where we would like to tax the existing capital stock as much as possible to replicate a lump-sum tax). Then in the initial period \( \tau^l_0 = \bar{\tau} \). Let \( \mu \) be the multiplier on the implementability constraint. Define

\[ W(c, L) \equiv u(c, L) + \mu \left[ u_c (c, L) c + u_L (c, L) L \right] \]
Also recall from the household problem, using (8) and (10):

\[ u_c (c(s^t), L(s^t)) = \beta \sum_{s^{t+1}} \Pr(s^{t+1}|s^t) u_c (c(s^{t+1}), L(s^{t+1})) R(s^{t+1}) \]  

(15)

so using (12) we solve out for the optimal labor tax

\[ \tau^{t*}(s^t) = 1 - \frac{u_L (c(s^t), L(s^t))}{u_c (c(s^t), L(s^t))} \frac{W_c (c(s^t), L(s^t))}{W_L (c(s^t), L(s^t))} \]

(14)

Also recall from the household problem, using (9) and (7):

\[ W_c (c(s^t), L(s^t)) = \beta \sum_{s^{t+1}} \Pr(s^{t+1}|s^t) W_c (c(s^{t+1}), L(s^{t+1})) R^* (s^{t+1}), \]

(13)

where

\[ R^* (s^t) \equiv 1 + F_K \left( K(s^{t-1}), L(s^t), s^t, t \right) - \delta \]

Recall, from household problem, using (9) and (7):

\[ F_L \left( K(s^{t-1}), L(s^t), s^t, t \right) \left( 1 - \tau^t(s^t) \right) = -\frac{u_L (c(s^t), L(s^t))}{u_c (c(s^t), L(s^t))} \]

Problem becomes

\[
\max_{c(s^t), K(s^t)} \sum_{s^t} \beta^t \Pr(s^t) W_c (c(s^t), L(s^t)) - \mu u_c (c_0, L_0) [B_0 + R_0 K_0]
\]

s.t. \( c(s^t) + g(s_t) + K(s^t) = F \left( K(s^{t-1}), L(s^t), s^t, t \right) + (1 - \delta) K(s^{t-1}) \)

For \( t \neq 0 \), assuming the upper bound to capital taxes is not binding, the FOCs are:

\[ \beta \Pr(s^t) W_c (c(s^t), L(s^t)) - \gamma(s^t) = 0 \]

\[ \beta \Pr(s^t) W_L (c(s^t), L(s^t)) + \gamma(s^t) F_L \left( K(s^{t-1}), L(s^t), s^t, t \right) = 0 \]

\[ -\gamma(s^t) + \sum_{s_{t+1}} \gamma(s^t, s_{t+1}) \left[ F_K \left( K(s^t), L(s^{t+1}), s^{t+1}, t+1 \right) + (1 - \delta) \right] = 0 \]

Intratemporal:

\[ -\frac{W_L (c(s^t), L(s^t))}{W_c (c(s^t), L(s^t))} = F_L \left( K(s^{t-1}), L(s^t), s^t, t \right) \]  

(12)

Intertemporal:
There are many choices of \( \tau^k(s^{t+1}) \) (or, equivalently, \( R(s^{t+1}) \)) that make (15) compatible with (13). One particular solution is:

\[
R(s^{t+1}) = R^*(s^{t+1}) \frac{W_c(c(s^{t+1}), L(s^{t+1}))}{u_c(c(s^{t+1}), L(s^{t+1}))} \frac{u_c(c(s^t), L(s^t))}{W_c(c(s^t), L(s^t))}
\]

(16)

2.3.5 Capital Taxation

**Theorem 3.** Suppose that (i) there is no uncertainty, (ii) there is an interior steady state, and (iii) the upper bound to capital taxes is not binding in the steady state. Then in the steady state \( \tau^k = 0 \) is optimal.

**Proof.** Impose steady state and no uncertainty on (16) to obtain

\[
R(ss) = R^*(ss)
\]

\[
\Leftrightarrow 1 + (1 - \tau^k(ss))(F_K(K(ss), L(ss)) - \delta) = 1 + F_K(K(ss), L(ss), ss) - \delta
\]

which is achieved with \( \tau^k(ss) = 0 \).

This result is due to Chamley (1986), who in fact assumed condition (iii) in the proposition.

- uniform taxation of consumption at different dates in the steady state without conditions on preferences
- steady state capital supply perfectly elastic:

\[
1 = \beta R(ss)
\]

\[
= \beta \left[ 1 + (1 - \tau^k)(F_K(ss) - \delta) \right]
\]

from the Euler equation (15) in the steady state

- Special case:

\[
u(c, L) = \frac{c^{1-\sigma}}{1-\sigma} - v(L)
\]

Then

\[
W(c, L) = \frac{c^{1-\sigma}}{1-\sigma} - v(L) + \mu \left[ c^{-\sigma} - v'(L)L \right]
\]

\[
= \left( \frac{1}{1-\sigma} + \mu \right) c^{1-\sigma} - \left[ v(L) + \mu v'(L)L \right]
\]
\[
W_c = (1 + \mu (1 - \sigma)) e^{-\sigma} \\
= (1 + \mu (1 - \sigma)) u_c \\
\frac{W_c}{u_c} = 1 + \mu (1 - \sigma)
\]

Therefore (16) reduces to
\[
R(s^{t+1}) = R^*(s^{t+1}),
\]
so for these preferences \( \tau^k = 0 \) is optimal even with uncertainty and outside of steady state, for every period other than the first one.

