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Disclaimer

These are lecture notes from a course (Math 232a: Algebraic Geometry I) offered at Harvard University in the fall of 2011 by Xinwen Zhu. I have (lightly) edited them, but there are inevitably still typos and mistakes stemming from my own misunderstanding, for which I take full responsibility.
Chapter 1

Algebraic Varieties

1.1 Affine algebraic varieties

Let $k$ be an algebraically closed field.

*Definition 1.1.1.* We define affine $n$-space to be the (for now) set

$$\mathbb{A}^n_k = \{(a_1, \ldots, a_n) \mid a_i \in k\}.$$  

*Definition 1.1.2.* An algebraic set $X \subset \mathbb{A}^n_k$ is the set of zeros of a collection of polynomials $f_1, \ldots, f_m \in k[x_1, \ldots, x_n]$, i.e.

$$X = \{(a_1, \ldots, a_n) \in \mathbb{A}^n_k \mid f_1(a_1, \ldots, a_n) = \cdots = f_m(a_1, \ldots, a_n) = 0\}.$$  

*Remark 1.1.3.* Note that this depends only on the ideal $I = (f_1, \ldots, f_m) \subset k[x_1, \ldots, x_n]$.

*Lemma 1.1.4.* In the notation used above, the following relations hold.

1. $I_1 \subset I_2 \iff V(I_1) \supset V(I_2)$.
2. $V(I_1) \cup V(I_2) = V(I_1 \cap I_2) = V(I_1 I_2)$.
3. $\bigcap_\alpha V(I_\alpha) = V(\sum_\alpha I_\alpha)$.
4. $V(\{0\}) = \mathbb{A}^n_k$, $V(k[x_1, \ldots, x_n]) = \emptyset$.

*Definition 1.1.5.* The Zariski topology on $\mathbb{A}^n_k$ is the topology where the closed subsets are the algebraic sets.

*Example 1.1.6.* Consider the Zariski topology on $\mathbb{A}^1_k$. The algebraic sets are $\emptyset$, the finite subsets, and all of $\mathbb{A}^1_k$ (i.e. the topology is the cofinite topology). This follows from the observation that $k[x]$ is a principal ideal domain, and $f$ splits into linear factors.

*Example 1.1.7.* Consider the Zariski topology on $\mathbb{A}^2_k$. The algebraic sets include $\emptyset, \mathbb{A}^2_k$, and $V(f)$, but there are also non-principal ideals. However, it can be showed that every algebraic set is a finite union of these.
Remark 1.1.8. The Zariski topology on $\mathbb{A}^2$ is not equal to the product topology on $\mathbb{A}^1 \times \mathbb{A}^1$!

Theorem 1.1.9 (Hilbert’s Nullstellensatz). Let $X = V(I)$. Then
$$\{f \in k[x_1, \ldots, x_n] \mid f(p) = 0 \text{ } \forall p \in X\} = \sqrt{I}.$$ 

Corollary 1.1.10. There is an order-reversing bijection between radical ideals and algebraic sets: $I \mapsto V(I)$ and $X \mapsto I(X)$.

Example 1.1.11. Consider $V(xy) \subset \mathbb{A}^2$, which is the union of the two coordinate axes: $V(xy) = V(x) \cup V(y)$. This algebraic set is somehow built out of other pieces. To make this notion precise, we make a definition.

Definition 1.1.12. Let $X \neq \emptyset$ be a topological space. We say that $X$ is irreducible if $X$ cannot be written as the union of two proper closed subsets.

Note that this notion is trivial for the classical topology, since nearly every set can be written as the union of two proper closed subsets. However, it’s a good notation for algebraic geometry, since the Zariski topology is so coarse. For example, $\mathbb{A}^1_k$ is irreducible since the proper closed subsets are finite.

Definition 1.1.13. The irreducible components of an algebraic set are its maximal irreducible closed subsets.

Proposition 1.1.14. There is a bijection between irreducible algebraic sets of $\mathbb{A}^n_k$ (with the induced topology) and prime ideals of $k[x_1, \ldots, x_n]$. In addition, every algebraic set can be written as the union of its irreducible components.

Proof. We first show that if $V(I) \neq \emptyset$ is irreducible, then $I$ is prime. We may assume that $I$ is radical. Suppose $fg \in I$ and $f \notin I$. Then $V(I) \subset V(fg) = V(f) \cup V(g)$, so $V(I) \subset V(f)$ or $V(I) \subset V(g)$, showing that $f \in I$ or $g \in I$.

Conversely, suppose $I$ is prime; we want to show that $V(I)$ is irreducible. Suppose $V(I) = V(I_1) \cup V(I_2) = V(I_1I_2)$. Therefore, $I = I_1I_2$, so $I = I_1$ or $I = I_2$.

Remark 1.1.15. It is a fact that any radical ideal $I$ can be written as $I = p_1 \cap \ldots \cap p_r$, with $p_r$ minimal prime ideals containing $I$, so we may write
$$V(I) = V(p_1) \cup \ldots \cup V(p_r).$$

So the $V(p_i)$ are maximal irreducible components of $V(I)$.

Definition 1.1.16 (Tentative). An affine (algebraic) variety is an irreducible algebraic set in $\mathbb{A}^n$ with its induced topology. A quasi-affine variety is an open subset of an affine variety.

Example 1.1.17. We give some examples of affine algebraic sets.
(i) \( \mathbb{A}^n \).

(ii) Let \( l_1, \ldots, l_m \) be independent linear forms of \( x_1, \ldots, x_m \); and let \( a_1, \ldots, a_m \in k \). Then \( V(l_1 - a_1, \ldots, l_m - a_m) \subset \mathbb{A}^n \). This is called a linear variety of dimension \( n - m \).

(iii) The variety \( V(x_1 - a_1, \ldots, x_n - a_n) \) “corresponds to” the point \( (a_1, \ldots, a_n) \in \mathbb{A}_k^n \).

(iv) Let \( f \in k[x_1, \ldots, x_n] \) be irreducible. Then \( (f) \) is prime, and \( V(f) \) is an affine variety (usually called a hypersurface in \( \mathbb{A}^n \)).

(v) Given \( f_1(t), \ldots, f_n(t) \in k[t] \). Then \( V(x_1 - f_1, \ldots, x_n - f_n) \subset \mathbb{A}^{n+1}, x_i - f_i \in k[t, x_1, \ldots, x_n] \). For example, \( V(x - t^2, y - t^3) \subset \mathbb{A}^3 \).

1.2 Regular functions

Definition 1.2.1. Let \( X \subset \mathbb{A}^n \) be a quasi-affine variety. A function \( f : X \to k \) is called regular at \( p \in X \) if there exists an open neighborhood \( U \subset X \) containing \( p \) and polynomials \( f_1, f_2 \in k[x_1, \ldots, x_n] \) such that \( f_2(q) \neq 0 \) for all \( q \in U \), and \( f = \frac{f_1}{f_2} \) on \( U \). We say \( f \) is regular if it is regular at all \( p \in X \).

Observe that the set of regular functions on \( X \) form a ring, denoted by \( \mathcal{O}(X) \).

Lemma 1.2.2. Let \( f : X \to k \) be a regular function; regard it as a map \( f : X \to \mathbb{A}^1_k \). Then \( f \) is continuous.

Proof. We claim that it is enough to show that \( f^{-1}(0) \) is closed, since the closed subsets of \( \mathbb{A}^1_k \) are just finite unions of points or the empty set. It suffices to show that for every \( p \in X \), there is an open subset \( U \subset X \) containing \( p \) such that \( f^{-1}(0) \cap U \) is closed in \( U \).

We can choose \( U \) such that \( f = f_1/f_2 \) on \( U \), with \( f_1, f_2 \in k[x_1, \ldots, x_n] \). Since \( f_2 \) does not vanish on \( U \), \( f^{-1}(0) \cap U = f_1^{-1}(0) \cap U = V(f_1) \cap U \), which is indeed closed in \( U \) by definition. \( \square \)

Corollary 1.2.3. Let \( f \) be a regular function on a quasi-affine variety \( X \). If \( f = 0 \) on some nonempty open subset \( U \subset X \), then \( f = 0 \) on \( X \).

Proof. Let \( Z = \{ x \in X \mid f(p) = 0 \} \). This is a closed subset since \( f \) is continuous. So \( X = Z \cup X \setminus U \). Since \( X \) is irreducible, it is contained in \( X \setminus U \cup Z \), so \( X = Z \). \( \square \)

Corollary 1.2.4. If \( X \) is a quasi-affine variety, then \( \mathcal{O}(X) \) is an integral domain.

Proof. If \( fg = 0 \), then \( V(f) \cup V(g) = X \), so \( V(f) = X \) or \( V(g) = X \). \( \square \)
Definition 1.2.5. The field of rational functions on $X$ is the fraction field of $\mathcal{O}(X)$, denoted by $K = k(X)$.

The local ring at $p \in X$ is $\mathcal{O}_{X,p} = \lim_{\leftarrow p \in U} \mathcal{O}(U)$.

Let $f \in k(X)$ be a rational function, so we may write $f = \frac{f_1}{f_2}$, with $f_1, f_2 \in \mathcal{O}(X)$. Therefore, $f$ defines a regular function on $U = X - \{f_2 = 0\}$. Conversely, if $f \in \mathcal{O}(U)$, we claim that $f$ can be regarded as a rational function on $X$. To see this, pick $p \in U$, so there exists an open neighborhood $V$ containing $p$ such that $f = \frac{f_1}{f_2}$ on $V$, where $f_1$ and $f_2$ are polynomials and can be regarded as in $\mathcal{O}(X)$.

This construction is well-defined: for $p' \in V' \subset U$, suppose $f = \frac{f_1'}{f_2'}$ on $V'$. Then

$$\frac{f_1}{f_2} = \frac{f_1'}{f_2'}$$ on $V \cap V'$

so

$$f_1 f_2' = f_1' f_2$$ on $V \cap V'$.

Therefore, $f_1 f_2' = f_1' f_2$ on $X$, so

$$\frac{f_1}{f_2} = \frac{f_1'}{f_2'}$$ in $K$.

In conclusion, for each open set $U \subset X$ and each $p \in U$, we have inclusions

$$\mathcal{O}(X) \subset \mathcal{O}(U) \subset \mathcal{O}_{X,p} \subset k(X).$$

Suppose $X$ is given as $V(p)$. We want to explicitly determine $\mathcal{O}(X)$ or $K = k(X)$.

Theorem 1.2.6. Let $X = V(p) \subset \mathbb{A}^n$ be an affine variety. Then

(i) There exists a natural isomorphism

$$A(X) := k[X_1, \ldots, X_n]/p \simeq \mathcal{O}(X).$$

(ii) There is a one-one correspondence

\[\{ \text{points on } X \} \longleftrightarrow \{ \text{maximal ideals of } A(X) \}\]

\[p \longleftrightarrow m_p\]

(iii) $A(X)_{m_p} \simeq \mathcal{O}_{X,p}$.

Proof. Part (ii) follows from recalling that $Y \subset X$ as nonempty irreducible algebraic sets is equivalent to $I(X) \subset I(Y)$. Therefore, the points must correspond to maximal ideals.
There is a natural map $A(X) \to \mathcal{O}(X)$, which is injective by the Nullstellensatz. On the other hand, there is an inclusion $\mathcal{O}(X) \to \text{Frac}(A(X)) = k(X)$. So we have the commutative diagram

$$
\begin{array}{ccc}
A(X) & \rightarrow & \text{Frac}(A(X)) \\
\downarrow & & \downarrow \\
A(X)_{m} & \rightarrow & k
\end{array}
$$

Recall that $A(X) = \bigcap_{m \text{ maximal}} A(X)_{m} \hookrightarrow \bigcap_{p \in X} \mathcal{O}_{X,p} = \mathcal{O}(X)$.

But the map $A(X)_{m} \rightarrow \mathcal{O}_{X,p}$ is also a surjection since $f \in \mathcal{O}_{X,p}$ can be written in the form $f = \frac{f_1}{f_2}$ on some open set $U$ containing $p$, with $f_2 \notin p$. Therefore, $A(X)$ surjects onto $\mathcal{O}(X)$ as well. \qed

### 1.3 Morphisms of algebraic varieties

**Definition 1.3.1.** Let $X, Y$ be two quasi-affine varieties. A **morphism** $\varphi : X \rightarrow Y$ is a continuous map such that for any open subset $V \subset Y$, if $f : V \rightarrow k$ is regular then $f \circ \varphi : \varphi^{-1}(V) \rightarrow k$ is regular.

A morphism $\varphi : X \rightarrow Y$ is an **isomorphism** if there is a morphism $\psi : Y \rightarrow X$ such that $\psi \circ \varphi = \varphi \circ \psi = \text{id}$.

**Definition 1.3.2.** A quasi-affine variety is said to be **affine** if it is isomorphic to an affine variety.

**Example 1.3.3.** $\mathbb{A}^1 - \{0\}$ is a quasi-affine variety, but is also an affine variety because it is isomorphic to $V(xy - 1) \subset \mathbb{A}^2$.

**Lemma 1.3.4.** Let $\varphi : X \rightarrow Y \subset \mathbb{A}^n$ be a map (of sets) of two quasi-affine varieties. Then $\varphi$ is a morphism if and only if $x_i \circ \varphi$ is regular.

Here $x_i$ represents the function given by projection to the $i$th coordinate.

**Proof.** If $\varphi$ is a morphism, then $x_i \circ \varphi$ is regular since $x_i$ is regular. Conversely, suppose $x_i \circ \varphi$ is regular for each $x_i$. For each $f \in k[x_1, \ldots, x_n]$, we have

$$f \circ \varphi = f(x_1 \circ \varphi, \ldots, x_n \circ \varphi) \in \mathcal{O}(X).$$

It follows that $\varphi$ is continuous, since for any closed subset $Z = \bigcap V(f_{\alpha})$, we have

$$\varphi^{-1}(Z) = \bigcap_{\alpha} V(f_{\alpha}(x_i \circ \varphi, \ldots, x_n \circ \varphi)).$$
If \( V \subset Y \), \( p \in \varphi^{-1}(V) \), and \( f \in \mathcal{O}(V) \), we have \( f = \frac{f_1}{f_2} \) in some neighborhood of \( \varphi(p) \) contained in \( V \), where \( f_1, f_2 \in k[x_1, \ldots, x_n] \), and \( f_2(p) \neq 0 \). Therefore, in some neighborhood of \( p \),

\[
 f \circ \varphi = \frac{f_1 \circ \varphi}{f_2 \circ \varphi}
\]
is regular at \( p \).

Example 1.3.5. Consider the map (of sets) \( \mathbb{G}_m = \mathbb{A}^1 - \{0\} \to V(xy - 1) \subset \mathbb{A}^2 \) sending \( t \mapsto (t, t^{-1}) \). Since each coordinate is a regular function, this is a morphism. The inverse morphism is given by \( (x, y) \mapsto x \).

Proposition 1.3.6. Let \( X \) be quasi-affine and \( Y \) be affine. Then there is a natural isomorphism

\[
 \text{Hom}(X, Y) \cong \text{Hom}_{k\text{-alg}}(\mathcal{O}(Y), \mathcal{O}(X)).
\]

Proof. If \( \varphi : X \to Y \) is a morphism, then we obtain \( \varphi^* : \mathcal{O}(Y) \to \mathcal{O}(X) \) given by \( f \mapsto f \circ \varphi \).

Now suppose that we are given a \( \psi \in \text{Hom}_{k\text{-alg}}(\mathcal{O}(Y), \mathcal{O}(X)) \). Let \( Y \subset \mathbb{A}^n \), and define a map \( \varphi : X \to \mathbb{A}^n \) by \( p \mapsto (\psi(x_1)(p), \ldots, \psi(x_n)(p)) \in Y \). Note that \( \varphi(p) \in Y \) since \( \varphi \) factors through \( I(Y) \) by definition.

Corollary 1.3.7. There is an equivalence of categories between affine varieties and finitely generated integral \( k \)-algebras.

Example 1.3.8. The quasi-affine variety \( \mathbb{A}^2 - \{(0,0)\} \) has ring of regular functions \( k[x, y] \), so the coordinate ring is not sufficient to describe quasi-affine varieties.
Chapter 2

Dimension Theory

2.1 Tangent Space

Definition 2.1.1. Let $A \to B$ be a homomorphism of commutative algebras, and $M$ a $B$-module. We define the *derivations* of $B$ into $M$ to be

\[ \text{Der}_A(B, M) = \{ D \mid D : B \to M \text{ map of abelian groups} \} \]

such that:

(i) $D(b_1 b_2) = b_1 D(b_2) + b_2 D(b_1)$,

(ii) $D(a) = 0$ for all $a \in A$.

Observe that $\text{Der}_A(B, M)$ is a $B$-module.

Example 2.1.2. Let $A = k$, $B = k[x_1, \ldots, x_n]$, and $M = k$. Then

\[ \text{Der}_k(B, k) = \{ D = \sum \lambda_i \frac{\partial}{\partial x_i} \mid \lambda_1, \ldots, \lambda_n \in k \} \]

where $\lambda_i$ is the image of $x_i$. By definition,

\[ \frac{\partial}{\partial x_i}(x_j) = \delta_{ij}. \]

Now consider $A = k$, $B = \mathcal{O}_{X, p}$. Then we have a map $B \to \mathcal{O}_{X, p}/m_p \simeq k$.

Definition 2.1.3. The *Zariski tangent space* of $X$ at $p$ is defined to be

\[ T_p X = \text{Der}_k(\mathcal{O}_{X, p}, k). \]

A priori $T_p X$ is a $\mathcal{O}_{X, p}$-module, and $m$ acts trivially, so it is an $\mathcal{O}_{X, p}/m \simeq k$ vector space.

Lemma 2.1.4. $T_p X \simeq (m/m^2)^*$. 

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Proof. Suppose we have a derivation \( D : \mathcal{O}_{X,p} \to k \). Then \( D \) factors through \( m^2 \). Since \( \mathcal{O}_{X,p} \cong k \oplus m \), and \( D \) is zero on \( k \), \( D \) depends only on its value on \( m \).

Exercise 2.1.5. Complete the proof.

Exercise 2.1.6. Show that \( m/m^2 \) is a finite-dimensional \( k \)-vector space.

**Theorem 2.1.7.** Let \( X \) be a quasi-affine variety. Let \( K = k(x) \) be its function field.

(i) For all \( p \in X \), we have

\[
\dim_K \text{Der}_k(K, K) \leq \dim_k T_p X.
\]

(ii) There exists a nonempty open set \( U \subset X \) such that

\[
\dim_k T_p X = \dim_K \text{Der}_k(K, K) \text{ for all } p \in U.
\]

**Definition 2.1.8.** A point \( p \in X \) is called smooth (or non-singular) if \( \dim_k T_p X = \dim X \). Otherwise, \( p \) is called a singular point. \( X \) is called smooth (or non-singular) if all points \( p \in X \) are smooth.

**Proof.** After replacing \( X \) with an open subset, we can assume \( X \) is affine. We have

\[
\text{Der}_k(K, K) \cong \text{Der}_k(\mathcal{O}(X), K).
\]

If \( X = V(p) \subset \mathbb{A}^n \), then \( \mathcal{O}(X) = k[x_1, \ldots, x_n]/p \), then

\[
\text{Der}_k(\mathcal{O}(X), K) = \{ D = \sum \lambda_i \frac{\partial}{\partial x_i} \lambda_i \in K \mid D(f) = 0 \text{ for all } f \in p \}.
\]

Choosing a set of generators \( (f_1, \ldots, f_m) = p \), we identify the above with

\[
\text{Der}_k(\mathcal{O}(X), K) = \{ D = \sum \lambda_i \frac{\partial}{\partial x_i} \lambda_i \in K \mid \sum \lambda_i \frac{\partial f_i}{\partial x_i} = 0 \}.
\]

This is the kernel of the map \( J : K^n \to K^n \) given by the Jacobian \( J \). So

\[
\dim_K \text{Der}_k(K, K) = n - \text{rank}_K J.
\]

By the same reasoning,

\[
\dim_k T_p X = n - \text{rank}_K J(p).
\]

So we need to show that (i) \( \text{rank}_K J(p) \leq \text{rank}_K J \) and (ii) there exists a non-empty subset \( U \) such that \( \text{rank}_K J(p) = \text{rank}_K J \) on \( U \).
The first point (i) is clear, as the rank can only go down under specialization. For the second point, assume that $\text{rank}_K J = r$. Then there exist $n \times n$ matrices $A, B$ with entries in $K$ such that

$$AJB = \begin{pmatrix} I_r \\ * \end{pmatrix}.$$ 

Let $A = A_0/\alpha$, $B = B_0/\beta$, for $\alpha, \beta \in \mathcal{O}(X)$ such that $A_0, B_0$ have entries in $\mathcal{O}(X)$. Let $f = \alpha \beta \det A_0 \det B_0 \in \mathcal{O}(X)$, so for all $p \in U = X - V(f)$ we have $\text{rank}_k J(p) = r$.

\[\square\]

**Lemma 2.1.9.** The function $p \mapsto \dim_k T_p X$ is upper semi-continuous, i.e.

$$X_l = \{ p \in X \mid \dim T_p X \geq l \}$$

is closed.

**Proof.** We know

$$\dim T_p X = n - \text{rank}_k \left( \frac{\partial f_i}{\partial x_j}(p) \right).$$

So $X_l$ is $V(p + \text{ideal generated by } (n - l + 1) \times (n - l + 1) \text{ minors of } J)$.  

\[\square\]

**Example 2.1.10.** Suppose $\text{char } k \neq 2$. Consider the curves defined by

- $y^2 = x^3$,
- $y^2 = x^3 + x^2$, and
- $y^2 = x^3 + x$.

These are plane curves, so intuitively they should have dimension 1.

- For the first curve $f = y^2 - x^3$, we have $f_x = -3x^2$ and $f_y = 2y$, so
  $$J = (-3x^2, 2y).$$

Then $J(0, 0) = (0, 0)$; the Zariski tangent space at the origin is two-dimensional. Thus the curve is singular at the origin.

- For the second curve $f = y^2 - x^3 - x^2$, $f_x = -3x^2 - 2x$ and $f_y = 2y$, so
  $$J = (-x^2 - 2x, 2y).$$

Then $J(0, 0) = (0, 0)$; the Zariski tangent space at the origin is two-dimensional, so this curve is also singular at the origin.

- For the third curve $f = y^2 - x^3 - x$, we have $f_x = 3x^2 - 1$ and $f_y = 2y$, so
  $$J = (3x^2 - 1, 2y).$$

In this case the rank of $J$ is 1 at every point, and the curve is smooth.
2.2 Dimension theory of rings

Definition 2.2.1. Let $K/k$ be a (finitely generated) field extension. We say that $K$ is separably generated over $k$ if there exists $L$ satisfying $k \subset L \subset K$, $L/k$ is purely transcendental, and $K/L$ is a finite, separable extension.

Theorem 2.2.2. If $K/k$ is finitely separably generated, then
\[ \dim_K \text{Der}_k(K, K) = \text{tr.deg}_k K. \]

Theorem 2.2.3. Let $k$ be a perfect field (e.g. algebraically closed). Then every finitely generated $K/k$ is separably generated.

Proposition 2.2.4. Let $X$ be an affine variety in $\mathbb{A}^n$. Then $\dim X = n - 1$ if and only if $X = V(f)$ for some irreducible $f \in k[x_1, \ldots, x_n]$.

Remark 2.2.5. $\dim X = n - 2$, then it is not necessarily true that $I(X)$ is generated by two elements. We’ll encounter an example of this in the homework.

Proof. Suppose $X = V(f)$. We want to show that $\dim X = k - 1$. By definition,
\[ \dim X = n - \text{rank} \left( \frac{\partial f}{\partial x_i} \right) \geq n - 1. \]

So we want to show that
\[ \text{rank} \left( \frac{\partial f}{\partial x_i} \right) \neq 0 \text{ in } \mathcal{O}_X \]

Well, $\frac{\partial f}{\partial x_i}$ is 0 only if it lies in $(f)$, but $f$ is irreducible so this implies $\frac{\partial f}{\partial x_i} = 0$ for all $i$. This is only possible when $k$ has characteristic $p > 0$ and
\[ f = g(x_1^p, \ldots, x_n^p) = g(x_1, \ldots, x_n)^p. \]

But $f$ is irreducible by assumption, so this is impossible.

Now assume that $\dim X = n - 1$. Let $g \in \mathfrak{p} = I(X)$. We can assume that $g$ is irreducible, so $\dim V(g) = n - 1$. Therefore, the proposition follows from the proceeding lemma.

Lemma 2.2.6. Let $A$ be an integral domain over $k$ and $\mathfrak{p} \subset A$ be a prime ideal. Then
\[ \text{tr.deg}_k A/\mathfrak{p} \leq \text{tr.deg}_k A \]
with equality holding if and only if $\mathfrak{p} = 0$ or both sides are infinity.

Remark 2.2.7. Note that this lemma also implies that the dimension of a hypersurface is $n - 1$. 

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Proof. Assume to the contrary that $\text{tr. deg}_k A/p = \text{tr. deg}_k A = n$. Then there exist $x_1, \ldots, x_n \in A$ such that $\bar{x}_1, \ldots, \bar{x}_n$ are algebraically independent over $k$.

Choose some non-zero element $y \in p$. There exists $P \in k[Y, X_1, \ldots, X_n]$ such that $P(y, x_1, \ldots, x_n) = 0$. We may assume that $P$ is irreducible, and not a power of $Y$. Therefore,

$$P(0, \bar{x}_1, \ldots, \bar{x}_n) = 0$$

is an algebraic relation in $A/p$. □

2.3 Local structure of smooth points

Theorem 2.3.1. Let $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$ have no constant terms and independent linear terms. Let $p = (f_1, \ldots, f_r)\mathcal{O}_{k^n,0} \cap k[x_1, \ldots, x_n]$. Then:

(i) $p$ is a prime ideal,

(ii) $X = V(p)$ has dimension $n - r$, and $0 \in X$ is smooth.

(iii) $V(f_1, \ldots, f_r) = X \cup Y$, where $Y$ is some algebraic set such that $0 \notin Y$.

The theorem tells us that locally around a smooth point, an $n - r$-dimensional affine variety in affine $n$-space is cut out by $r$ equations.

Corollary 2.3.2. Let $X$ be an affine variety of dimension $n - r$ and suppose $0 \in X$ is smooth. Then there exists $f_1, \ldots, f_r \in I(X)$ such that

$$I(X) = (f_1, \ldots, f_r)\mathcal{O}_{k^n,0} \cap k[x_1, \ldots, x_n].$$

Proof. We just need to find $r$ polynomials in $I(X)$ with independent linear terms, at which point we may apply the Theorem 2.3.1. Let $m = (x_1, \ldots, x_n) \subset k[x_1, \ldots, x_n]$ be the maximal ideal corresponding to the origin, and $m_0$ the maximal ideal of $\mathcal{O}(X)$ corresponding to 0. These fit into a short exact sequence

$$0 \rightarrow p \rightarrow m \rightarrow m_0 \rightarrow 0.$$ 

Looking “to second order” gives an exact sequence

$$0 \rightarrow (p + m^2)/m^2 \rightarrow m/m^2 \rightarrow m_0/m_0^2 \rightarrow 0.$$ 

But

$$(p + m^2)/m^2 \simeq p/(p \cap m^2),$$

so we choose a basis ($f_0, \ldots, f_r$) for $p/(p \cap m^2)$ and lift it to $(p + m^2)/m^2$, which then has the property that the linear terms are independent. We then set

$$q = (f_1, \ldots, f_r)\mathcal{O}_{k^n,0} \cap k[x_1, \ldots, x_n]$$

$$= \{ \sum g_i f_i \mid k_i(0) \neq 0, \sum g_i f_i \subset k[x_1, \ldots, x_m] \} \subset I(X).$$

So $Y = V(q) \supset X$, while $\dim Y = \dim X = n - r$, implying that $X = Y$. □
Recall that if \((A, m, k)\) is a local ring, then we define the completion
\[
\hat{A} = \lim_{\leftarrow} A / m^n.
\]

**Example 2.3.3.** We have seen that
\[
\mathcal{O}_{A^{n,0}} = k[x_1, \ldots, x_n](x_1, \ldots, x_n).
\]
Then
\[
\hat{\mathcal{O}}_{A^{n,0}} = k[[x_1, \ldots, x_n]].
\]

**Proof of Theorem 2.3.1.** Let
\[
p = (f_1, \ldots, f_r)\mathcal{O}_{A^{n,0}} \cap k[x_1, \ldots, x_n],
p' = (f_1, \ldots, f_r)\mathcal{O}_{A^{n,0}},
p'' = (f_1, \ldots, f_r)k[[x_1, \ldots, x_n]].
\]
It suffices to show:
(i) \(p''\) is prime in \(k[[x_1, \ldots, x_n]]\).
(ii) \(p'' \cap \mathcal{O}_{A^{n,0}} = p'\).

By definition,
\[
p \subset p'' \cap \mathcal{O}_{A^{n,0}} = \left\{ \sum g_i f_i \mid g_i \in k[[x_1, \ldots, x_n]], \sum g_i f_i \in k[x_1, \ldots, x_n](x_1, \ldots, x_n) \right\}
\subset \bigcap_{n=0}^{\infty} (p' + m^N).
\]

We have an exact sequence
\[
0 \to m^N \to \mathcal{O}_{A^{n,0}} \to \mathcal{O}_{A^{n,0}} / m^N \to 0,
\]
where \(m = (x_1, \ldots, x_n)\) is the maximal ideal of \(\mathcal{O}_{A^{n,0}}\). Thus
\[
\mathcal{O}_{A^{n,0}} / m^N \simeq k[x_1, \ldots, x_n]/(x_1, \ldots, x_n)^N
\]
This gives a splitting of \(k\)-vector spaces
\[
\mathcal{O}_{A^{n,0}} \simeq m^N \oplus \{\text{polynomials of deg < } N\}.
\]
Now, any element \(f \in p'' \cap \mathcal{O}_{A^{n,0}}\) may be written as \(f = \sum h_i f_i\) with \(h_i \in k[[x_1, \ldots, x_n]]\). We may write \(h_i = h_i' + (\text{deg } \geq N\) terms). Then
\[
f - \sum h_i' f_i = (f_i - \sum h_i' f_i) + m^N \mathcal{O}_{A^{n,0}} \in m^N \mathcal{O}_{A^{n,0}}.
\]
Therefore,
\[
p'' \cap \mathcal{O}_{A^{n,0}} \subset \bigcap_{N \geq 0} (p' + m^N) = p'
\]
by Krull’s intersection theorem, stated below.
Theorem 2.3.4 (Krull intersection theorem). If \((A, m, k)\) is Noetherian and local, then for any ideal \(I\),
\[
I = \bigcap_{N=0}^{\infty} (I + m^N).
\]

We will show below that \(k[[x_1, \ldots, x_n]]/(f_1, \ldots, f_r) \simeq k[[x_{r+1}, \ldots, x_n]]\). This will imply that \(p''\) is prime, and
\[
\mathcal{O}_{\mathbb{A}^n, 0}/p' \to \hat{\mathcal{O}}_{\mathbb{A}^n, 0}/p''
\]
is injective. In particular, \(x_{r+1}, \ldots, x_n\) are algebraically independent in \(\mathcal{O}_{\mathbb{A}^n, 0}\) since their images are algebraically independent. Therefore, \(\dim V(p) \geq n - r\).

We have \(T_p V(p) = (m_p/m_p^2)^\ast\). We have a short exact sequence

\[
0 \to p/(p + (x_1, \ldots, x_n)) \to m/m^2 \to m_p/m_p^2 \to 0.
\]

So \(T_p V(p)\) is cut out by the linear terms of \(f_1, \ldots, f_r\) in \(T_p \mathbb{A}^n\). Since the dimension of the tangent space is always at least the dimension of the variety, we conclude that \(\dim V(p) = n - r\) and \(p\) is smooth.

Finally, we wish to show that \(V(f_1, \ldots, f_r) = V(p) \cup Y\) where 0 \(\not\in Y\). Let \(g_1, \ldots, g_s\) be a set of generators of \(p\). Then

\[
g_i \in (f_1, \ldots, f_r) \mathcal{O}_{\mathbb{A}^n, 0} \cap k[x_1, \ldots, x_n]
\]
so there exists \(h_i \notin (x_1, \ldots, x_n)\), i.e. not vanishing at 0, such that \(h_i g_i \in (f_1, \ldots, f_r)\).

Let \(h = h_1 h_2 \ldots h_s\), so \(hp \subset (f_1, \ldots, f_r)\). Then

\[
V(f_1, \ldots, f_r) \subset V(hp) = V(h) \cup V(p).
\]

So \(V(f_1, \ldots, f_r) = V(f_1, \ldots, f_r, h) \cup V(p)\). We may take the first term in the union to be \(Y\), since \(h(0) \neq 0\) implies 0 \(\notin Y\).

It only remains to complete the proof of the claim above, which is accomplished by the following two results.

\[\square\]

Proposition 2.3.5 (Formal implicit function theorem). Let \(f = \sum a_i x_i + \text{higher terms} \in k[[x_1, \ldots, x_n]]\) such that \(a_i \neq 0\). Then every \(g \in k[[x_1, \ldots, x_n]]\) can be uniquely written as

\[
g = uf + h(x_2, \ldots, x_n).
\]

Proof. We aren’t going to prove this, but it’s actually pretty trivial: just attempt to solve for the power series term-by-term. \[\square\]

Corollary 2.3.6. \(k[[x_1, \ldots, x_n]]/(f) \simeq k[[x_2, \ldots, x_n]]\). By induction, we conclude the following.
Corollary 2.3.7. Let \( f_1, \ldots, f_r \) be \( r \) power series with \( f_i = \sum a_{ij}x_j + \) higher terms. If \( \det(a_{ij})_{1 \leq i,j \leq r} \neq 0 \) then
\[
k[[x_1, \ldots, x_n]]/(f_1, \ldots, f_r) \simeq k[[x_{r+1}, \ldots, x_n]].
\]
This wraps up our proof of the theorem. Let’s recall the corollary from last time.

Corollary 2.3.8. Let \( p = (0, \ldots, 0) \in X = V(p) \) be a smooth point of \( X \), and \( \dim X = n - r \). Choose \( f_1, \ldots, f_r \in p \) with independent linear terms (i.e. cutting out the tangent space at \( 0 \)). [Note that this is possible because we have the short exact sequence
\[
0 \to p/(p + (x_1, \ldots, x_n)) \to m/m^2 \to m_p/m_p^2 \to 0.
\]
The third term has dimension \( n - r \), the second \( n \), so the first has dimension \( r \). This amounts to choosing \( r \) polynomials that cut out the tangent space at \( p \).] Then \( p = (f_1, \ldots, f_r) \mathcal{O}_{\mathbb{A}^n, 0} \cap k[[x_1, \ldots, x_n]] \).

In other words, in the neighborhood of smooth point on an affine variety of dimension \( n - r \), the variety is cut out by \( r \) equations. If we trace through our proof of the theorem, we obtain the following corollary.

Corollary 2.3.9. Let \( p \in X \) be a smooth point on a quasi-affine variety of dimension \( r \). Then
\[
\hat{\mathcal{O}}_{\mathbb{A}^n, 0} \simeq k[[x_1, \ldots, x_r]].
\]
This is analogous to differential geometry, where a manifold is locally of dimension \( r \). Note, however, that we must work with completions of the local rings.
Chapter 3

Projective varieties

3.1 Abstract algebraic varieties

So far, we’ve established the notion of affine varieties. Just like Euclidean space is used to build up general manifolds, affine varieties are used to build up abstract varieties. We now discuss these more general notions. Fix an algebraically closed field \( k \).

**Definition 3.1.1.** A (pre)variety over \( k \) is a connected irreducible topological space together with a covering \( \mathcal{U} = \{ U_\alpha \} \), for each alpha a homeomorphism \( \varphi_\alpha : U_\alpha \to X_\alpha \) with \( X_\alpha \) a quasi-affine variety, and for each \( \alpha, \beta \) a morphism of quasi-affine varieties

\[
\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta).
\]

In addition, we require \( \mathcal{U} \) to be maximal, i.e. if \( V \subset X \) is open, \( \psi : V \xrightarrow{\sim} Y \) is a map to a quasi-affine variety such that \( \varphi_\alpha \circ \psi^{-1} : \psi(U_\alpha \cap V) \to \varphi_\alpha(U_\alpha \cap V) \) is a morphism for all \( \alpha \) implies \( V \in \mathcal{U} \). This is just a technical point to make this choice of atlas canonical.

**Definition 3.1.2.** A function \( f : X \to k \) is called regular if for all \( \alpha \) the composition \( f \circ \varphi_\alpha^{-1} : X_\alpha \to k \) is regular.

**Definition 3.1.3.** We denote by \( \mathcal{O}(X) \) the ring of regular functions on \( X \).

\[
\mathcal{O}_{X,p} = \varprojlim \mathcal{O}(U),
\]

where the direct limit ranges over open sets \( U \) containing \( p \). Finally, we define \( k(X) \) to be \( k(X_\alpha) \) for any \( \alpha \). In particular, \( k(X) \) is not necessarily the fraction field of \( \mathcal{O}(X) \).

We also define dimension, tangent space, and smoothness, etc. affine-locally, since these should be local notions.

**Definition 3.1.4.** Let \( X \) be a pre-variety. An irreducible closed subset \( Y \subset X \) with the canonical variety structure is called a closed subvariety.
In this case, we need to check that the restrictions of the transition maps are still morphisms, but this is clear by using the criterion that the maps are morphisms if and only if the coordinate functions are morphisms.

Definition 3.1.5. Let $X, Y$ be two varieties. A continuous map $\varphi : X \to Y$ is called a morphism if for any $p \in X$ and $U$ containing $\varphi(p)$

$$f \in \mathcal{O}(U) \implies f \circ \varphi \in \mathcal{O}(\varphi^{-1}(U)).$$

This stuff is too abstract. Let's move on to projective varieties - the most important class of varieties in geometry.

3.2 Projective space

Definition 3.2.1. The projective $n$-space $\mathbb{P}^n$ is the set

$$(k^{n+1} - \{0\})/\sim$$

where $(a_0, \ldots, a_n) \sim (b_0, \ldots, b_n)$ if and only if $b_j = \lambda a_j$.

Remark 3.2.2. Alternatively, we may view $\mathbb{P}^n$ as the set of 1-dimensional subspaces in $k^{n+1}$. This is an important perspective!

An element $p \in \mathbb{P}^n$ is called a point, although it is really an equivalence class. Any $(a_0, \ldots, a_n) \in p$ is called a set of homogeneous coordinates of $p$.

Definition 3.2.3. An algebraic set $Z$ in $\mathbb{P}^n$ is the set of zeros of a set of homogeneous polynomials $f_1, \ldots, f_m$.

$$Z = V(f_1, \ldots, f_m) = \{p \in \mathbb{P}^n | f_1(p) = \ldots = f_m(p) = 0\}$$

where we evaluate the $f_j$ on a set of homogeneous coordinates of $p$.

Note that in general the value of a polynomial at a point $p \in \mathbb{P}^n$ is not well-defined, since $p$ has many different sets of homogeneous coordinates. However, the notion of zeros of a homogeneous polynomial is well-defined.

Let $I = (f_1, \ldots, f_r)$ where the $f_i$ are homogeneous and $V(I) = V(f_1, \ldots, f_r)$. Then $I$ is a homogeneous ideal of $k[x_0, \ldots, x_n]$, i.e. if $f \in I$, $f = \sum_d f_d$ with $f_d$ the homogeneous piece of degree $d$, then $f_d \in I$.

Lemma 3.2.4. (i) $V(k[x_1, \ldots, x_n]) = V(x_1, \ldots, x_n) = \emptyset$ and $V(0) = \mathbb{P}^n$.

(ii) $V(\bigcup I) = \bigcap I$.

(iii) $V(I_1 \cap I_2) = V(I_1) \cup V(I_2)$.

Proof. Left as an easy exercise. $\square$

Definition 3.2.5. The Zariski topology on $\mathbb{P}^n$ has as its closed subsets the algebraic sets.
Now we mention the extension of some results on affine varieties to projective space.

**Theorem 3.2.6** (Hilbert’s Nullstellensatz). For all homogeneous ideals \( I \) and all homogeneous polynomials \( f \) satisfying \( \deg f \geq 1, f \in \sqrt{I} \) if and only if \( f \) vanishes on \( V(I) \).

**Corollary 3.2.7.** There is a 1−1 correspondence between algebraic sets in \( \mathbb{P}^n \) and homogeneous radical ideals contained in \( S_+ := (x_0, \ldots, x_n) \).

**Proposition 3.2.8.** Let \( I \) be a homogeneous ideal in \( S = k[x_1, \ldots, x_n] \), regarding \( S \) as a graded ring. If \( \sqrt{I} = p_1 \cap \ldots \cap p_r \) with \( p_i \) the minimal primes containing \( I \), then each \( p_i \) is a homogeneous ideal.