- This is similar to the separability and homotheticity requirement for the uniform linear taxation result in the static or finite horizon models (linear tax version of Atkinson-Stiglitz)
- In general, in steady state, the optimal tax rate fluctuates around zero (Zhu, 1992)
- Atkeson/Chari/Kehoe (1999): result holds under much more general conditions than shown here:
  - durable consumption goods
  - endogenous growth
  - open economy (fixed interest rate)
  - OLG models (with some caveats, see our discussion of estate taxation later)
- Straub-Werning (2015): At optimum, upper bound \( \tau \) to capital taxes may bind forever. Can invalidate Chamley’s result.

### 2.4 Tax smoothing

Special case: iso-elastic labor supply

\[
v(L) = \alpha \frac{L^\gamma}{\gamma}
\]

Then
\[
W(c, L) = \left( \frac{1}{1-\sigma} + \mu \right) c^{1-\sigma} - \alpha \left( \frac{1}{\gamma} + \mu \right) L^\gamma
\]
Same functional form as \( u(\cdot) \) but with more disutility of labour (as long as \( \gamma > 1 - \sigma \)). Furthermore

\[
\frac{W_c}{u_c} = 1 + \mu (1 - \sigma) \\
\frac{W_L}{u_L} = 1 + \mu \gamma
\]

so (14) becomes

\[
\tau_{l^*}(s^t) = 1 - \frac{1 + \mu (1 - \sigma)}{1 + \mu \gamma} \quad \forall s^t
\]

- Tax perfectly smooth over time and states of the world
- Smooth out distortions over time
- Role of debt and state-contingent securities
- What happens if there is an expenditure shock with high \( g \)? State contingent debt. State contingent capital tax.
- This result was first shown by Lucas/Stokey (1983) for an economy without capital, but with state-contingent debt (equivalent to a state-contingent tax to the return on bonds). Outside the special case of iso-elastic labor supply, the optimal labor tax rate follows the same stochastic process as \( g \).
- In contrast to Barro (1979): labor tax should follow a random walk to smooth out expenditure shocks due to convex deadweight loss. Finance shock by debt and pay back through permanently increased tax.
- Chari, Christiano and Kehoe (1994) consider the present model with both capital and state contingent debt and find that the optimal labor income tax has tiny fluctuations (which vanish with iso-elastic labor supply). Labor tax smoothing can be achieved both by using state-contingent debt and the state-contingent capital tax.
- In particular, the optimal policy is to set the ex ante (expected) tax on capital income to zero, but vary the ex post rate. This leaves investment incentives undistorted, while the ex post rate can be used like a lump-sum tax to finance the spending shocks (see handout).
- Implications for estimating effect of taxes on investment and saving: If we find a high variability of ex post capital tax rates but ex ante rates are roughly constant, we expect saving to be roughly constant. In the data, we then observe varying tax
rates and constant saving and might falsely conclude that taxes have no effect on saving. Would need to measure ex ante (expected) tax rates.

- Aiyagari et al. (2002) study the case where there is no state-contingent debt and no capital. Ex post capital taxation is not possible. Find that then, labor tax rate (almost) follows random walk, as in Barro (1979)

- Farhi (2010) introduces capital in AMSS-economy, but government is slow and the tax rate on capital can only be changed with some delay after a spending shock. Then agents observe a high \( g \) shock and expect an increase in the capital tax rate. Hence all the effect of ex post taxation is gone. Similar results to AMSS.

- bottom line: possibility of issuing state-contingent debt or ex post capital taxation is key for tax smoothing results

### 2.5 Initial period taxation and time inconsistency

Return to government’s problem and assume \( \tau^k \) can be chosen freely:

\[
\max_{c(s^t), L(s^t), K(s^t), \tau^k_0} \sum_{s^t} \beta^t \Pr(s^t) u\left(c(s^t), L(s^t)\right)
\]

\[
s.t.: c(s^t) + g(s^t) + K(s^t) = F\left(K(s^{t-1}), L(s^t), s^t, t\right) + (1 - \delta)K(s^{t-1})
\]

\[
\sum_{t, s^t} \beta^t \Pr(s^t) \left[ u_c\left(c(s^t), L(s^t)\right) c(s^t) + u_L\left(c(s^t), L(s^t)\right) L(s^t)\right] = u_c\left(c_0, L_0\right) \left[B_0 + \left(1 + (1 - \tau^k_0)(r_0 - \delta)\right)K_0\right]
\]

FOC w.r.t. \( \tau^k_0 \):

\[-\mu u_c\left(c_0, L_0\right) (r_0 - \delta)K_0 = 0 \]

- problem is linear (and increasing) in \( \tau^k_0 \) (note \( \mu < 0 \))

- Tax initial capital until that is enough to pay for all government expenditure

- May require \( \tau^k_0 > 1 \)

- Replicable with consumption tax, so no consumption tax is w.l.o.g. only if \( \tau^k_0 \) is allowed

- Achieve first-best allocation

- Non-distortionary, so no time inconsistency problem arises
• Suppose we impose $\tau^k \leq \bar{\tau}$ and this is not enough to satisfy government budget constraint. Then $\tau^k_0 = \bar{\tau}$ is optimal, but optimal plan is not time consistent.

• Plan to tax initial capital highly, but future capital at zero.

• If cannot commit, reoptimize each period, with high taxes and lower welfare

• Werning (2007) introduces heterogeneity, allows for lump-sum tax. Shows that Chamley-Judd and tax smoothing results go through. Initial capital taxation and time inconsistency are more subtle: depends on asset distribution and redistributive motives.