**Corollary 3.2.9.** \( V(p) \) is irreducible if and only if \( p \) is a homogeneous prime in \( S_+ \). Furthermore, every algebraic set in \( \mathbb{P}^n \) can be uniquely written as the union of its irreducible components.

**Definition 3.2.10.** A projective variety is an irreducible algebraic set in \( \mathbb{P}^n \). A quasi-projective variety is an open subset of a projective variety.

**Proposition 3.2.11.** \( \mathbb{P}^n \) together with the Zariski topology has a natural variety structure over \( k \).

**Corollary 3.2.12.** Every quasi-projective variety has a natural variety structure over \( k \).

**Proof of Theorem 3.2.11.** Let \( H_i = \{(a_0, \ldots, a_n) \mid a_i = 0\} / \sim = V(x_i) \). This a closed subset by definition, so \( U_i := \mathbb{P}^n \setminus H_i \) is open. Note also the \( U_i \) cover \( \mathbb{P}^n \) since every point in \( \mathbb{P}^n \) has at least one non-zero coordinate by definition. Define

\[
\varphi_i : U_i \rightarrow \mathbb{A}^n
\]

\[
(a_0 : \ldots : a_n) \mapsto \left( \frac{a_0}{a_i}, \ldots, \frac{\hat{a_i}}{a_i}, \ldots, \frac{a_n}{a_i} \right).
\]

[The hat means the element is omitted.] Now we need to show:

(i) \( \varphi_i \) is a homeomorphism.

(ii) \( \varphi_i \circ \varphi_j^{-1} \) is a morphism.

Without loss of generality suppose \( i = 0 \). For (i), since this is a bijection, we just need to show that it is closed. Let \( I \subset S \) be a homogeneous ideal. Denote by \( I^{th} \) the ideal of polynomials of the form \( f(1, x_1, \ldots, x_n) \) where \( f \in I \). We want to consider \( \varphi_i : U_0 \cap V(I) \rightarrow \mathbb{A}^n \). But

\[
U_0 \cap V(I) = \{(a_0, \ldots, a_n) \mid f(a_0, \ldots, a_n) = 0 \ \forall \ f \in I\},
\]
which is mapped to
\[
\left\{ \left( \frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0} \right) \mid g \left( \frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0} \right) = 0 \forall g \in I^h \right\}.
\]
On the other hand, let \( I \subset k[x_1, \ldots, x_n] \) be any ideal. We define the homogeneous ideal
\[
I^h = \left\{ x_0^{\deg f} g \left( \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0} \right) \mid g \in I \right\}.
\]
Then \( \varphi_0(U_0 \cap V(I^h)) = V(I) \). This completes the proof that \( \varphi_0 \) is a homeomorphism.

For (ii), assume that \( i = n, j = 0 \). We wish to show that the map
\[
\varphi_n \circ \varphi_0^{-1} : \mathbb{A}^n \setminus \{ x_n \neq 0 \} \to \mathbb{A}^n \setminus \{ x_n \neq 0 \}
\]
\[
(a_1, \ldots, a_n) \mapsto \left( \frac{1}{a_n}, \frac{a_1}{a_n}, \ldots, \frac{a_{n-1}}{a_n} \right)
\]
is an algebraic morphism. For this, it suffices to check that the pullback of the coordinate functions is regular. This is obvious, since the pullback of \( x_i \) is \( \frac{x_i+1}{x_n} \), which is regular on \( U_n \).

In particular, every projective variety \( V(p) \subset \mathbb{P}^n \) is a variety.

**Example 3.2.13.** Consider \( V(xy - z^2) \subset \mathbb{P}^2 \). Then \( U_x = \{(x, y, z) \mid x \neq 0\} \simeq \mathbb{A}^2 \).

On this chart,
\[
V(xy - z^2) \cap U_x = V(y - z^2) \subset \mathbb{A}^2.
\]
On the chart \( U_z = \{(x, y, z) \mid z \neq 0\} \simeq \mathbb{A}^2 \),
\[
V(xy - z^2) \cap U_z = V(xy - 1) \subset \mathbb{A}^2.
\]

### 3.3 Regular functions on projective varieties

Let \( Y = V(p) \subset \mathbb{P}^n \) be a projective variety. We denote by \( S(Y) = S/p \) the homogeneous coordinate ring of \( Y \). This is a graded ring
\[
S(Y) = \bigoplus_d S(Y)_d.
\]
In the affine setting, we saw that there was an equivalence of categories between affine varieties and affine \( k \)-algebras via the coordinate ring. However, this is not the case with projective varieties: there are non-isomorphic projective varieties with isomorphic coordinate rings.

Let \( p' \) be a homogeneous prime ideal of \( S(Y) \). We denote by \( S(Y)_{p'} \) the localization of \( S(Y) \) at \( p' \), i.e.
\[
S(Y)_{p'} = \left\{ \frac{f}{g} \mid f, g \text{ homogeneous}, \deg f = \deg g, \ g \notin p' \right\}.
\]
Proposition 3.3.1. Let \( Y = V(p) \) be a projective variety, \( p \in Y \), and \( m_p \) the homogeneous maximal ideal corresponding to \( p \). Then

(i) \( \mathcal{O}_{Y,p} \cong S(Y)_{m_p} \).

(ii) \( K = k(Y) = S(Y)_{(0)} \).

(iii) \( \mathcal{O}(Y) = k \).

Proof. Let \( Y_i = Y \cap U_i \). This is an affine variety since it is a closed, irreducible subset of an affine variety. There is a natural isomorphism

\[ \varphi^*_i : A(Y_i) \rightarrow S(Y)_{x_i} \]

The map sends \( f(x_1, \ldots, x_n) \mapsto f \left( \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0} \right) \). This induces an inverse map of spectra \( p S_{x_0} \mapsto p^{ih} \), and \( Y_0 = V(p^{ih}) \). (Here the “ih” stands for “inhomogeneous.” This means to replace \( f(x_0, x_1, \ldots, x_n) \) with \( f(1, x_1, \ldots, x_n) \).)

Now (i) and (ii) follow from the corresponding facts for affine varieties: without loss of generality, we may assume that \( p \in U_0 \cap Y = Y_0 \). Then by definition

\[ \mathcal{O}_{Y,p} = \mathcal{O}_{Y_0,p} = A(Y_i)_{m_p} \cong (S(Y)_{x_0})_{\varphi^*_i((m_i)^{ih})} \cong S(Y)_{m_p} \]

For part (iii), let \( f \in \mathcal{O}(Y) \subset k(Y) = S(Y)_{(0)} \subset L \), where \( L \) is the fraction field of \( S(Y) \). Since \( Y = \bigcup Y_i \), we get a map

\[ \mathcal{O}(Y) \rightarrow \prod \mathcal{O}(Y_i) = A(Y_i) \cong S(Y)_{x_i} \]

So for each \( i \), there exists \( n_i \) such that \( x_i^{n_i} f \in S(Y) \). Now let \( N \geq \sum N_i \). Then \( S(Y)_N f \subset S(Y)_N \). So for any \( m \), \( S(Y)_N f^m \subset S(Y)_N \).

Regarding \( f \) as a map between finite-dimension \( k \)-vector spaces \( S(Y)_N \rightarrow S(Y)_N \), there exists \( m \) such that \( f \) satisfies the polynomial relation

\[ f^m + a_1 f^{m-1} + \ldots + a_m = 0 : S(Y)_N \rightarrow S(Y)_N , \]

implies that \( f^m + a_1 f^{m-1} + \ldots + a_m = 0 \) in \( L \). In particular, this is a finite polynomial relation over \( k \), but \( k \) is algebraically closed, so \( f \in k \).

3.4 Products of Varieties

Definition 3.4.1. Let \( X, Y \) be two varieties. The product

\[ X \times_k Y \]

is a variety together with two morphisms

\[ X \times_k Y \xrightarrow{pr_X} X \text{ and } X \times_k Y \xrightarrow{pr_Y} Y, \]

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such that the natural map
\[
\text{Hom}(Z, X \times_k Y) \to \text{Hom}(Z, X) \times \text{Hom}(Z, Y)
\]
is a bijection.

**Theorem 3.4.2.** The product of $X$ and $Y$ exists, and is unique up to unique isomorphism.

**Proof.** The second property follows by general abstract nonsense. We will show that there is a natural variety structure on $X \times Y$ making it the product. We do this in two steps.

(i) If $X$ and $Y$ are affine, then $X \times_k Y$ exists.

(ii) Let $\{U_i\}$ be a cover of $X$ and each $U_i \times_k Y$ exists, then $X \times_k Y$ exists.

These two claims establish the theorem, since for any two varieties $X$ and $Y$, we have open covers by affines $X = \bigcup U_i$ and $Y = \bigcup V_j$. Then $U_i \times_k V_j$ exists, so $U_i \times_k Y$ exists, so $X \times_k Y$ exists.

For (i), recall that we have an equivalence of categories between affine varieties over $k$ and finitely generated integral $k$-algebras given by $X \mapsto \mathcal{O}(X) = A(X)$ and $A \mapsto \text{Spec} A$.

**Lemma 3.4.3.** Let $Z$ be any variety, and $X$ be affine. Then
\[
\text{Hom}(Z, X) \simeq \text{Hom}_{k-\text{alg}}(\mathcal{O}(X), \mathcal{O}(Z)).
\]

**Proof.** Cover $Z$ by quasi-affines: $Z = \bigcup U_i$. Then
\[
\text{Hom}(Z, X) \to \prod_I \text{Hom}(U_i, X) \Rightarrow \prod_I \text{Hom}(U_i \cap U_j, X)
\]
(In categorical language, $\text{Hom}(Z, X)$ is the equalizer of the second map.) Concretely, this means that giving a morphism $\varphi : Z \to X$ is equivalent to giving morphisms $\varphi_i : U_i \to X$ such that $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$. On the other hand, we have
\[
\mathcal{O}(Z) \to \prod_I \mathcal{O}(U_i) \Rightarrow \prod_{I \times I} \mathcal{O}(U_i \cap U_j)
\]
by definition. Since the $U_i$ are quasi-affine, the first sequence is equivalent to
\[
\prod_I \text{Hom}_{k-\text{alg}}(\mathcal{O}(X), \mathcal{O}(U_i)) \Rightarrow \prod_{I \times I} \text{Hom}_{k-\text{alg}}(\mathcal{O}(X), \mathcal{O}(U_i \cap U_j)).
\]
and using the above, we deduce that $\text{Hom}(\mathcal{O}(X), \mathcal{O}(Z))$ is the equalizer, so we have an isomorphism. \qed
Now we return to proving (i). Let $X,Y$ be affine. Let $A = \mathcal{O}(X)$ and $B = \mathcal{O}(Y)$. Then $A \otimes_k B$ is a finitely generated $k$-algebra, since if $A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and $B = k[y_1, \ldots, y_s]/(g_1, \ldots, g_r)$, then

$$A \otimes_k B = k[x_1, \ldots, x_n, y_1, \ldots, y_s]/(f_1, \ldots, f_m, g_1, \ldots, g_r).$$

Consider $\text{Spec } A \otimes_k B = V(f_1, \ldots, f_m, g_1, \ldots, g_r) \subset \mathbb{A}^{n+s}$. We claim that $\text{Spec } (A \otimes_k B)$ is $X \times_k Y$. Well,

$$\text{Hom}(Z, \text{Spec } (A \otimes_k B)) \simeq \text{Hom}_{k\text{-alg}}(A \otimes_k B, \mathcal{O}(Z)) = \text{Hom}_{k\text{-alg}}(A, \mathcal{O}(Z)) \times \text{Hom}_{k\text{-alg}}(B, \mathcal{O}(Z)) = \text{Hom}(Z, X) \times \text{Hom}(Z, Y).$$

The rest of the proof is delegated to homework.

**Example 3.4.4.** This tells us that $\mathbb{A}^n \times_k \mathbb{A}^m = \mathbb{A}^{n+m}$. (But recall that the topology is not the product topology).

As an extended example, we study the product variety $\mathbb{P}^n \times_k \mathbb{P}^m$. We have a natural open covering $\mathbb{P}^n = \bigcup U_i$ and $\mathbb{P}^m = \bigcup V_j$.

**Lemma 3.4.5.** If $X \times_k Y$ exists and $U \subset X$ is an open subset, then $U \times_k Y$ exists and is equal to $U \times Y \subset X \times_k Y$ with the induced topology.

$$\mathbb{P}^n \times \mathbb{P}^m = \bigcup (U_i \times_k \mathbb{P}^m) = \bigcup (U_i \times_k U_j)$$

has an open cover by affines of the isomorphic to $\mathbb{A}^{n+m}$.

**Lemma 3.4.6.** The topology on $\mathbb{P}^n \times_k \mathbb{P}^m$ can be described as follows: closed subsets are $V(f_1, \ldots, f_N)$ where $f_i \in k[x_0, \ldots, x_n, y_0, \ldots, y_m]$ that are bihomogeneous, i.e.

$$f_i = \sum c_I x_0^{a_0} \ldots x_n^{a_n} y_0^{b_0} \ldots y_m^{b_m}$$

where $a_0 + \ldots + a_n = d$ and $b_0 + \ldots + b_m = e$. (In this case we say the bidegree is $(d, e)$).

**Proof.** We have maps $\varphi_i \times \varphi_j : U_i \times V_j \to \mathbb{A}^n \times \mathbb{A}^m \simeq \mathbb{A}^{n+m}$. We claim that

$$(\varphi_i \times \varphi_j)(U_i \times V_j) \cap V(f_1, \ldots, f_N)$$

are exactly the algebraic sets in $\mathbb{A}^{n+m}$. Given this, the proof is the same as that showing $\varphi_i : U_i \to \mathbb{A}^n$ is a homeomorphism.

This establishes that $\mathbb{P}^n \times_k \mathbb{P}^m$ is a variety, but we want to show that it is a projective variety, i.e. can be realized as a subset of some $\mathbb{P}^N$. 27
Definition 3.4.7. Let $\varphi : X \to Y$ be a morphism of (pre)-varieties. We say $\varphi$ is a closed embedding if $\varphi$ is one-to-one onto an irreducible closed subset $\varphi(X) \subset Y$ (giving $\varphi(Y)$ and induced variety structure) and $\varphi : X \to \varphi(X)$ is an isomorphism.

Theorem 3.4.8. Let $N = mn + m + n$. Then there is a natural closed embedding

$$S : \mathbb{P}^n \times_k \mathbb{P}^m \to \mathbb{P}^N.$$  

This is called the Segre map.

Corollary 3.4.9. Let $X, Y$ be projective. Then $X \times_k Y$ is projective.

Proof. Immediate from the theorem and the following lemma. \qed

Lemma 3.4.10. Let $\varphi' : X' \to X$ and $\psi : Y' \to Y$ be closed embeddings. Then $\varphi \times \psi : X' \times_k Y' \to X \times Y$ is a closed embedding.

Proof. Exercise. \qed

Proof of Theorem 3.4.8. The map $s$ is given by

$$([a_0 : \ldots : a_n], [b_0 : \ldots : b_m]) \mapsto [\ldots : a_ib_j : \ldots].$$

First we need to check that the map is a morphism. It suffices to check this on an open cover of affines. Put coordinates $[x_0 : \ldots : x_n]$ on $\mathbb{P}^n$ and coordinates $[y_0 : \ldots : y_m]$, and denote the coordinates in $\mathbb{P}^N$ by $[\ldots : z_{ij} : \ldots]$. Set

$$U_{ij} = \mathbb{P}^n - H_{ij} = \{z_{ij} \neq 0\}.$$  

Then we have a diagram

$$\begin{array}{ccc}
\mathbb{A}^n 	imes \mathbb{A}^m & \longrightarrow & \mathbb{A}^{n+m} \\
\downarrow \varphi_i \times \varphi_j & & \downarrow \varphi_{ij} \\
U_i \times U_j & \longrightarrow & U_{ij}
\end{array}$$

For convenience, we assume that $i = j = 0$. Then the map on the bottom is given by

$$((x_1, \ldots, x_n), (y_1 : \ldots : y_m)) \mapsto (x_1, \ldots, x_n, y_1, \ldots, y_m, x_1y_1).$$

This shows that $s$ is a morphism. Next, we show that $s$ is injective. Suppose $s(a, b) = s(a', b')$, i.e. $a_ib_j = a'_ib'_j$ for each $i, j$. This means that

$$a_ib_j = \lambda a'_i b'_j \forall i, j.$$  

There exists $i_0, j_0$ such that $a_{i_0} \neq 0$ and $b_{j_0} \neq 0$, so we may assume that $(a, b) \in U_{i_0} \times V_{j_0}$. Then

$$a_{i_0}b_{j_0} = \lambda a'_{i_0} b'_{j_0}$$
implies that \((a', b') \in U_{i_0} \times V_{j_0}\). Let \(\mu = \frac{a_{i_0}}{a_{i_0}}\) and \(\nu = \frac{b_{j_0}}{b_{j_0}}\), so \(\lambda = \mu \nu\). Then

\[ a_i b_{j_0} = \lambda a_i' b_{j_0}' \implies a_i = \mu a_i' \implies a = a'. \]

Similarly, \(b = b'\). This verifies that \(s\) is injective.

Let \(\mathfrak{p} = \ker(k[z_{ij}] \to k[x_0, \ldots, x_n, y_0, \ldots, y_m])\) sending \(z_{ij} \mapsto x_i y_j\). Then \(\mathfrak{p}\) is prime, and \(\mathfrak{p} \supset \mathfrak{p}' = \{z_{ij} z_{kl} - z_{il} z_{kj}\}\). In fact, one can show that \(\mathfrak{p} = \mathfrak{p}'\). Obviously,

\[ s(\mathbb{P}^n \times \mathbb{P}^m) \subset V(\mathfrak{p}) \subset \mathfrak{B}'. \]

We now prove that \(S(\mathbb{P}^n \times \mathbb{P}^m) = V(\mathfrak{p})\). Let

\[ (c_{ij}) \in V(\mathfrak{p}') \implies c_{ij} c_{kl} = c_{il} c_{kj}. \]

There exists \(c_{i_0 j_0} \neq 0\), so \((c_{ij}) \in U_{i_0 j_0}\). Let \(a_i = \left(\frac{c_{i j_0}}{c_{i_0 j_0}}\right) \in \mathbb{P}^n\) and \(b_j = \left(\frac{c_{i j_0}}{c_{i_0 j_0}}\right) \in \mathbb{P}^m\).

Then

\[ S(a, b) = (a_i b_j) = \left(\frac{c_{ij_0} c_{i_0 j}}{c_{i_0 j_0}}\right) = \left(\frac{c_{i j}}{c_{i_0 j}}\right) = (c_{ij}) \in \mathbb{P}^N. \]

This also lets us conclude that \(s^{-1}(U_{i j}) = U_i \times_k V_j\).

Finally, we show that \(s\) is an isomorphism. We want a map

\[ s^{-1} : V(\mathfrak{p}) \cap U_{i_0 j_0} \to U_{i_0} \times V_{j_0}. \]

Let’s consider the diagram

\[
\begin{array}{ccc}
V(\mathfrak{p}) \cap U_{i_0 j_0} & \longrightarrow & U_{i_0} \times V_{j_0} \\
\downarrow & & \downarrow \\
U_{i_0 j_0} & \longrightarrow & U_{i_0} \times V_{j_0}
\end{array}
\]

The bottom map is given by projection to the first \(n + m\) coordinates:

\[(a_1, \ldots, a_n, b_1, \ldots, b_m, b_{i j}) \mapsto (a_1, \ldots, a_n, b_1, \ldots, b_m).\]

\[ \square \]

### 3.5 The Grassmannian

**Example 3.5.1.** (Linear subvarieties)

- A **hyperplane** in \(\mathbb{P}^n\) is defined by a linear polynomial \(V(a_0 x_0 + \ldots + a_n x_n)\).
- A **linear variety** in \(\mathbb{P}^n\) is the intersection of hyperplanes.
- A linear variety of dimension \(r\) is also called an \(r\)-plane.
• A 1-plane is called a line.

**Lemma 3.5.2.** (i) Let $X \subset \mathbb{P}^n$ be an $r$-plane. Then $X \cong \mathbb{P}^r$ and $I(X)$ can be generated by $n-r$ linear polynomials.

(ii) There is a natural bijection between $(r+1)$-dimensional sub-vector spaces in $k^{n+1}$ and $r$-planes in $\mathbb{P}^n$:

$$L \mapsto \mathbb{P}L.$$

**Proof.** Left as exercise. □

**Definition 3.5.3.** The Grassmanian $G(r,n)$ is the set of all $r$-dimensional subspaces in $k^n$, (equivalently, the set of all $(r-1)$-planes $\mathbb{P}^{n-1}$).

A couple of remarks on notation:

(i) We also use the notation $G(r-1,n-1) := G(r,n)$.

(ii) Let $V$ be a finite-dimensional $k$-vector space. Then $G(r,V)$ is the set of $r$-dimensional subspaces in $V$. In particular, $G(1,V)$ is denoted $\mathbb{P}(V)$.

**Theorem 3.5.4.** $G(r,V)$ is naturally a projective variety of dimension $r(n-r)$, where $n = \dim V$.

**Proof.** Let $e_1, \ldots, e_n$ be a basis for $V$. For each $I \subset \{1, \ldots, n\}$, let $k^I = \text{Span}\{e_i, i \in I\} \subset V$. Now, fix $I \subset \{1, \ldots, n\}$ with $|I| = n-r$. Let

$$V_I = \{L \subset V \mid \dim L = r, L \cap k^I = 0\} \subset G(r,V).$$

We claim that

$$G(r,V) = \bigcup_{I \subset \{1, \ldots, n\}, |I| = n-r} V_I.$$

This is clear because for each $r$-dimensional space in $G(r,V)$ we can just choose $I$ to represent a complementary subspace. There is a natural bijection

$$\varphi_I : V_I \cong \mathbb{A}^{r(n-r)}.$$

In fact,

$$V_I \cong \text{Hom}(V/k^I, k^I) \cong M_{(n-r)}(k) \cong \mathbb{A}^{r(n-r)m}.$$ 

In one direction, we can take any $\varphi \in \text{Hom}(V/k^I, k^I) \cong M_{(n-r)}(k)$ and consider its graph $\Gamma \varphi \subset V$. In the other direction, given $L \subset V$, we may project to $V/k^I$. This is an isomorphism since $L \cap k^I = 0$. 

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Explicitly, this means each $L \subset V_I$ uniquely determines

$$(v_1, \ldots, v_r) = (e_1, \ldots, e_n) \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ a_{r+1,1} & \cdots & a_{r+1,r} & & \\ & \vdots & & \ddots & \\ a_{n,1} & \cdots & a_{n,r} & & \end{pmatrix}.$$ 

It is not hard to show that $\varphi_I \circ \varphi_I^{-1}$ is a morphism, giving a variety structure on $G(r, n)$. Details are left to the reader.

We now construct a closed embedding $\psi : G(r, n) \to \mathbb{P}^N$. Let $W = \bigwedge^r V$. Then $\dim W = \binom{n}{r}$. For each $L \in G(r, n)$, we choose a basis $v_1, \ldots, v_r \in L$ and associate to it $v_1 \wedge \ldots \wedge v_r \in w$. This is unique up to scalars since $\bigwedge^r L$ is one-dimensional. Now define

$$\psi : G(r, n) \to \mathbb{P}(W)$$

$$L \mapsto v_1 \wedge \ldots \wedge v_r = [L].$$

This is called the Plucker embedding.

Let $I = \{r + 1, \ldots, n\}$. We have $\psi : V_I \to \mathbb{P}(W)$. The coordinates on $\mathbb{P}(W)$ are $(a_J)$, where $J \subset \{1, \ldots, n\}$ and $|J| = r$. Using the basis

$$(v_1, \ldots, v_r) = (e_1, \ldots, e_n) \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ a_{r+1,1} & \cdots & a_{r+1,r} & & \\ & \vdots & & \ddots & \\ a_{n,1} & \cdots & a_{n,r} & & \end{pmatrix},$$

we see that $\psi$ maps to the $r \times r$ minors of the matrix

$$\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ a_{r+1,1} & \cdots & a_{r+1,r} & & \\ & \vdots & & \ddots & \\ a_{n,1} & \cdots & a_{n,r} & & \end{pmatrix}.$$ 

This is transparently a polynomial map, therefore a morphism.

Next we show that $\psi$ is injective. But this follows from the fact that we can recover $L$ as

$$L = \{v \in V \mid v \wedge [L] = 0 \in \bigwedge^{r+1} V\}.$$
Let \( w \in W \). When can we write \( w = v_1 \wedge \ldots \wedge v_r \)? (Such a \( w \) is called decomposable.) An obvious condition is that the map

\[
- \wedge w : V \to \bigwedge^{r+1} V
\]

has kernel with dimension \( \geq r \). The following result in linear algebra shows that this condition is also sufficient.

**Lemma 3.5.5.** Let \( w \in W \) be non-zero. Then

\[
\dim \ker(\wedge w) \leq r
\]

with equality holding if and only if \( w = v_1 \wedge \ldots \wedge v_r \).

**Proof.** Left as exercise. \qed

So we have a map \( \delta : W \to \text{Hom}(V, \bigwedge^{r+1} V) \supset \{ \varphi \mid \text{rank } \varphi \leq n - r \} \) sending

\[
w \mapsto (v \mapsto \bigwedge^r v \wedge w).
\]

This induces a map on projectivizations \( \tilde{\delta} : \mathbb{P}(W) \to \mathbb{P}(\text{Hom}(V, \bigwedge^{r+1} V)) \). The latter set contains

\[
Z_{n-r}(V, \bigwedge^r V) := \{ \varphi : V \to \bigwedge^r V \mid \text{rank } \varphi \leq n - r \}.
\]

In general, let \( V_1 \) and \( V_2 \) be two vector spaces and let

\[
Z_d(V_1, V_2) = \{ \varphi : V_1 \to V_2 \mid \text{rank } \varphi \leq d \} \subset \mathbb{P}(\text{Hom}(V_1, V_2)).
\]

This is an algebraic set in \( \mathbb{P}(\text{Hom}(V_1, V_2)) \) determined by the vanishing of the \((d + 1) \times (d + 1)\) minors; such a variety is called a “determinantal variety.” Anyway, we have \( \psi(G(r, n)) = \delta^{-1} Z_{n-r}(V, \bigwedge^{r+1} V) \).

**Remark 3.5.6.** In this language, the Segre embedding of \( \mathbb{P}^n \times \mathbb{P}^m \) is just the determinantal variety \( Z_1(k^{n+1}, k^{m+1}) \).

Now we want to show that \( \psi \) is a homeomorphism, and in fact an isomorphism of varieties. This is something we can check locally, so consider the restriction

\[
\psi : V_I \to U_I := \mathbb{P}(W) - \{ a_I = 0 \}.
\]

For example, suppose \( I = \{ r+1, \ldots, n \} \). The map sends

\[
\begin{pmatrix}
1 \\
\vdots \\
a_{r+1,1} & \ldots & a_{r+1,r} \\
\vdots \\
a_{n,1} & \ldots & a_{n,r}
\end{pmatrix} \to a_J = \{ r \times r \text{ minor for } J \}.
\]
Suppose $J = \{1, 2, \dotsc, r\} \cup j$ for some $r + 1 \leq j \leq n$. Then $a_J = a_{ji}$, so the projection to this coordinate on $U_i$ is an inverse morphism back to $V_i$. \hfill \square

We remark that the ideal of $\psi(G(r, n))$ can be generated by quadratic polynomials, called the “Plucker” relations.
Chapter 4

Morphisms

4.1 Rational maps and correspondences

Definition 4.1.1. A correspondence from a variety $X$ to a variety $Y$ is a relation given by a closed subset $Z \subset X \times Y$.

- $Z$ is said to be a rational map if $Z$ is irreducible and there is an open subset $X_0 \subset X$ such that $\text{pr}_X(Z \cap \text{pr}_X^{-1}(X_0)) \rightarrow X_0$ is an isomorphism.
- $Z$ is said to be birational if $Z$ is rational and $Z^{-1} = \{(y, x) \in Y \times X \} \subset Y \times X$ is also rational.

We will also use $\sim$ to denote a rational map.

Example 4.1.2. Let $X$ be projective, and let $f \in k(X)$ be a rational function. Then $f$ gives rise to a rational map $f : X \rightarrow \mathbb{P}^1$. Recall that $k(\mathbb{P}^1) = k(t)$, a rational map $f : X \rightarrow \mathbb{P}^1$ is a map $k(t) \rightarrow k(X)$. Now we see that $f \in k(X)$ uniquely specifies the map $k(t) \rightarrow K(X)$ sending $t \mapsto f$, so

$$k(X) \setminus \{0\} = \text{Hom}_{k-\text{alg}}(k(t), K(X)).$$

Since $X$ is projective, we may write any $f \in K(X)$ as

$$f = \frac{g}{h} \quad g, h \in k[x_0, \ldots, x_n] \text{ homogeneous of the same degree.}$$

Now we define $X_0 = X - (\{g = 0\} \cap \{h = 0\})$. This is a non-empty open subset of $X$. Then we define

$$F : X_0 \rightarrow \mathbb{P}^1$$

by sending $p \mapsto [g(p) : h(p)]$. It is obvious after restricting to local affines that this is a morphism. Let $y_0, y_1$ be the homogeneous coordinates on $\mathbb{P}^1$. Consider
$V(y_0h - y_1g)$; this is a bihomogeneous polynomial of bidegree $(d,1)$ defining a closed subset of $\mathbb{P}^n \times \mathbb{P}^1$.

Let $Z = V(y_0h - y_1g) \cap X \times \mathbb{P}^1$. Over $X_0$, $\text{pr}^{-1}(X_0) \cap Z$ is just the graph of $F$, i.e.

$$\text{pr}^{-1}(X_0) \cap Z = \{(x,F(x)) \mid x \in X\}.$$  

In fact, $\text{pr} : Z \cap \text{pr}^{-1}(X_0) \to X_0$ is an isomorphism since there is an inverse morphism by the universal property.

Now let $Z^*$ be the closure of $Z \cap \text{pr}^{-1}(X_0)$ in $X \times \mathbb{P}^1$. Then $Z^*$ is irreducible because it is isomorphic to $X_0$, and $Z^* \cap \text{pr}^{-1}(X_0) = Z \cap \text{pr}^{-1}(X_0)$ since $Z \cap \text{pr}^{-1}(X_0)$ is already closed in $X \times \mathbb{P}^1$. Therefore, $Z^*$ is a rational map from $X$ to $\mathbb{P}^1$.

Later, we will see that for any two varieties $X$ and $Y$, any inclusion $k(Y) \subset k(X)$ gives a rational map $X \dasharrow Y$.

### 4.2 Blowing up

Let $O = [0 : \ldots : 1] \in \mathbb{P}^n$ and let $p : \mathbb{P}^n - \{O\} \to \mathbb{P}^{n-1}$ be defined by

$$[a_0 : \ldots : a_n] \mapsto [a_0 : \ldots : a_{n-1}].$$

Then $p$ is a morphism called projection from $O$. Geometrically, it works as follows: for a point $q \in \mathbb{P}^n - \{O\}$, take the line through $q$ and $\{O\}$; this intersects $\mathbb{P}^{n-1}$ in a unique point. Note that $p^{-1}(p(x))$ is the line joining $O$ and $x$, less $O$.

Let $Z = V(\{x_iy_j - y_ix_j\}_{0 \leq i,j \leq n-1}) \subset \mathbb{P}^n \times \mathbb{P}^{n-1}$. We have a correspondence

$$Z = V(x_iy_j - y_ix_j)$$

Then $Z = \Gamma_p \cup \text{pr}^{-1}(O) = \Gamma_p \cup \mathbb{P}^{n-1}$. This $\mathbb{P}^{n-1}$ is called the exceptional divisor $E$.

We claim that $Z$ is irreducible. This is because

$$Z \cap \text{pr}^{-1}(\mathbb{P}^n - \{O\}) = \Gamma_p = \{(x,p(x)) \in (\mathbb{P}^n - \{O\}) \times \mathbb{P}^{n-1}\}.$$  

But the graph is isomorphic to $\mathbb{P}^n - \{O\}$, so $\Gamma_p$ is irreducible. To prove the claim, it suffices to show that $Z = \Gamma_p$.

For $q \in \mathbb{P}^{n-1}$, let $\ell_q = p^{-1}(q) \cup \{O\}$, which is the line in $\mathbb{P}^n$ through $O$ and $q$. Then

$$\ell_q \times \{q\} \subset \mathbb{P}^n \times \{q\} \subset \mathbb{P}^n \times \mathbb{P}^{n-1}$$
and 
\[(l_q - \{O\}) \times \{q\} \subset \Gamma_{p} \subset Z.\]

But \((O, q) \in l_q \times \{q\} \subset \Gamma_{p} \subset Z\), so \(\{O\} \times \mathbb{P}^{n-1} = E \subset \Gamma_{p}\), implying that \(Z \subset \Gamma_{p}\). So \(Z\) is irreducible; since it is a closed subspace of projective space, it is a projective variety. We call it the blow-up of \(\mathbb{P}^{n}\) at \(O\). Note that projection to \(\mathbb{P}^{n}\) is one-to-one from \(Z - E\) to \(\mathbb{P}^{n}\) (since it is the graph of the projection map at these points) and \(\text{pr}^{-1}(O) = E \simeq \mathbb{P}^{n-1}\).

**Definition 4.2.1.** Let \(X \subset \mathbb{P}^{n}\) be quasi-projective and \(O \in X\). The blow-up of \(X\) at \(O\) is 
\[\text{Bl}_O X := \text{pr}^{-1}(X - \{O\}) \subset Z.\]

**Example 4.2.2.** Let \(X : y^2 = x^3 + x^2 \subset \mathbb{A}^2 \subset \mathbb{P}^2\). Let \(O = [0 : 0 : 1] \in X\). Let \(\tilde{X}\) be the blowup of \(X\) at \(O\). Let \((u, v)\) be the coordinates on \(\mathbb{P}^1\). Then \(Z = V(xu - yv) \subset \mathbb{P}^2 \times \mathbb{P}^1\). So \(\text{pr}^{-1}(X) \cap Z \supset \tilde{X}\) is given by the solutions of the system 
\[
\begin{align*}
xu - yv &= 0 \\
y^2 - x^3 - x^2 &= 0
\end{align*}
\]
Restricting to the open subset \(\mathbb{A}^2 \times \mathbb{A}^1 \subset \mathbb{A}^2 \times \mathbb{P}^1\) where \(v \neq 0\), this system is 
\[
\begin{align*}
xu - y &= 0 \\
y^2 - x^3 - x^2 &= 0 \implies x^2(u^2 - x - 1) &= 0
\end{align*}
\]
This is reducible, given by \(V(y - xu, x^2(u^2 - x - 1)) = V(y - xu, u^2 - x - 1) \cup Y(y - xu, x)\). The second term is the exceptional locus \(E \cap (\mathbb{A}^2 \times \mathbb{A}^1)\). The first thing is \(\tilde{X}\).

**Example 4.2.3.** Last time we defined a rational map from \(X\) to \(Y\) to be a correspondence \(Z \subset X \times Y\) which is bijective over some open subset of \(X\). This is right in characteristic 0, but not quite enough in characteristic \(p\). Let \(\text{char } k > 0\). Consider the Frobenius morphism \(F : \mathbb{A}^1 \to \mathbb{A}^1\). Then \(\Gamma_F \subset \mathbb{A}^1 \times \mathbb{A}^1\) is a correspondence, which is bijective over an open subset of each \(\mathbb{A}^1\), but we want to consider it as not birational (it does not induce an isomorphism on function fields).

Now we summarize the geometric picture in our discussion of projections and blowups. Let \(O = (0, \ldots, 0, 1) \in \mathbb{P}^n\). There is a projection \(p : \mathbb{P}^n - \{0\} \to \mathbb{P}^{n-1}\), which gives a correspondence 
\[Z = V(x_i y_j - y_i x_j)\]

The variety \(Z\) is called \(\text{Bl}_O(\mathbb{P}^n)\). If \(X\) is a subvariety of \(\mathbb{P}^n\), then the closure of \(\text{pr}^{-1}(X - \{O\})\) is called the blowup of \(X\), \(\text{Bl}_O(X)\). The picture is that **the blowup separates the lines through a singular point**.
Resolution of singularities. The basic problem is: given $X$ an algebraic variety, can we find $Y \to X$ a birational, surjective morphism with $Y$ smooth? Hironaka proved that this can be done over a field of characteristic 0 (and he got a Fields medal for this work), but the answer is unknown over fields of positive characteristic.

Example 4.2.4 (Incidence correspondence). Let $C \subset G(r, n) \times \mathbb{P}^n$ be defined as

$$C = \{(L, x) \mid x \in L\}.$$  

We claim that $C$ is a correspondence; furthermore, $C$ is a smooth projective variety of dimension $(r + 1)(n - r) + r$.

Proof. Let $\mathbb{P}^n = \mathbb{P}(V)$, where $\dim V = n + 1$. Let $e_0, \ldots, e_n$ be a basis of $V$. Consider the map

$$G(r, n) \times \mathbb{P}^n \xrightarrow{\psi \times \text{id}} \mathbb{P}(W) \times \mathbb{P}^n,$$

where $W = \bigwedge^{r+1} V$ sending $(L, x) \mapsto ([L], x)$. The coordinates on $W$ are $\sum a_I e_i \in w$, $I \subset \{0, \ldots, n\}$, where $|I| = r + 1$. The $x \in L \iff [L] \wedge x = 0$, which we can express in coordinates as

$$\sum a_I e_I \wedge \sum x_i e_i = 0.$$  

This is a bihomogeneous polynomial $F(a_I, x)$ in the $a_I$ and $x_i$ of bi-degree $(1, 1)$. So

$$C = V(F) \cap (\psi(G(r, n))) \times \mathbb{P}^n.$$  

We claim that for each $I \subset \{0, 1, \ldots, n\}$ of cardinality $n - r$,

$$\text{pr}^{-1}(V_I) \cap C \simeq V_I \times \mathbb{P}^r,$$

where the $V_I$ are the open sets in the cover of $\psi(G(r, n))$. Note that this gives the desired dimension formula, since $\dim V_I \times \mathbb{P}^r = \dim V_I + \dim \mathbb{P}^r$.

$I$ corresponds to a vector space $k^I \subset V$ of dimension $n - r$. So $k^I / \sim$ corresponds to $M \subset \mathbb{P}^n$, an $n - r - 1$ plane in $\mathbb{P}^n$. After changing coordinates, we may assume that $M = \{(0, \ldots, 0, a_r, \ldots, a_{n-r-1})\}$. We define a map by projecting away from $M$:

$$P_M : \mathbb{P}^n - M \to \mathbb{P}^r$$

$$(a_0, \ldots, a_n) \mapsto (a_0, \ldots, a_r)$$

$P_M^{-1}(y)$ is an $(n - r)$-plane.

Now we define a map

$$C \cap \text{pr}^{-1}(V_I) \to V_I \times \mathbb{P}^r$$

$$(L, x) \mapsto (L, P_M(x))$$

This is well-defined because $L \cap k^I = \{0\}$, so $x \in \mathbb{P}(L) \implies x \notin M$.  \qed
4.3 Rational maps and function fields

Definition 4.3.1. A morphism \( f : X \to Y \) is called dominant if \( f(X) \) contains a dense subset of \( Y \) (in fact, we can assume that it contains an open dense subset). A rational map \( Z \subset X \times Y \) is called dominant if \( \text{pr}_Y : Z \to Y \) is dominant.

In the homework, we defined another notion of rational maps

\[
\text{Maps}(X,Y)_{\text{rat}} = \{ (U,f) \mid U \subset X \text{ open subset, } f : U \to Y \text{ a morphism} \}.
\]

On the homework you show that there is an injection \( \text{Maps}(X,Y)_{\text{rat}} \to \text{Maps}(X,Y)_{\text{rat}} \).

In the other direction, if \( Z \subset X \times Y \) is a rational map, there is an isomorphism \( \text{pr}^{-1}(X_0) \cap Z \cong X_0 \). Composing with projection to \( Y \) gives a morphism \( X_0 \to Y \).

Therefore, these two notions are equivalent.

Under this correspondence, we have that \( Z \) is dominant \( \iff \) \( f : U \to Y \) is dominant. Now if \( f : X \dashrightarrow Y \), then we obtain a map \( f^* : k(Y) \to k(X) \) as follows.

If \( g \in k(Y) \), then \( g \in \mathcal{O}(V) \) for \( V \) an open subset of \( Y \), so \( g \circ f \in \mathcal{O}(f^{-1}(V)) \subset k(X) \).

Proposition 4.3.2. Consider the functor \( F \) defined by \( F(X) = k(X) \), and \( F(f : X \dashrightarrow Y) = (f^* : k(Y) \to k(X)) \). Then \( F \) induces an equivalence of categories between the category of algebraic varieties with morphisms the dominant rational maps, and the category of finitely generated field extensions of \( k \), with morphisms \( k - \text{alg homomorphism} \).

Proof. First, we have to show that

\[
\text{Maps}(X,Y)_{\text{rat}} \cong \text{Hom}_{k-\text{alg}}(k(Y),k(X))
\]

is a bijection. To do this, we construct an inverse. Let \( \theta : k(Y) \to k(X) \). Let \( Y_0 \subset Y \) be affine open and \( Y_0 \subset \mathbb{A}^n \), with coordinate functions \( (y_1, \ldots, y_n) \). Similarly, let \( X_0 \subset X \) be affine. So \( \theta(y_i) \) is a rational function on \( X_0 \), and can be written as the rational of two regular functions:

\[
\theta(y_i) = \frac{f_i}{f_0}, \quad f_i \in \mathcal{O}(X_0), f_0 \neq 0.
\]

Let \( U = X_0 - \{f_0 = 0\} \). This is a non-empty open subset of \( X \) since \( f_0 \neq 0 \), so \( \theta(y_i) \in \mathcal{O}(U) \). So we may regard \( \theta \) as a map \( \mathcal{O}(Y_0) \to \mathcal{O}(U) \), where \( Y_0 \) is affine, so by the equivalence of categories between affine varieties and affine \( k \)-algebras, this induces a map \( \varphi : U \to Y_0 \). Furthermore, \( \varphi \) is dominant because if the closure in \( Y_0 \) of the image is not all of \( Y_0 \), then there is a regular function \( f \) on \( Y_0 \) which is sent to 0 by \( \varphi \). But this is impossible under a non-zero field homomorphism.

Let \( K = k(x_1, \ldots, x_n) \) be a finitely generated field extension of \( k \). We want to study how \( K \) arises as the function field of some variety. Let \( A = k[x_1, \ldots, x_n] \subset K \), and let \( X = \text{Spec} \, A \). Then \( k(X) = K \). This seemingly trivial observation has some interesting corollaries.
Corollary 4.3.3. Every algebraic variety $X$ is birational to a hypersurface in $\mathbb{P}^n$.

Proof. Let $K = k(X)$, finitely generated over $k$. Then $K/k$ is separably generated, i.e. there exists $x_1, \ldots, x_r \in K$ transcendental over $k$ such that $K/k(x_1, \ldots, x_r)$ is finite separable. By the primitive element theorem, there exists $y \in K$ such that $K = k(x_1, \ldots, x_r, y)$, where $y$ satisfies its minimal polynomial equation

$$f(x_1, \ldots, x_r, y) = y^n + a_1(x_1, \ldots, x_r)y^{n-1} + \ldots + a_n(x_1, \ldots, x_r) = 0.$$ 

This $f$ is irreducible by minimality. From the construction, $V(f) \subset \mathbb{A}^{r+1}$ is an affine variety of dimension $r$ and $k(V(f)) = K$. Therefore, $X$ is birational to $V(f)$. $\square$

4.4 Complete varieties and projections

Definition 4.4.1. An algebraic variety is called complete if for any algebraic variety $Y$, the projection map $X \times Y \to Y$ is closed.

Theorem 4.4.2. The projection $pr_2 : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$ is a closed map, i.e. every $Z \subset \mathbb{P}^n \times \mathbb{P}^m$ is closed implies $pr_2(Z)$ is closed.

Remark 4.4.3. In fact, one can show that $\mathbb{P}^n$ is complete.

Example 4.4.4. The map $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ given by projection to the second coordinate is not closed, since $V(xy - 1)$ maps to $\mathbb{A}^1 \setminus \{0\}$.

Corollary 4.4.5. Let $X, Y$ be projective, and let $Z \subset X \times Y$ be a correspondence. Then $pr_Y(Z)$ is closed.

Corollary 4.4.6. Let $A \subset X \times Y$ and $B \subset Y \times Z$ be correspondences. Then

$$C = B \circ A := \{(x, z) \in X \times Z \mid \exists y \in Y, (x, y) \in A, (y, z) \in B\}$$

is a correspondence.

Proof. We have projection maps

$$\begin{array}{ccc}
X \times Y \times Z & \xrightarrow{pr_{12}} & X \times Y \\
| & & | \\
X \times Y & \xrightarrow{pr_{13}} & X \times Z \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
Y \times Z & \xrightarrow{pr_{23}} & Y \times Z
\end{array}$$

So $C = pr_{13}(pr_{12}^{-1}(A) \cap pr_{23}^{-1}(B))$ is closed. $\square$

Corollary 4.4.7. Let $Z \subset \mathbb{G}(r, n)$ be a closed subvariety. Then

$$\bigcup_{L \subset \mathbb{P}^n} L$$

is a projective variety.
Proof. We have the incidence correspondence

\[ C \subset G(r, n) \]

Then \( \bigcup_{L \in Z} L \subset \mathbb{P}^n = \text{pr}_2(\text{pr}_1^{-1}(Z)). \]

**Corollary 4.4.8.** Let \( X \subset \mathbb{P}^n \setminus \{O\} \). Then \( P_O(X) \) is a projective variety in \( \mathbb{P}^{n-1} \) and \( P_O : X \to P_O(X) = X' \) has finite fibers. In addition, \( \dim X = \dim P_O(X) \) and \( S(X) = S^n/I(X) \) is a finite module over \( S(X') = S^{n-1}/I(X') \).

Proof. If \( y \in \mathbb{P}^{n-1} \), then \( P_O^{-1}(y) \simeq \mathbb{A}^1 \), so \( P_O^{-1}(y) \cap X \) is closed in \( \mathbb{A}^1 \). Therefore, \( P_O^{-1}(y) \cap X \neq P_O^{-1}(y) \implies P_O^{-1}(y) \cap X \) is finite (since the only closed subsets of \( \mathbb{A}^1 \) are finite points).

Because \( O \notin X \), there exists \( f \in I(X) \) depending on \( x_n \), which we can assume is of the form

\[ f = x_n^d + \ldots + a_1(x_0, \ldots, x_{n-1})x_n^{d-1} + \ldots + a_d(x_0, \ldots, x_{n-1}). \]

So \( S(X) \) is generated over \( S(X') \) by \( 1, x_n, \ldots, x_n^{d-1} \), i.e. \( S(X) \) is finite over \( S(X') \). This also shows that

\[ \text{tr.deg}_k S(X) = \dim C(X) = \dim C(X') + 1 = \text{tr.deg}_k S(X'). \]

**Corollary 4.4.9** (Noether’s normalization lemma). Let \( X^r \subset \mathbb{P}^n \) be a projective variety of dimension \( r \). Then there exists an \( n - r - 1 \) plane \( L \) such that \( X \cap L = \emptyset \) and \( \text{pr}_L : X^r \to \mathbb{P}^r \) has finite fibers and \( S(X^r) \) is finite over \( S^r \).

Proof. We induct on \( n - r \). If \( n - r = 1 \), project from a point to get \( X \to P_O(X) \subset \mathbb{P}^{n-1} \). Since \( P_O(X) \) is a closed subset of \( \mathbb{P}^{n-1} \) of dimension \( n - 1 \), it is equal to \( \mathbb{P}^{n-1} \).

In general, choose \( O \notin X \). Project \( P_O : X \to P_O(X) = X' \subset \mathbb{P}^{n-1} \). Choose \( L' \subset \mathbb{P}^{n-1} \) an \( (n - 1) - (r_1) \) plane such that \( P_{L'} : X' \to \mathbb{P}^r \) with \( P_{L'} \circ P_O = P_{L'O} = P_L \).
Proof of Theorem 4.4.2. It’s enough to show that \( \text{pr}_2 : \mathbb{P}^n \times \mathbb{A}^m \to \mathbb{A}^m \) is closed. Let \( Z = V(f_1, \ldots, f_r) \) be an algebraic set of \( \mathbb{P}^n \times \mathbb{A}^m \). Let \( f_i = f_i(x_0, \ldots, x_n, y_1, \ldots, y_n) \), so \( f_i \) is homogeneous of degree \( d_i \) in the \( x \)'s. If \( \text{pr}_2(z) \) does not contain \( b \), then \( \{ f_i(x, b) \} \) do not have common zeros in \( \mathbb{P}^n \). By Hilbert’s Nullstellensatz, there exists \( N \) such that \( (x_0, \ldots, x_n)^N \subset (f_1(x, b), \ldots, f_r(x, b)) \). Let
\[
U_N = \{ b \in \mathbb{A}^m \mid (x_0, \ldots, x_n)^N \subset (f_1(x, b), \ldots, f_r(x, b)) \}.
\]
In other words, \( \mathbb{A}^m - \text{pr}_2(Z) = \bigcup_{N \geq 1} U_N \), so it’s enough to show that \( U_N \) is open. Let \( S = k[x_0, \ldots, x_n] = \bigoplus S_d \) be the natural grading on \( S \). For \( b \in \mathbb{A}^m \), define
\[
T_b^{(N)} : S_{N-d_1} \oplus \cdots \oplus S_{N-d_r} \to S_N
\]
\[
(g_1, \ldots, g_r) \mapsto \sum g_i f_i(x, b).
\]
So \( T_b^{(N)} \) is a \( k \)-linear map between two \( k \)-vector spaces. In terms of bases, the map is given as a matrix whose entries are polynomials in \( b \). We can view this as a map
\[
T^{(N)} : \mathbb{A}^m \to \text{Hom}_k(S_{N-d_1} \oplus \cdots \oplus S_{N-d_r}, S_N)
\]
and \( b \in U_N \iff T_b^{(N)} \) is surjective. We observe:

(i) \( T^{(N)} \) is a morphism of affine varieties.

(ii) Let \( D = \dim_k S_N - 1 \). Then \( Z_D \) is a determinantal variety, so the affine cone \( C(Z_D) \subset \text{Hom}_k(S_{N-d_1} \oplus \cdots \oplus S_{N-d_r}, S_N) \) is closed.

(iii) Thus \( \mathbb{A}^m - U_N = (T^{(N)})^{-1}(C(Z_D)) \) is closed.

\[
\square
\]

Proposition 4.4.10. Let \( \varphi : X^r \to Y^s \) be a morphism of quasiprojective varieties \( (r \geq s) \). Let \( y \in \varphi(X^r) \) and \( W \subset \varphi^{-1}(y) \) be an irreducible component of the fiber. Then \( \dim W \geq r - s \).

Proof. We have a finite map \( Y^s \to \mathbb{P}^s \) by Noether normalization. This fits into a composition
\[
X^r \to Y^s \to \mathbb{P}^s,
\]
so we may assume that \( Y^s = \mathbb{P}^s \) since the second map has finite fibers. By restricting to an affine open, we can assume \( Y = \mathbb{A}^s \) and \( y = 0 \). Let \( f_i = y_i \circ \varphi \), where \( y_i \) are the coordinate functions on \( \mathbb{A}^s \), so \( \varphi^{-1}(0) = V(f_1, \ldots, f_s) \). Now it suffices to show: if \( X^r \subset \mathbb{A}^n \) is affine, then every irreducible component of \( X^r \cap V(f) \) has dimension \( \geq r - 1 \).

We can assume that \( X^r \not\subset V(f) \); we will show that each irreducible component of \( X \cap V(f) \) has dimension \( r - 1 \). Replace \( X \) and \( V(f) \) by their closures in \( \mathbb{P}^n \). Well
$V(f) = V(F)$, where $F$ is the homogenization of $f$. Now, after taking the $d$-uple embedding we can assume $\deg F = 1$.

Now we show by induction on $n - r$ that if $X \subset \mathbb{P}^n$ is a projective variety, then every irreducible component of $X \cap H$ has dimension $r - 1$. It is clear if $n - r = 0$. If $n - r = 1$, then $X = V(f)$ is a hypersurface in $\mathbb{P}^n$. We can assume that $H = V(x_n)$, so $X \cap H$ is a hypersurface in $\mathbb{P}^{n-1}$.

If $n - r \geq 2$, let $O \in H, O \notin X \cap H$. Now $P_O : \mathbb{P}^n \to \mathbb{P}^{n-1}$ sends $H \cong \mathbb{P}^{n-1} \to P_O(H) \cong \mathbb{P}^{n-2}$. Observe that

$$P_O(X \cap H) = P_O(X) \cap P_O(H - \{O\}).$$

The inclusion $\subset$ is obvious; for the other direction, let $y \in P_O(X) \cap P_O(H - \{O\})$. There exists $x \in X$ such that $P_O(x) = y$, and $P_O^{-1}(P_O(H)) = H - O$, implying that $x \in H$.

Let $W \subset X \cap H$ be an irreducible component. Of course, $P_O(W)$ is irreducible in $P_O(X) \cap P_O(H)$. If $P_O(W)$ is a component of $P_O(X) \cap P_O(H - \{O\})$, then we are done by induction. We claim that we can find $O \in H - X \cap H$ so that this will be the case. Suppose $X \cap H = W \cup W^*$, where $W^*$ is the union of the other components. It is enough to show that there exists $O$ such that $P_O(W) \not\subset P_O(W^*)$.

To this end, pick $x \in W \setminus W^*$. Consider $P^{-1}_O(P_x(W^*)) = C(P_x(W^*))$. But $\dim W_i \leq \dim X - 1 = r - 1$. Therefore, $\dim C(P_x(W_i)) \leq r < n - 1 = \dim H$. So there exists $O \in H - C(P_x(W^*)) - W$. Now consider the projection from this point: we claim that $P_O(x) \not\in P_O(W^*)$, since otherwise $P_O(x) \in P_O(W^*) \implies O$ is on the line joining $x$ and a point in $W^*$, i.e. $O$ is in the cone over the image of projection from $x$.

**4.5 Separable morphisms**

**Definition 4.5.1.** Let $\varphi : X \to Y$ be a dominant morphism. We say $\varphi$ is separable if $k(X)$ is separably generated over $\varphi^*(k(Y))$.

**Example 4.5.2.** Let $F_p : X \to X$ be the Frobenius morphism. Then $F$ is dominant but not separable, since the map in the other direction is $k(X) \to k(X)$ sending $x \mapsto x^p$.

**Proposition 4.5.3.** Let $\varphi : X^r \to Y^r$ be a separable morphism. Then there exists an open subset $Y_0 \subset Y$ such that $\varphi^{-1}(y)$ is finite and $\# \varphi^{-1}(y) = [k(X) : k(Y)]$ for all $y \in Y_0$.

Before we give the proof, let us recall the primitive element theorem.

**Theorem 4.5.4** (Primitive Element theorem). Let $L/K$ be a finite separable extension. Then $L$ is generated over $K$ by one element. In addition, if $\beta_1, \ldots, \beta_r$ form a set of generators, then some linear combination $\alpha = \sum \lambda_i \beta_i \in K$. 

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Proof of Proposition 4.5.3. We begin with the special case where \( k(Y) = k(X) \). We can assume \( X, Y \) are affine, so \( Y \subset \mathbb{A}^m \) and \( X \subset \mathbb{A}^n \). Let \( x_1, \ldots, x_m \) be the coordinate functions on \( \mathbb{A}^m \). Write \( (\varphi^*)^{-1}(x_i) = \frac{f_i}{f_0}, \ f_i \in \mathcal{O}(Y) \). Consider \( Y_0 = Y - V(f_0) \). Then

\[
\varphi^{-1}(y) = \left( \frac{f_1(y)}{f_0(y)}, \ldots, \frac{f_n(y)}{f_0(y)} \right), \ y \in Y_0.
\]

In general, we can assume that \( k(X) = k(Y)[f] \), where \( f \in \mathcal{O}(X) \). Consider the map

\[
X \xrightarrow{\psi} Y \times \mathbb{A}^1 \xrightarrow{pr_1} Y
\]

and let \( Z = \overline{\psi(X)} \). Then we have a map of function fields in the other direction, \( k(Y) \to k(Z) \to k(X) \). But since \( k(Z) \) contains \( f \), it also surjects to \( k(X) \), so it is isomorphic to \( k(X) \).

\[
k(Y) \subset k(Z) \simeq k(X).
\]

There exists \( Z_0 \subset Z \) open such that \( \#\psi^{-1}(Z) = 1 \) for \( z \in Z_0 \). Therefore, we can replace \( X \) by \( Z \) and assume \( X \subset Y \times \mathbb{A}^1 \). Let \( P(t) \) be the minimal polynomial of \( f \), so

\[
P(t) = t^n + a_1 t^{n-1} + \ldots + a_n, \quad a_i \in k(Y).
\]

By replacing \( Y \) by an open subset, we may assume that \( a_i \in \mathcal{O}(Y) \). We have by assumption that \( k(X) = k(Y)[f]/P(f) \), so by replacing \( X \) by an open subset we may also assume hat \( \mathcal{O}(X) = \mathcal{O}(Y)[f]/P(f) \). In other words,

\[
X = \{(y, t) \in Y \times \mathbb{A}^1 \mid \sum a_i(y)t^{n-i} = 0 \}.
\]

For fixed \( y \), this is a degree \( n \) polynomial, and therefore has \( n \) solutions with multiplicity, corresponding to the points of the fiber \( \#\varphi^{-1}(y) \). We must prove that there is an open set over which there are exactly \( n \) solutions (without multiplicity).

Recall that by separability, \( p(t) \) and \( p'(t) \) have distinct roots in \( k(y)[t] \). So \( p(t) \) and \( p'(t) \) do not have common zeros in \( k(y)[t] \), so there exists \( a(t), b(t) \in k(y)[t] \) such that

\[
a(t)p(t) + b(t)p'(t) = 1.
\]

Clearing denominators, we have

\[
a(t)p(t) + b(t)p'(t) = c(y).
\]

Therefore, over the open subset \( Y_0 = \{ y \in Y \mid c(y) \neq 0 \} \), \( p(t) \) and \( p'(t) \) do not have common zeros. Over this open subset, \( \#\varphi^{-1}(y) = n = [k(X) : k(Y)] \). \( \square \)
4.6 Smooth morphisms

Let \( \varphi : X \to Y \) be a morphism, \( \varphi(x) = y \). Then there is a map \( \varphi^* : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x} \)
pulling back \( m_{Y,y} \mapsto m_{X,x} \). In particular, \( \varphi \) induces \( d\varphi_x : T_x X \to T_y Y \), since \( T_x X = m_x / m_x^2 \).

**Definition 4.6.1.** The morphism \( \varphi \) is smooth at \( x \) if (i) \( x \) (resp. \( \varphi(x) \)) is smooth in \( X \) (resp. in \( Y \)) and (ii) \( d\varphi_x : T_x X \to T_y Y \) is surjective.

**Proposition 4.6.2.** Let \( \varphi : X \to Y \) be a dominant morphism. Then the following are equivalent.

(i) \( \varphi \) is separable.

(ii) There exists \( X_0 \subset X \) open such that \( \varphi \) is smooth at \( x \in X_0 \).

(iii) There exists a point \( x \in X \) such that \( \varphi \) is smooth at \( x \).

**Proof.** First we show that \( (iii) \implies (ii) \). We may assume that \( X \) and \( Y \) are affine. By replacing \( X \) by the graph of \( \varphi \) in \( \mathbb{A}^n \times \mathbb{A}^m \), we assume that \( X \subset \mathbb{A}^n \times \mathbb{A}^m \) and \( \varphi \) is the projection to the second coordinate.

Let \( g_1, \ldots, g_p \) be generators of \( I(Y) \). Then \( \varphi^*(g_1), \ldots, \varphi^*(g_p), f_1, \ldots, f_q \) generate \( I(X) \) for some \( f_1, \ldots, f_q \). Let \( x \in X \) and \( y \in Y \). Then

\[
T_x X = \left\{ (\xi_1, \ldots, \xi_{n+m}) \in k^{n+m} \mid \left( \begin{array}{c} 0 \\ \frac{\partial g}{\partial x_i} \\ \frac{\partial f_j}{\partial x_i} \end{array} \right) \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.
\]

The projection to \( T_y Y \) maps \( (\xi_1, \ldots, \xi_{n+m}) \mapsto (0, \ldots, 0, \xi_{n+1}, \ldots, \xi_{n+m}) \). The kernel of \( d\varphi_x \) consists of

\[
\{(\xi_1, \ldots, \xi_n) \mid \sum_{j=1}^n \frac{\partial f_i}{\partial x_j} \xi_j = 0\} = \ker \left( \frac{\partial f_i}{\partial x_j} \right).
\]

Now, \( \varphi \) is smooth at \( x \iff \dim \ker d\varphi_x = r - s \). However, since the kernel always has dimension at least \( \geq r - s \), this is the same as saying \( \ker d\varphi_x = r - s \iff \rank \frac{\partial f_i}{\partial x_j} \geq n - (r - s) \). This defines an open subset (determined by the non-vanishing of minors). Therefore, \( (iii) \implies (ii) \).

Next we show \( (i) \implies (ii) \). We have

\[
\text{Der}_{k(Y)}(k(X), k(X)) \cong \text{Der}_{k(Y)}(k(Y) \otimes_{\mathcal{O}(Y)} \mathcal{O}(X), k(X)).
\]
This is
\[ \{ D \in \text{Der}_k(Y)(k(Y)[x_1, \ldots, x_n], k(X)) \mid Df_i = 0 \} \cong \{(\xi_1, \ldots, \xi_n) \in k(X)^n \mid \sum \xi_i \frac{\partial f_i}{\partial x_j} = 0\} \]
since \( k(Y) \otimes_{\mathcal{O}(Y)} \mathcal{O}(X) \cong k(Y)[x_1, \ldots, x_n]/(f_1, \ldots, f_q) \). Therefore,
\[ \dim_{k(X)} \text{Der}_k(Y)(k(X), k(X)) = n - \text{rank}_{k(X)} \left( \frac{\partial f_i}{\partial x_j} \right). \]

We know that there exists \( X_0 \subset X \) open such that
\[ \text{rank}_{k(X)} \left( \frac{\partial f_i}{\partial x_j} \right) = \text{rank}_{k(X)} \left( \frac{\partial f_i}{\partial x_j} \right) \quad \forall x \in X_0. \]

According to (i), \( k(X)/k(Y) \) is separable. By a fact from commutative algebra, \( k(X)/k(Y) \) is separable generated if and only if
\[ \dim_{k(X)} \text{Der}_k(Y)(k(X), k(X)) = r - s, \]
where \( r = \text{tr.deg} k(X) \) and \( s = \text{tr.deg} k(Y) \). Therefore, there exists an open \( X_0 \subset X \) such that
\[ \text{rank}_{k(X)} \left( \frac{\partial f_i}{\partial x_j} \right) = n - r + s. \]

This shows that (i) \( \implies \) (ii). Conversely, if \( \varphi \) is smooth on an open subset then \( \text{rank}_{k(X)} \left( \frac{\partial f_i}{\partial x_j} \right) = n - r + s \) on an open subset, and we can run the argument in reverse. \( \square \)

**Remark 4.6.3.** If \( \varphi : X \to Y \) is smooth over \( X_0 \subset X \), then \( \varphi \) is dominant, so this assumption was not necessary. Indeed, suppose that \( \varphi : X \to \varphi(x) \subset Y \) is a proper closed subset, so we have a factorization
\[ X \xrightarrow{\psi} Z = \varphi(X) \xrightarrow{\iota} Y. \]

Then \( d\varphi_x = dt_{\psi(x)} \circ d\psi_x \). Choose \( z \in Z \), \( Z \) smooth at \( z \), and \( z \in \varphi(x) \), so we have a factorization of the derivative map
\[ d\varphi_x : T_x X \to T_z Z \to T_z Y \]
so \( \dim T_z Z < T_z Y \), so the map on tangent spaces is not surjective, so \( d\varphi_x \) is not smooth.

**Corollary 4.6.4 (Generic smoothness).** Let \( \text{char} k = 0 \) and \( \varphi : X \to Y \) be dominant. Then there exists some (non-empty) Zariski-open subset \( Y_0 \subset Y \) such that \( \varphi^{-1}(Y_0) : \varphi^{-1}(Y_0) \setminus \text{Sing}(X) \rightarrow Y_0 \) is smooth.
Proof. We can assume that $X$ and $Y$ are smooth. In characteristic 0, any dominant morphism is separable. Let $X_0 \subset X$ be the open subset such that $\varphi$ is smooth at all $x \in X_0$. Let $Z = X \setminus X_0$. We need to show that $\psi = \varphi|_Z : Z \to Y$ is not dominant. Suppose for the sake of contradiction that $\psi : Z \to Y$ is dominant, so there exists $Z_0 \subset Z$ open such that $\psi|_{Z_0} : Z_0 \to Y$ is smooth. So we have a factorization

$$Z \hookrightarrow X \xrightarrow{\varphi} Y$$

inducing a factorization of the derivative maps

$$d\psi_z : T_zZ \xrightarrow{d\varphi_z} T_{\psi(z)}Y \xrightarrow{d\psi} T_zX$$

But by the definition of $z$, $d\varphi_z$ is not surjective, contradiction.

\[\square\]

**Proposition 4.6.5.** Let $\varphi : X^r \to Y^s$ be smooth at $x \in X$. Then there exists a unique component $Z$ of $\varphi^{-1}(\varphi(x))$ passing through $x$. In addition, $Z$ is smooth at $x$ and $\dim Z = \dim X - \dim Y$.

**Example 4.6.6.** Consider $X = V(xy-t) \subset \mathbb{A}^3$ projecting down to $\mathbb{A}^1$ by $(x,y,t) \mapsto t$. For all $t \neq 0$, the fiber is a hyperbola, and there is a unique component passing through a given point in the fiber. This breaks down at $t = 0$, where the fiber is the union of two axes.

Proof. We may assume that $0 \in X \subset \mathbb{A}^m \times \mathbb{A}^n$ mapping down to $0 \in Y \subset \mathbb{A}^m$. Choose $g_1, \ldots, g_{m-s} \in I(Y)$ with independent linear terms. Let $f_1, \ldots, f_{n-r+s} \in I(X)$ such that $(f_1, \ldots, f_{n-r+s}, g_1, \ldots, g_{m-s})$ have independent linear terms.

Note that $\dim_k \ker d\varphi = r - s$ implies that $\text{rank } \left( \frac{\partial f_i}{\partial x_j} \right)(0,0) = n - r + s$, so $f_1(x,0), \ldots, f_{n-r+s}(x,0)$ also have independent linear terms.

As we showed in Theorem 2.3.1, $V(f_1, \ldots, f_{n-r+s}, g_1, \ldots, g_{m-s}) = X \cup X'$ where $0 \notin X'$. Consider

$$V(f_1, \ldots, f_{n-r+s}, g_1, \ldots, g_{m-s}, y_1, \ldots, y_n) = pr_2^{-1}(0) \cap (X \cup X') = \varphi^{-1}(0) \cup (pr_2^{-1}(0) \cap X'),$$

the latter component not containing 0. This algebraic set may also be written as

$$V(f_1(x,0), \ldots, f_{n-r+s}(x,0)) \subset \mathbb{A}^n \times \{0\} = \mathbb{A}^n,$$

but by Theorem 2.3.1 again,

$$V(f_1(x,0), \ldots, f_{n-r+s}(x,0)) = Z \cup Z' \quad 0 \notin Z'.$$

So $Z \subset \varphi^{-1}(0)$ is the unique component passing through 0. $Z$ is smooth at 0, and $\dim Z = n - (n - r + s) = r - s$.  

\[\square\]
Theorem 4.6.7 (Zariski’s Main Theorem, smooth case). Let \( \varphi : X \to Y \) be a birational morphism of quasi-projective varieties. Assume \( Y \) is smooth. Let

\[ Y_0 = \{ y \in Y \mid \# \varphi^{-1}(y) = 1 \}. \]

Then

(i) \( \varphi|_{\varphi^{-1}(Y_0)} : \varphi^{-1}(Y_0) \to Y_0 \) is an isomorphism.

(ii) For any \( y \in Y \setminus Y_0 \) and \( x \in \varphi^{-1}(y) \), there exists \( E \subset X \) a subvariety of dimension \( r - 1 \) passing through \( x \) such that \( \varphi(E) \) has dimension \( \leq r - 2 \). In particular, for any \( y \in Y \setminus Y_0 \) there exists a component in \( \varphi^{-1}(y) \) of positive dimension.

To summarize, given a birational morphism with smooth target, the morphism restricted to the locus where it is bijective is an isomorphism. Furthermore, if the pre-image of \( y \in Y \) is not a single point, it has a component of positive dimension, called the exceptional divisor.

Proof. Let \( x \in X \) and \( y \in Y \). There is an induced map \( \varphi^* : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x} \) (in fact, this is injective, since the map on function fields is an isomorphism). Let

\[ Y_1 = \{ y \in Y \mid \exists x \in \varphi^{-1}(y) \text{ such that } \varphi^* \mathcal{O}_{Y,y} \simeq \mathcal{O}_{X,x} \}. \]

\[ Y_2 = \{ y \in Y \mid \varphi^* \mathcal{O}_{Y,y} \not\simeq \mathcal{O}_{X,x} \forall x \in \varphi^{-1}(y) \}. \]

We claim that \( Y_1 \subset Y_0 \) and \( Y_2 \subset Y \setminus Y_0 \) (this immediately implies that we have equality in both inclusions). Let \( y \in Y_1 \), so \( \mathcal{O}_{Y,y} \simeq \mathcal{O}_{X,x} \). If \( x' \in \varphi^{-1}(y) \) is another point in the fiber, then \( \mathcal{O}_{X,x} = \varphi^* \mathcal{O}_{Y,y} \hookrightarrow \mathcal{O}_{X,x'} \). Then the first part of the claim follows from this lemma:

Lemma 4.6.8. Let \( X \) be quasi-projective, \( x, x' \) two points such that

\[ \mathcal{O}_{X,x} \subset \mathcal{O}_{X,x'} \subset k(X). \]

Then \( x = x' \).

We already showed on the homework that an isomorphism of local rings implies an isomorphism of open neighborhoods, so the proof of this Lemma will finish off part (i).

Proof. We can assume that \( X \subset \mathbb{P}^n \) is projective. Choose a hyperplane not passing through \( x \) and \( x' \), we can assume that \( X \) is affine. So points of \( X \) correspond to maximal ideals of \( \mathcal{O}(X) \); and inclusion of the form \( \mathcal{O}_{X,x} \subset \mathcal{O}_{X,x'} \implies m_x \supset m_{x'} \). \( \square \)

Now we must establish the second part of the claim. We can choose \( f \in \mathcal{O}_{X,x} \) such that \( f = \varphi^*(\frac{a}{b}) \) for \( a, b \in \mathcal{O}_{Y,y} \), and \( b(y) = 0 \) since \( \varphi^*(\mathcal{O}_{Y,y}) \neq \mathcal{O}_{X,x} \).
Theorem 4.6.9. If \( y \in Y \) is smooth, then \( \mathcal{O}_{Y,y} \) is a UFD.

The proof of this theorem is omitted; it follows from generalities on commutative algebra.

So we may assume that \( a \) and \( b \) are coprime in \( \mathcal{O}_{Y,y} \). Let \( b = \beta b' \), where \( \beta \) is a prime element. Then

\[
p = \mathcal{O}(Y) \cap \beta \mathcal{O}_{Y,y}
\]

is a prime ideal.

Let \( E \) be a component of \( V(\varphi^* \beta) \cap X \) containing \( x \). Then \( \dim E = r - 1 \). So

\[
\varphi^*(a) = f \varphi^*(b) = f \varphi^*(\beta) \varphi^*(b') \implies \varphi^*(a) \text{ vanishes on } E.
\]

So \( a \) vanishes on \( \overline{\varphi(E)} \). But \( a, b \) coprime implies that \( a \notin p \). Therefore, \( \overline{\varphi(E)} \subset V(p) \) is a strict inclusion, so \( \dim \overline{\varphi(E)} \leq r - 2 \).

\[ \square \]

Example 4.6.10. Consider the blow-up of \( \mathbb{A}^2 \), \( Z = V(xu - yv) \subset \mathbb{A}^2 \times \mathbb{P}^1 \).

\[
Z \to \mathbb{A}^2 \quad \text{where } (x, y) \times (u, v) \mapsto (x, y).
\]

We know already that this is an isomorphism outside the origin \( O = (0, 0) \), \( P = (0, 0, 0, 1) \in E \), etc. but let’s trace through the proof of Zariski’s main theorem in this case.

We have an inclusion \( \varphi^*: \mathcal{O}_{\mathbb{A}^2, O} \to \mathcal{O}_{Z, P} \). We can find a function in \( \mathcal{O}_{Z, P} \) not in the image of \( \varphi \). Indeed, consider the open subset \( Z - \{ u = 0 \} = V(xu - y) \subset \mathbb{A}^2 \times \mathbb{A}^1 \). Then \( u \), which is morally \( \varphi^*(\frac{y}{x}) \), is not in the image of \( \mathcal{O}_{\mathbb{A}^2, O} \). Then \( E = V(\varphi^* x) \subset Z - \{ v = 0 \} \) is defined by \( x = 0 \) and \( xu - y = 0 \), i.e. \( x = y = 0 \). We see that \( E \) is a projective line, and the image of \( E \) lies in the intersection of the \( x \) and \( y \) axes, i.e. \( O \).

Remark 4.6.11. We saw on the homework that the image of \( \varphi: Z - \{ v = 0 \} = V(xu - y) \to \mathbb{A}^2 \) is \( \mathbb{A}^2 \) with the \( y \) axis removed and the origin put back in.
Chapter 5

Divisors and Line Bundles

5.1 Divisors

We now turn to the theory of divisors on a smooth algebraic variety $X$.

**Lemma 5.1.1.** Let $X^r$ be an affine variety of dimension $r$, $x \in X$. Let $f \in \mathcal{O}_{X,x}$ be an irreducible element, so $f\mathcal{O}_{X,x} \cap \mathcal{O}(X) = \mathfrak{p}$ is a prime ideal. Then $X' = V(\mathfrak{p})$ is a subvariety of dimension $r - 1$ containing $x$.

Conversely, suppose $X' \subset X$ is a subvariety of dimension $r - 1$ and $x \in X'$. Then there exists $f \in \mathcal{O}_{X,x}$ irreducible such that $X' = V(\mathfrak{p})$, where $\mathfrak{p} = f\mathcal{O}_{X,x} \cap \mathcal{O}(X)$. In other words, $I(X')\mathcal{O}_{X,x} = f \mathcal{O}_{X,x}$.

*Proof.* By taking an open subset of $X$ if necessary, we may assume that $f \in \mathcal{O}(X)$. Then there exists an open subset $U \subset X$ such that $V(f) \cap U = V(\mathfrak{p}) \cap U$. But each irreducible component of $V(f)$ has dimension $r - 1$, so $\dim X' = r - 1$.

Conversely, choose $f \in I(X')$. Then $f$ is not a unit in $\mathcal{O}_{X,x}$, so we may factor $f$ into a product of primes. So, restricting to an open subset, we may assume that $f$ is prime in $\mathcal{O}_{X,x}$. Then $X' \subset V(f)$, where $\dim X' = r - 1$, so $X'$ is an irreducible component of $V(f)$, implying that $X' = V(\mathfrak{p})$.

*Definition 5.1.2.* Let $X' \subset X^r$ be a closed subvariety of dimension $r - 1$, $x \in X'$. An element of $\mathcal{O}_{X,x}$ is called a local equation of $X'$ at $x$ if $f\mathcal{O}_{X,x} = I(X')\mathcal{O}_{X,x}$.

We have just seen that every co-dimension 1 subvariety has a local equation.

*Definition 5.1.3.* Let $X$ be a smooth algebraic variety. The divisor group $\text{Div}(X)$ is the free abelian group generated by the co-dimension 1 subvarieties.

In other words, elements of the divisor group are finite formal linear combinations of co-dimension 1 subvarieties.
Lemma 5.1.4. There exists a natural group homomorphism

\[ k(X)^* \to \text{Div}(X) \]
\[ f \mapsto (f). \]

**Proof.** We write \((f) = \sum n_Z Z\) and we need to define \(n_Z \in \mathbb{Z}\). Choose any \(x \in Z\); in \(\mathcal{O}_{X,x}\) choose a local equation \(f_Z\). Then we may write

\[ f = gh, \quad g, h \in \mathcal{O}_{X,x}, \gcd(gh, f_Z) = 1. \]

Then we define \(n_Z = \text{ord}_{Z,x}(f) := r\). To see that this is well-defined, note that different choices of the local equation differ by units, and the function \(x \mapsto \text{ord}_{Z,x}(f)\) is locally constant.

In a neighborhood \(U\) of \(x\), we have \(U \cap Z = V(f_Z) \cap U\). Therefore, \(f_Z\) is a local equation for every \(y \in Z \cap U\). By shrinking \(U\) if necessary, we can assume that \(g\) and \(h\) are regular on \(U\) so that the above equation still holds in the local ring of any other \(y \in U\). Therefore, \(\text{ord}_{Z,y}(f) = r = \text{ord}_{Z,x}(f)\) for all \(y \in U \cap Z\).

Next, we have to show that for all but finitely many \(Z\), \(\text{ord}_Z(f) = 0\). By taking an open subset, we may assume that \(X\) is affine (since the complement has only finitely many irreducible components). Write \(f = \frac{g}{h}\) where \(g, h \in \mathcal{O}(X)\), so \(\text{ord}_Z(f) \neq 0 \implies Z \subset V(gh)\). But \(V(gh)\) has only finitely many irreducible components. \(\square\)

Intuitively, \(\text{ord}_Z(f)\) measures the order of zero or pole that \(f\) has on \(Z\). From the proof, we see that the following identities hold, as we expect.

1. \(\text{ord}_Z(fg) = \text{ord}_Z(f) + \text{ord}_Z(g)\).
2. \(\text{ord}_Z(f + g) \geq \min\{\text{ord}_Z(f), \text{ord}_Z(g)\}\) if \(f + g \neq 0\).

**Definition 5.1.5.** \(\text{Div}^0(X) = \text{Im}(k(X)^* \to \text{Div}(X))\) is called the group of **principal divisors**. \(\text{Pic}(X) = \text{Div}(X)/\text{Div}^0(X)\) is called the **Picard group** (or the divisor class group) of \(X\).

This is an extremely important group attached to an algebraic variety, like cohomology for topological spaces.

**Example 5.1.6.** Let \(X = \mathbb{A}^n\). Then \(\text{Pic}(X) = 0\), since every subvariety of codimension 1 is a hypersurface. So

\[ \sum n_i Z_i = \sum n_i V(f_i) = \sum n_i(f_i) = (\prod f_i^{n_i}). \]

More generally, we have the following proposition.

**Proposition 5.1.7.** Let \(X\) be a smooth affine variety. Then \(\mathcal{O}(X)\) is a UFD if and only if \(\text{Pic}(X) = 0\).
Proof. Assume \( \mathcal{O}(X) \) is a UFD. Let \( Z \subset X \) be codimension 1, choose \( f \in I(Z) \) (so \( f \) is not a unit). By replacing \( f \) by one of its prime factors, we may assume that \( f \) is irreducible. Therefore, \( (f) \) is prime, hence \( Z \subset V(f) \) is a closed subvariety of an irreducible variety, and they have the same dimension, implying that \( Z = V(f) \).

Conversely, suppose that \( \text{Pic}(X) = 0 \). Then \( Z = (f) \) for \( f \in k(X) \). We claim that in fact, \( f \in \mathcal{O}(X) \). To this end, it suffices to show that \( f \in \bigcap_{x \in X} \mathcal{O}_{X,x} \). By definition, in each such local ring

\[
f = \frac{g}{h} f_Z,
\]

but \( h \) must be a unit, since otherwise \( f \) has non-zero order at some other closed subvariety of co-dimension 1.

If \( g \in \mathcal{O}(X) \) is irreducible, then \( V(g) \) is irreducible because if \( Z \subset V(g) \), then \( Z = V(f) \), so \( f \mid g \). This means that if \( g \mid fh \), then

\[
V(g) \subset V(fh) = V(f) \cap V(h) \implies V(g) \subset V(f) \text{ or } V(g) \subset V(h) \implies g \mid f \text{ or } g \mid h.
\]

\( \square \)

Example 5.1.8. Let \( X = \mathbb{P}^n \). We have a map \( \text{deg} : \text{Div}(X) \to \mathbb{Z} \) given by

\[
Z = V(f) \mapsto \text{deg } f = \deg.
\]

This is clearly a surjective map, and \( \text{Div}^0(X) \subset \ker(\text{deg}) \) since elements of \( \text{Div}^0(X) \) can be written as \( \frac{f}{g} \), where \( f \) and \( g \) are homogeneous polynomials of the same degree.

Exercise 5.1.9. Check that the divisor of \( \prod f_i^{n_i} \prod g_j^{m_j} \) is \( \sum n_i V(f_i) - \sum m_j V(g_j) \).

Also note that the divisor of a constant function is 0. So we have a short exact sequence

\[
0 \to k^* \to k(X)^* \to \text{Div}(X) \xrightarrow{\text{deg}} \mathbb{Z} \to 0.
\]

Now let \( D = \sum n_i Z_i \) such that \( \text{deg } D = 0 \). Since all codimension 1 subvarieties of \( \mathbb{P}^n \) are hypersurfaces, \( Z_i = V(f_i) \) for each \( i \). So

\[
\prod f_i^{n_i} = \frac{\prod_{n_k > 0} f_k^{n_k}}{\prod_{n_j < 0} f_j^{-n_j}} = \frac{f}{g} \implies D = \left( \frac{f}{g} \right)
\]

Definition 5.1.10. A prime divisor is \( D = Z \), where \( Z \) is a subvariety of codimension 1.

Definition 5.1.11. An effective divisor is \( D = \sum n_i Z_i, \ n_i \geq 0 \).

Lemma 5.1.12. Let \( f \in k(X)^* \). Then \( (f) \) is effective if and only if \( f \in \mathcal{O}(X) - 0 \).

Proof. We essentially proved this already. We can assume that \( X \) is affine, where the coordinate ring is the intersection of the local rings. But \( \text{ord}_Z(f) \geq 0 \implies f \in \mathcal{O}_{X,x} \) for all \( x \in X \).

\( \square \)
Corollary 5.1.13. Let $X$ be smooth, $F \subset X$ a closed codimension $\geq 2$ subvariety. Then $\mathcal{O}(X) \simeq \mathcal{O}(X - F)$.

Proof. This follows from the above lemma and the observation that the divisor group is unchanged by removing a codimension 2 subvariety. \qed

Let $U \subset X$ be open. Then we get a restriction map $\text{Div}(X) \to \text{Div}(U)$. If $X = U \cup V$, then we have an exact sequence

$$0 \to \text{Div}(X) \to \text{Div}(U) \oplus \text{Div}(V) \to \text{Div}(U \cap V).$$

5.2 Linear Systems

Definition 5.2.1. Let $X$ be a projective variety. Let $D \in \text{Div}(X)$. Define

$$L(D) = \{ f \in k(X) \mid f = 0 \text{ or } (f) + D \geq 0 \}.$$

Note that $L(D)$ has a natural $k$-vector space structure, since $\text{ord}_Z(f + g) \geq \min\{\text{ord}_Z(f), \text{ord}_Z(g)\}$. We also define $|D|$ to be the set of all effective divisors of the form $(f) + D$ for some $f \in k(X)$. This corresponds to the one-dimensional subspaces of $L(D)$, giving a natural identification

$$|D| \simeq \mathbb{P}(L(D)).$$

This is because

$$(f) + D = (g) + D \implies (f) = (g) \implies (\frac{f}{g}) = 0 \implies f \in \mathcal{O}(X)^* = k^*$$

since $X$ is projective. Later, we will see that $\dim L(D) =: \ell(D) < \infty$. Therefore, $|D|$ has a natural structure as a projective space.

Definition 5.2.2. A linear subvariety of $|D|$ is called a linear system.

Definition 5.2.3. A linear system $\mathcal{D}$ is called complete if it is of the form $|D|$ for some $D$.

Definition 5.2.4. Let $L$ be a linear system on $X$. The base points of $L$ are defined to be

$$\bigcap_{D \in L} \text{Supp}(D) \quad \text{where if } D = \sum n_i Z_i \text{ then } \text{Supp}(D) = \bigcup_{n_i \neq 0} Z_i.$$

One of the most important questions in additive function theory is to calculate the quantity $\ell(D) = \dim L(D)$.

Let $X \subset \mathbb{P}^n$ be a projective variety, $S(X)$ its homogeneous coordinate ring, with the natural grading by degree $S(X) = \bigoplus S(X)_d$. Let $f \in S(X)_d$, $H_i = V(X_i)$. A
priori $f$ is not a well-defined function on $X$, but $f_{X_i}$ defines a regular function on $X \cap (\mathbb{P}^n - H_i)$. Therefore, 

$\left( \frac{f}{X_i^d} \right)$ is an effective divisor on $X \cap (\mathbb{P}^n - H_i)$.

Moreover, note that 

$$
\left( \frac{f}{X_i^d} \right)_{|X \cap (\mathbb{P}^n - H_i - H_j)} = \left( \frac{f}{X_j^d} \right)_{|X \cap (\mathbb{P}^n - H_i - H_j)}
$$

since $X_i$ and $X_j$ are both nonvanishing on the intersection. Therefore, there exists an effective divisor $(f) := X \cdot V(f)$ on $X$ whose restriction to $X \cap (\mathbb{P}^n - H_i)$ is $(\frac{f}{X_i^d})$.

Another way to reach this definition is to consider $X \cap V(f)$. This is a union of codimension 1 components 

$$
X \cap V(f) = \bigcup Z_i.
$$

Then we define $X \cdot V(f) = \sum n_i Z_i$, where $n_i$ is determined as follows. Pick $X_{k(i)}$ such that $Z_i \not\subset H_{k(i)}$. Then set 

$$
n_i := \text{ord}_{Z_i} \frac{f}{X_{k(i)}^d}.
$$

In this way, we define a map 

$$
S(X)_d \to \text{Div}(X).
$$

We claim that the image is a linear system. This is because (exercise) 

$$(f) = (g) + \left( \frac{f}{g} \right), \quad \frac{f}{g} \in k(X).$$

Fix $g$. Every other $(f)$ is 

$$(f_1) = (g) + \left( \frac{f_1}{g} \right)$$

$$(f_2) = (g) + \left( \frac{f_2}{g} \right)$$

$$(f_1 + f_2) = (g) + \left( \frac{f_1 + f_2}{g} \right).$$

To summarize, this linear system, denoted by $L_X(d)$, is isomorphic to $\mathbb{P}(S(X)_d)$.

**Theorem 5.2.5.** For $d \gg 0$, $L_X(d)$ is complete.

Now we return to the theorem we stated before.
Theorem 5.2.6. For \( d \gg 0 \), \( L_X(d) \) is complete.

Proof. Let \( D \in |L_X(d)| \). We know that \( L_X(d) \) contains \( (x_i^d) = dH_i \), so

\[
D - dH_i = (f_i) \quad f_i \in k(X).
\]

Then \( \frac{D}{f_j} = (D - dH_i) - (D - dH_j) = d(H_j - H_i) \). Therefore, in the fraction field \( L \) of \( S(X) \), we have \( x_i^d f_i = x_j^d f_j \) up to some constant which we normalize to be 1. Thus \( F := x_i^d f_i \) is well-defined.

Note that \( (f_i)|_{\mathbb{P}^n-H_i} \) is effective, so \( f_i \in \mathcal{O}(X - H_i \cap X) \), i.e. \( x_i^n f_i \in S(X) \). Therefore, \( x_i^{N-d} F = x_i^{n-d} x_i^d f_i \in S(X)_N \).

Now define \( S'_d \subset L \) by

\[
S'_d = \{ s \in L \mid \exists N \text{ such that } x_i^N s \in S(X)_{N+d} \forall i \}
\]

and set \( S' = \bigoplus_{d \geq 0} S'_d \). So \( S' \) is a graded ring containing \( S(X) \). Also, \( F \in S' \), and to say \( D \in L_X(d) \) is equivalent to saying that \( F \in S(X) \), since \( F = x_i^d f_i \) and \( D = (f_i) + dH_i \). So to show that the linear system is complete for all sufficiently large \( d \), it suffices to show that \( S'_d = S(X)_d \) for \( d \gg 0 \).

Fix \( s \in S'_d, d \geq 0 \). We have \( x_i^N s \in S(X)_{N+d} \) for some \( N \), and taking \( \ell_0 = N(n+1) \) we get

\[
S(X)_{\ell_0} \subset S(X)_{\ell+d} \text{ for } \ell \geq \ell_0.
\]

Therefore,

\[
S(X)_{\ell_0} S^q \subset S(X) \forall q \geq 0.
\]

So

\[
x_0^{\ell_0} s^q \in S(X) \forall q \geq 0 \implies s^q \in \frac{1}{x_0^{\ell_0}} S(X) \forall q \geq 0.
\]

Consider \( M = \frac{1}{x_0^{\ell_0}} S(X) \subset L \). This is a f.g. \( S(X) \)-module, so \( s \) satisfies a polynomial relation

\[
s^q + a_1 s^{q+1} + \ldots + a_q = 0 \quad a_i \in S(X),
\]

i.e. \( s \) is integral over \( S(X) \). Therefore, \( S' \) is integral over \( S(X) \).

Theorem 5.2.7. Let \( A \) be a finitely generated integral \( k \)-algebra, \( K = \text{Frac}(A) \). Let \( L/K \) be a finite field extension. Then the integral closure \( A' \) of \( A \) in \( L \) is finite over \( A \) and \( A' \) is also a finitely generated \( k \)-algebra.

Using this fact from commutative algebra, we know that \( S' \) is finite over \( S(X) \). Let \( s_1, \ldots, s_m \) be a set of homogeneous generators of degree \( d_i \). There exists \( \ell_0 \) such that \( x_i^s j \in S(X)_{\ell+d} \) for any \( \ell \geq \ell_0 \). Therefore, \( S'_d = S(X)_d \) if \( d \geq \max \{d_i \} + \ell_0 \).

Note that if \( D_1 \sim D_2 \), then \( \mathcal{L}(D_1) \simeq \mathcal{L}(D_2) \), since

\[
D_1 - D_2 = (f) \implies \mathcal{L}(D_1) \xrightarrow{f} \mathcal{L}(D_2)
\]

is an isomorphism.
Corollary 5.2.8. We have \( \ell(D) < \infty \).

Proof. If \(|D| \neq \emptyset\), we can assume that \( D = \sum n_i Z_i \) is effective. Choose \( F_i \) homogeneous of degree \( d_i \) such that \( Z_i \subset X \cap V(F_i) \) but \( X \not\subset V(F_i) \). Let \( F = \prod F_i^{n_i} \), a homogeneous polynomial of degree \( e = \sum n_i d_i \) of degree \( e = \sum n_i d_i \). Then

\[
(F) = G = D + D', \text{ where } D' \text{ is effective.}
\]

Therefore, \( \mathcal{L}(D) \subset \mathcal{L}(G) \), so \( \ell(D) \leq \ell(G) \). By taking some multiple of \( G \), we may assume that \( \mathcal{L}(G) \) is complete by the theorem. Therefore, \( \ell(G) = \dim L_X(e) < \infty \). \( \square \)

Let \( \varphi : X \dashrightarrow \mathbb{P}^n \) be a rational map such that \( \varphi(X) \) is not contained in any hyperplane. We will construct a linear system \( L_\varphi \) on \( X \) as follows. By definition, we have a diagram

\[
\begin{array}{ccc}
Z = \Gamma_\varphi & \xrightarrow{pr_1} & X \\
& \xleftarrow{pr_2} & \xrightarrow{\varphi} \mathbb{P}^n
\end{array}
\]

where \( pr_1 : Z \to X \) is birational. Let \( F \subset X \) be the subvariety such that \( pr_1 : pr_1^{-1}(F) \to F \) is not an isomorphism over \( F \). By Zariski’s Main Theorem, \( \text{codim } F \geq 2 \) (since \( \dim pr_1^{-1}(F) \leq r - 1 \)).

Now let \( H = V(\ell) \subset \mathbb{P}^n \) be a hyperplane. We define a divisor \( \varphi^*(H) \) on \( X \) as follows: since \( F \) has codimension at least two, divisors on \( X \) are the same as divisors on \( X - F \). Define \( \varphi^*H \) on \( X - F - \varphi^{-1}(H_i) \) as \( (\varphi^*(\frac{x_i}{x_j})) \). One can check the this agrees on the intersections \( X - F - \varphi^{-1}(H_i \cup H_j) \), so this extends to a divisor on \( X - F \).

It is easy to see that if \( H = V(\ell) \) and \( H' = V(\ell') \), then \( \varphi^*H = \varphi^*H' + (\varphi^*(\frac{x_i}{x_j})) \). Therefore, \( \varphi^*H \) form a linear system, denoted by \( L_\varphi \simeq \mathbb{G}(n - 1, n) = (\mathbb{P}^n)^* \).

Proposition 5.2.9. The base point locus of \( L_\varphi \) is \( F \).

Proof. If \( x \notin F \), then \( \varphi(x) \in \mathbb{P}^n \) and there exists \( H \) such that \( \varphi(x) \notin H \). Therefore, \( x \notin \varphi^{-1}(H) = \text{Supp}(\varphi^*H|_{X - F}) \).

Conversely, let \( H \subset \mathbb{P}^n \) be a hyperplane. We need to show that \( F \subset \text{Supp}(\varphi^*H) \).

By changing coordinates if necessary, we may assume that \( H = H_0 \). Then \( \varphi^*\frac{x_i}{x_0} \) is a rational function on \( X \) with divisor \( \varphi^*H_i - \varphi^*H_0 \). Therefore, \( \varphi^*\frac{x_i}{x_0} \) is a regular function on \( X - \text{Supp}(\varphi^*H_0) \), so \( \varphi|_{X - \text{Supp}(\varphi^*H_0)} \) is a morphism given by

\[
p \mapsto (\varphi^*(\frac{x_i}{x_0}(p)), \ldots, \varphi^*(\frac{x_1}{x_0}(p))).
\]

But we know that this rational map is not regular on \( F \), so \( F \subset \text{Supp}(\varphi^*(H_0)) \). \( \square \)

Theorem 5.2.10. There is a \( 1 \to 1 \) correspondence between
(a) A linear system \( L \) on \( X \), with basepoints of codimension \( \geq 2 \), and an isomorphism \( \psi : L \simeq (\mathbb{P}^n)^* \).

(b) A rational map \( \varphi : X \dashrightarrow \mathbb{P}^n \) such that \( \varphi(X) \) is not contained in any hyperplane.

In addition, this correspondence is \( \text{PGL}_{n+1} \) equivariant.

In other words, given a linear system \( L \) on \( X \) with basepoints of codimension \( \geq 2 \), there exists a rational map \( \varphi : X \dashrightarrow \mathbb{P}^n \) such that \( \varphi(X) \) is not contained in any hyperplane.

Remark 5.2.11. There is a natural map \( V \rightarrow V^{**} \) when \( V \) is finite dimension. Well, \( \mathbb{P}(V^*) \) is the projectivization of the dual space, so its elements correspond naturally to hyperplanes in \( \mathbb{P}(V) \), i.e. \( \mathbb{P}(V)^* \).

Suppose \( V,W \) are two vector spaces. We can take their projectivizations: \( \mathbb{P}V \) and \( \mathbb{P}W \). If there is an isomorphism \( \psi : \mathbb{P}V \simeq \mathbb{P}W \), then there exists a unique lift \( \tilde{\psi} : V \xrightarrow{\simeq} W \) which is unique up to scaling by a constant.

Proof. We have already shown how (b) gives rise to (a). Conversely, let \( X \) be smooth and projective, \( L \) a linear system on \( X \) with base points of codimension \( \geq 2 \). Then \( L = \mathbb{P}V \) for some \( V \subset \mathcal{L}(D), \ D \in L \). Fix an isomorphism \( \psi : L \simeq (\mathbb{P}^n)^* \), which gives a lift \( \tilde{\psi} : V \rightarrow k\{x_0, \ldots, x_n\} \), where the \( x_i \) are coordinates on \( \mathbb{P}^n \), which is unique up to scaling by a constant. Let \( D_i = \psi^{-1}(H_i), \ H_i = V(x_i) \). The point is that we can regard \( D_i \) as “coordinates” on \( X \), since they correspond to functions “up to constants.” So we write \( D_i - D_0 = (f_i) \), where \( f_i \) is a rational function, regular \( X - \text{Supp}(D_0) \). Define a map

\[
x \mapsto (f_0(x), f_1(x), \ldots, f_m(x)).
\]

It is evident that this construction is equivariant with respect to \( \text{PGL}_{n+1}(\mathbb{C}) \).

Example 5.2.12. Let \( X = \mathbb{P}^n \), \( L \subset L_X(1) \), where \( L_X(1) \) is the linear system of hyperplanes on \( \mathbb{P}^n \), i.e. \( (\mathbb{P}^n)^* \). Then the rational map \( \varphi_L : X \dashrightarrow L^* \) is the projection \( P_M \) away from \( M = \bigcap_{H \in L} H \).

Conversely, given given \( M \subset \mathbb{P}^n \) of dimension \( n - r - 1 \), we may define

\[
L = \{H \in (\mathbb{P}^n)^* \mid H \supset M\}.
\]

Then \( \varphi_L = P_M \). More generally, given \( L_1 \subset L_2 \subset (\mathbb{P}^n)^* \), we have the diagram
where \( M = \bigcap_{H \in L} H \subset L_x^* \).

**Example 5.2.13.** Let \( X = \mathbb{P}^n \), \( L = L_X(d) \). Then \( \varphi_L : X \rightarrow L^* \) is the \( d \)-uple embedding.

### 5.3 Differentials

Now we turn to the construction of a divisor class called the **canonical class**.

**Definition 5.3.1.** Let \( A \rightarrow B \) be a ring homomorphism. The Kähler differentials is a pair \( (\Omega_{B/A}, d) \), where \( \Omega_{B/A} \) is a \( B \)-module and \( d \in \text{Der}_A(B, \Omega_{B/A}) \) are characterized by the universal property that for any \( B \)-module \( M \), the natural map \( \text{Hom}_B(\Omega_{B/A}, M) \rightarrow \text{Der}_A(B, M) \) given by \( \varphi \mapsto \varphi \circ d \) is an isomorphism.

In categorical language, the Kahler differential represents the functor of \( B \)-modules sending \( M \mapsto \text{Der}_A(B, M) \).

**Theorem 5.3.2.** \( (\Omega_{B/A}, d) \) exists and is unique up to unique isomorphism.

**Proof.** Uniqueness follows from the usual abstract nonsense. For existence, we construct \( \Omega_{B/A} \) as the free \( B \)-module on the symbols \( db \) for \( b \in B \), modded out by the relations \( da = 0 \), \( d(b_1 + b_2) = db_1 + db_2 \) and \( d(b_1b_2) = b_1db_2 + b_2db_1 \). Then \( d : B \rightarrow \Omega_{B/A} \) is defined by \( d(b) = db \).

For any \( D : B \rightarrow M \), we defined \( \varphi : \Omega_{B/A} \rightarrow M \) by sending \( \varphi(db) = D(b) \).

**Example 5.3.3.** If \( A = k \), \( B = k[x_1, \ldots, x_n] \) then \( \Omega_{B/A} \cong Bdx_1 \oplus \cdots \oplus Bdx_n \).

**Corollary 5.3.4.** If \( B/A \) is finitely generated as an \( A \)-algebra, with generators \( b_1, \ldots, b_r \), then \( \Omega_{B/A} \) is finitely generated over \( B \) with generators \( db_1, \ldots, db_r \).

**Lemma 5.3.5.** Let \( S \subset B \) be a multiplicative set. Then \( S^{-1}\Omega_{B/A} \cong \Omega_{S^{-1}B/A} \).

**Proof.** \( \text{Der}_A(S^{-1}B, M) = \text{Der}_A(B, M) \) for any \( S^{-1}B \)-module \( M \).

**Corollary 5.3.6.** Let \( X/k \) be an algebraic variety. Then \( \Omega_{\mathcal{O}_{X,x}/k} \) is a finitely generated \( \mathcal{O}_{X,x} \)-module.

**Notation.** Let \( X/k \) be affine. We denote \( \Omega_X := \Omega_{\mathcal{O}(X)/k} \) and \( \Omega_{X,x} := \Omega_{\mathcal{O}_{X,x}/k} \).

**Remark 5.3.7.** If \( X \) is not affine, we denote \( \Omega_X \) to be the functor from the category of affine opens on \( X \) to abelian groups, \( U \mapsto \Omega_U \). In other words, \( \Omega_X \) is a sheaf on \( X \).

**Proposition 5.3.8.** Let \( X/k \) be an algebraic variety. Then \( X \) is smooth at \( x \) if and only if \( \Omega_{X,x} \) is a free module over \( \mathcal{O}_{X,x} \) of rank \( \dim X = r \).
Proof. We claim that \( \Omega_{X,x} \otimes \mathcal{O}_{X,x} k \simeq T_x^* X = (T_x X)^* \). In other words,

\[
\text{Hom}_k(\Omega_{X,x} \otimes \mathcal{O}_{X,x} k, k) \simeq \text{Hom}_{\mathcal{O}_{X,x}}(\Omega_{X,x}, k) \simeq \text{Der}_k(\mathcal{O}_{X,x}, k) = T_x X.
\]

If \( \Omega_{X,x} \) is free of rank \( r \), then \( \dim_k \Omega_{X,x} \otimes \mathcal{O}_{X,x} k = r = \dim T_x X \), so \( X \) is smooth at \( x \).

Conversely, suppose \( X \) is smooth at \( x \), so \( \dim_k \Omega_{X,x} \otimes \mathcal{O}_{X,x} k = r \). Let \( K = k(X) \); we claim that \( \dim_K \Omega_{X,x} \otimes K = r \). This is because

\[
\text{Hom}_K(\Omega_{X,x} \otimes \mathcal{O}_{X,x} K, K) = \text{Hom}_{\mathcal{O}_{X,x}}(\Omega_{K/k}, K) = \text{Der}_k(K, K) = r.
\]

We will be done after showing the following lemma.

**Lemma 5.3.9.** Let \((A, m, k)\) be a Noetherian local ring and \( M \) be a finite \( A \)-module, \( K = \text{Frac}(A) \). If

\[
\dim_k M \otimes_A k = r = \dim_K M \otimes_A K
\]

then \( M \) is a free \( A \)-module of rank \( r \).

Our proof of this lemma uses Nakayama’s lemma, which we will state but not prove.

**Lemma 5.3.10.** Let \((A, m, k)\) be a Noetherian local ring, \( M \) a finite \( A \)-module. If \( M \otimes_A k = 0 \), then \( M = 0 \).

**Proof of Lemma 5.3.9.** Let \( m_1, \ldots, m_r \in M \) be elements whose images in \( k \) form a basis of \( M \otimes_A k \). Consider the map of \( A \)-modules

\[
A^r \to M
\]

\[
e_i \mapsto m_i
\]

We claim that this is an isomorphism. Consider the exact sequence of the cokernel

\[
A^r \to M \to N \to 0
\]

then upon tensoring with \( k \), we get

\[
A^r \otimes k \to M \otimes k \to N \otimes k \to 0
\]

but \( N \otimes k = 0 \) \( \implies \) \( N = 0 \) by Nakayama’s lemma. Therefore, the map is surjective.

Now consider the exact sequence of the kernel:

\[
0 \to N \to A^r \to M \to 0.
\]

Tensoring with \( K \), we get the exact sequence

\[
0 \to N \otimes K \to A^r \otimes K \to M \otimes K \to 0
\]

implying that \( N \otimes K = 0 \) since the map on the right is a surjection of vector space of the same dimension. But since \( N \) is a submodule of a free module, it is torsion-free, so this is only possible if \( N = 0 \). \( \Box \)
In the proof, we showed $\Omega_{X,x} \otimes k \simeq T^*_x X = m_x/m_x^2$. Let’s give a more explicit construction of this isomorphism. Recall that we have a derivation $d : \mathcal{O}_{X,x} \to \Omega_{X,x}$. One can check that $m_x^2$ goes to 0 in the map to $\Omega_{X,x} \otimes k$, so the map factors through the quotient.

\[
\begin{array}{ccc}
m_x \subset \mathcal{O}_{X,x} & \overset{d}{\longrightarrow} & \Omega_{X,x} \\
\downarrow & & \downarrow \\
m_x/m_x^2 & \longrightarrow & \Omega_{X,x} \otimes k
\end{array}
\]

In particular, if $\{x_1, \ldots, x_r\} \subset m_x$ are such that $x_1, \ldots, x_r$ (mod $m_x^2$) are a basis of $m_x/m_x^2$ then $dx_1, \ldots, dx_r$ form a basis of $\Omega_{X,x}$ over $\mathcal{O}_{X,x}$. Such $x_1, \ldots, x_r$ are called local parameters of $X$ at $x$. So every $f \in \mathcal{O}_{X,x}$ may be written as

\[
df = \sum \frac{\partial f}{\partial x_i} dx_i
\]

where $\frac{\partial}{\partial x_i} \in \text{Hom}_{\mathcal{O}_{X,x}}(\Omega_{X,x}, \mathcal{O}_{X,x}) = \text{Der}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}, \mathcal{O}_{X,x})$ are the dual basis.

Let $X$ be affine. Observe that every element in $\Omega_X$ can be regarded as a map from $X$ to $\bigcup_{x \in X} T^*_x X$. For $x \in X$, we have

\[
\Omega_x \to \Omega_{X,x} \to \Omega_{X,x} \otimes k \simeq T^*_x X
\]

\[
\omega \mapsto \omega_\ast \mapsto \omega(x)
\]

Now we want to give another definition of $\Omega_x$ without all of this commutative algebra. Let $f \in \mathcal{O}(X)$; we know $df$ should be in $\Omega_X$. The map given by $df$ in this case is

\[
x \mapsto f - f(x) \pmod{m_x^2}.
\]

So we define $\Omega_X$ to be the maps $\omega : X \to \bigcup_{x \in X} T^*_x X$ such that for every $p \in X$, there exists an open neighborhood $V$ containing $p$ such that there exist $f_i, g_i \in \mathcal{O}(V)$ with $\omega = \sum_i f_idg_i$. This is another way of understanding the Kahler differentials: as a map to the disjoint union of cotangent spaces given locally by regular functions. Moreover, this makes sense for arbitrary varieties. Our old definition clearly has a natural inclusion into this set; it is left as an exercise to check that they are the same.

### 5.4 The canonical bundle

**Definition 5.4.1.** Let $X^r$ be a smooth variety of dimension $r$. The canonical form is

\[
\omega_{X,x} := \wedge_{\mathcal{O}_{X,x}}^r \Omega_{X,x}
\]

is a free $\mathcal{O}_{X,x}$-module of rank 1, with a basis given by $dx_1 \wedge \ldots \wedge dx_r$. 

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Definition 5.4.2. We define $\omega_{K/k} := \wedge r \Omega_{K/k}$. This is a 1-dimensional $K$-vector space; elements in $\omega_{K/k}$ are called rational $r$-forms.

Our goal is to associate to each element in $\omega_{K/k}$ a divisor. Let $w \in \omega_{K/k}$ be non-zero. Recall that $\omega_{K/k} \cong \omega_{X,x} \otimes_{\mathcal{O}_{X,x}} K$, so $\omega = f dx_1 \wedge \ldots \wedge dx_r$, where $f \in K$ and $x_1, \ldots, x_r$ are local parameters for $Z_i$. Define

$$(\omega) = \sum n_i Z_i, \quad n_i = \text{ord}_{Z_i} f.$$ 

We have made many non-canonical choices here: the point $x$ and the local parameters $x_1, \ldots, x_r$. However, one can check everything is well-defined; this is left as an exercise.

It is clear from this construction that for $f \in K$, $(f \omega) = (f) + (\omega)$. Therefore $\{(\omega) \mid \omega \in \omega_{K/k} - \{0\}\}$ is a well-defined divisor class, called the canonical divisor class (usually denoted $K_X$). Likewise, we define the $n$-fold canonical class as follows: let $\omega \in \omega_{K/k}^{\otimes n}$. This is a one-dimensional space over $K$, whose elements look like

$$\omega = f(dx_1 \wedge \ldots \wedge dx_r)^n.$$ 

We then define

$$(\omega) = \sum n_i Z_i \quad n_i = \text{ord}_{Z_i} f \quad \text{for} \ x \in Z_i.$$ 

Notice that if $\omega_1 \in \omega_{K/k}^{n_1}$ and $\omega_2 \in \omega_{K/k}^{n_2}$, then $(\omega_1 \otimes \omega_2) = (\omega_1) + (\omega_2)$. In particular, $\omega \in \omega_{K/k}^{n} - \{0\}$ implies that $(\omega) \sim nK_X$.

Now we turn to a discussion of linear systems. Assume $X$ is projective. For $\omega \in \omega_{K/k}^{\otimes n}$, we may define

$$\mathcal{O}((\omega)) = \{ f \in K \mid (f) + (\omega) = (f \omega) \geq 0 \} = \{ \omega \in \omega_{K/k}^{\otimes n} \mid (\omega) \geq 0 \}.$$ 

This does not depend on the choice of $\omega$, so we denote it by $\mathcal{L}(nK_X)$.

Definition 5.4.3. Let $P_n(X) = \ell(nK_X)$ is the $n$th plurigenera of $X$. In particular, $P_1(X) =: P_g$ is called the geometric genus of $X$.

Note that $\bigoplus_{n \geq 0} \mathcal{L}(nK_X)$ has a natural $k$-algebra structure. This is called the canonical ring of $X$. The following theorem was just proved.

Theorem 5.4.4 (Siu, Hacon, McKernan, ...). The canonical ring is a finitely generated $k$-algebra.

Example 5.4.5. Let $X = \mathbb{P}^n$. We claim that $K_X = -(n+1)H$. In particular, this implies that $P_g(\mathbb{P}^n) = 0$.

Let $x_0, \ldots, x_n$ be homogeneous coordinates on on $\mathbb{P}^n$. Let $y_i = \frac{x_i}{x_0}$. Let

$$\omega = dy_1 \wedge \ldots \wedge dy_n \in \omega_{K/k}.$$
Therefore, \((\omega)|_{\mathbb{P}^n - H_0} = 0\), so \((\omega) = mH_0\). Now let \(z_i = \frac{x_i}{x_n}\), so \(y_n = \frac{1}{z_0}\) and \(y_i = \frac{z_i}{y_0}\) for \(1 \leq i \leq n - 1\). In these coordinates,
\[
\omega = d\left(\frac{z_1}{z_0}\right) \wedge \ldots \wedge d\left(\frac{z_{n-1}}{z_0}\right) \wedge d\left(\frac{1}{z_0}\right) = \pm \frac{1}{z_0^n+1} dz_0 \wedge \ldots \wedge dz_{n-1}.
\]
So we see that \((\omega) = -(n+1)H_0\).

**Example 5.4.6.** Let \(X = V(y^2z - x^3 - xz^2) \subset \mathbb{P}^2\), char \(k \neq 2\). We claim that \(K_X = 0\), so \(P_g(X) = 1\).

On \((\mathbb{P}^2 - H_z) \cap X\), the curve is defined by \(V(y^2 - x^3 - x)\). Here
\[
\Omega_{K/k} = \langle dx, dy \mid 2ydy - (3x^2 + 1)dx \rangle.
\]
So \(\Omega_{X,(a,b)} \otimes k\) as a vector space over \(k\) is \(\langle dx, dy \mid 2bdy - (3a^2 + 1)dx \rangle\). Therefore, \(dx \neq 0\) if \(b \neq 0\), so \(x\) is a local parameter at \((a,b)\) if \(b \neq 0\). Let
\[
\omega = \frac{dx}{y} = \frac{2dy}{3x^2 + 1}.
\]

From the first expression, it is clear that the divisor is supported possibly on \(y = 0\) (in this chart) and the line at infinity. If at \((a,b)\), \(3a^2 + 1 \neq 0\), then \(dy \neq 0\) so \(y\) is a local parameter and \(\omega\) is not supported at such points. But by the non-singularity of the curve, these two cover all of this affine chart.

Now we look at \(X \cap H_z = p = (0,1,0)\). On the affine \(y \neq 0\), the curve is \(V(z - x^3 - xz^2)\). So
\[
\Omega_{X,p} \otimes k = \langle dx, dz \mid dz = 3x^2 dx + z^2 dx + 2xzdz \rangle = \langle dx, dz \mid dz \rangle
\]

So \(x\) is a local parameter at \(p\). The coordinate change is \(x \mapsto x/z\), and \(y \mapsto y/z\), so in the new local coordinates
\[
\omega = \frac{dx}{y} = \frac{d(x/z)}{y/z} = \frac{zdx - xdz}{z}.
\]

From the relation, we know that \(dz = 3x^2 dx + z^2 dx + 2xzdz\), so
\[
dz = \frac{(3x^2 + z^2)dx}{1 - 2xz}.
\]

From the relation we know \(\frac{x^3}{z} = 1 - xz\), so
\[
\omega = dx - \frac{x}{z} \cdot \frac{3x^2 + z^2}{1 - 2xz} dx = dx - \frac{3x^3/z + xz}{1 - 2xz} \cdot dx
\]
\[
= dx - \frac{3 - 3xz + xz}{1 - 2xz} dx = dx - \frac{3 - 2xz}{1 - 2xz} dx
\]
\[
= \frac{-2}{1 - 2xz} dx.
\]
Proposition 5.4.7. If $X, Y$ are birational then $P_n(X) = P_n(Y)$.

Proof. If $X$ and $Y$ are birational, then by Zariski’s Main Theorem we have a morphism $U \rightarrow V$ defined away from codimension 2. Since codim $(X \setminus U) \geq 2$ and codim $(Y \setminus V) \geq 2$, $U$ and $V$ have the same divisor groups as $X$ and $Y$, respectively. Note that $k(X) \simeq k(Y)$, so $\omega_{k(X)/k} \simeq \omega_{k(Y)/k}$. These two facts together imply that there is a canonical bijection $\mathcal{L}(nK_X) \simeq \mathcal{L}(nK_Y)$. □

5.5 The ramification divisor

Let $\varphi : X \rightarrow Y$ be a morphism of algebraic varieties. We wish to define $\varphi^* : \Omega_Y \rightarrow \Omega_X$. One way is purely through commutative algebra, and another way is geometric. We will discuss the first way and sketch the second.

More generally, if we have a map of commutative rings

$$A \rightarrow B \xrightarrow{\varphi} C$$

then there is a natural map

$$\Omega_{B/A} \otimes_C C \rightarrow \Omega_{C/A} \xrightarrow{\varphi^*} \Omega_{C/B} \rightarrow 0.$$  

For any $C$-module $M$ (which is also a $B$-module via $\varphi$), $\text{Hom}_B(\Omega_{B/A}, M) \simeq \text{Der}_A(B, M)$. Furthermore, there is a natural map $\text{Der}_A(B, M) \leftarrow \text{Der}_A(C, M) \simeq \text{Hom}_C(\Omega_{C/A}, M)$ obtain by composing with $\varphi$. Taking $M = \Omega_{C/A}$, define $\varphi^* := \text{res}(\text{Id}) : \Omega_{B/A} \otimes_B C \rightarrow \Omega_{C/A}$.

Remark 5.5.1. Some remarks.

(i) From the construction, we see that $\varphi^*(df) = d(\varphi^* f)$.

(ii) The kernel of $\varphi^* : \text{Der}_A(C, M) \rightarrow \text{Der}_A(B, M)$ is precisely $\text{Der}_B(C, M)$.

Therefore, we have a right exact sequence

$$\Omega_{B/A} \otimes_B C \rightarrow \Omega_{C/A} \rightarrow \Omega_{C/B} \rightarrow 0.$$  

Here is the geometric way of seeing this map. Let $\omega \in \Omega_y$, $\omega : Y \rightarrow \bigsqcup_{y \in Y} T^*_y Y$. We want a map $\varphi^* \omega : X \rightarrow \bigsqcup_{x \in X} T^*_x X$. We let $\varphi^* \omega(x) = d\varphi^* \omega(\varphi(x))$ (note that $\omega(\varphi(x)) \in T^*_x Y$, and $d\varphi^* : T^*_{\varphi(x)} Y \rightarrow T^*_x X$). If $\omega = df$, where $f \in \mathcal{O}(Y)$, then $\varphi^*(df) = d(\varphi^* f) = d(f \circ \varphi)$. Therefore, if $\omega = \sum f_i dg_i$, then locally on $X$

$$\varphi^* \omega = \sum f_i \circ \varphi d(g_i \circ \varphi).$$

Now specialize to the case $A = k, B = k(Y), C = k(X)$, and $\varphi^* : k(Y) \rightarrow k(X)$ is induced by a dominant morphism $\varphi : X \rightarrow Y$. By the discussion above, we have a right exact sequence

$$\Omega_{k(Y)/k} \otimes k(X) \rightarrow \Omega_{k(X)/k} \rightarrow \Omega_{k(X)/k(Y)} \rightarrow 0.$$
If \( \varphi : X \to Y \) is separable, then

\[
\dim_{k(X)} \Omega_{k(X)/k(Y)} = \dim X - \dim Y.
\]

This is because \( \text{Der}_k(K, K) \) has dimension the transcendental degree of \( K \) over \( k \) if the extension is separable, and \( \text{Hom}_K(\Omega_{K/k}, k) \). But then we can see by counting the dimensions that the sequence is also left-exact. In particular, if \( \dim X = \dim Y \) then this tells us that \( \Omega_{k(Y)/k \otimes k(Y)} k(X) \simeq \Omega_{k(X)/k} \). Then we get an induced map

\[
\varphi^* : \omega_{k(Y)/k \otimes k(Y)} k(X) \simeq \omega_{k(X)/k}.
\]

Let \( \omega \in \omega_{k(Y)/k} \). Then \( \varphi^*(\omega \otimes 1) \in \omega_{k(X)/k} \).

In the homework we also introduced the divisor \( \varphi^{-1}(\omega) \). This was a map on divisor groups \( \varphi^{-1} : \text{Div}(Y) \to \text{Div}(X) \) defined as follows. If \( D = \sum n_i Y_i \in \text{Div}(Y) \), then \( \varphi^{-1}D = \sum n_i \varphi^{-1}(Y_i) \). Write \( \varphi^{-1}(Y_i) = \sum m_j X_j \). Choose \( x \in X_j \) and a local equation \( f_{Y_i} \) at \( \varphi(x) \), and define \( m_j = \varphi_{X_j} \varphi^*(f) \). It is left as an exercise on the homework to show that this process gives a well-defined divisor class and is a group homomorphism.

**Lemma 5.5.2.** \( B = \varphi^*(\omega \otimes 1) - \varphi^{-1}(\omega) \) is an effective divisor, called the branch divisor.

**Proof.** Write \( \omega = f dy_1 \wedge \ldots \wedge dy_x \), where \( (y_1, \ldots, y_r) \) are local parameters at \( y = \varphi(x) \), so \( (x_1, \ldots, x_r) \) are local parameters at \( x \). Then

\[
\varphi^* \omega = (f \circ \varphi) dy_1 \wedge \ldots \wedge dy_r = (f \circ \varphi) \det \left( \frac{\partial(y_i \circ \varphi)}{\partial x_j} \right) dx_1 \wedge \ldots \wedge dx_r.
\]

Around \( x \),

\[
\varphi^* \omega - \varphi^{-1}(\omega) = (f \circ \varphi \det \left( \frac{\partial(y_i \circ \varphi)}{\partial x_j} \right)) - (f \circ \varphi) = (\det \left( \frac{\partial(y_i \circ \varphi)}{\partial x_j} \right))
\]

In other words,\)

\[
\text{Supp}(B) = \{ x \in X \mid \text{det} \frac{\partial(y_i \circ \varphi)}{\partial x_j} \in \mathfrak{m}_x \}
\]

\[
= \{ x \in X \mid (d\varphi)^* : T_{\varphi(x)}^* Y \to T_{x}^* X \text{ not an isomorphism.} \}
\]

\[
= \{ x \in X \mid (d\varphi)^* \text{ not smooth at } x. \}
\]

**Example 5.5.3.** Let \( X = (y^2 z = x^3 + x z^2) \subset \mathbb{P}^2 \). The projection \([x, y, z] \mapsto [x, z]\) defines a map \( \varphi : X \to \mathbb{P}^1 \).

Let \([x, y, z]\) be homogeneous coordinates on \( \mathbb{P}^2 \), \([u, v]\) homogeneous coordinates on \( \mathbb{P}^1 \). Then \( \varphi \) sends \( X' := \{ z \neq 0 \} \to \mathbb{A}^1 := \{ v \neq 0 \} \). So \( \varphi^* : k[u] \to \)
\[ k[x, y]/(y^2 - x^3 - x) \text{ sending } u \mapsto x. \] Let \( du \in \Omega_{k[\mathbb{P}^3]/k}. \) Then \( \varphi^{-1}(du) = 0 \) on \( X'. \) On the other hand, \( (\varphi^*du) = (dx). \) Now, \( x \) is a local parameter on \( x - \{y = 0\} = \{(1, 0), (0, 0), (-1, 0)\}, \) at which \( y \) is a local parameter, and

\[ dx = \frac{2ydy}{3x^2 + 1}. \]

So \( (dx) = [1] + [0] + [-1] \) on \( X'. \) Finally, we need to check the point at infinity, \( p = (0, 1, 0). \) There is a difficulty here: the map isn’t even defined at \( p. \) To rectify this, consider the blowup of \( X \) at \( p. \)

The equations

\[
\begin{align*}
y^2z &= x^3 + xz^2 \\
xv &= zu
\end{align*}
\]

cut out \( \tilde{X} \) along with an exceptional component. In \( \mathbb{A}^2 \times \mathbb{A}^1 \) defined by \( y \neq 0 \) and \( v \neq 0, \) the system is

\[
\begin{align*}
z &= x^3 + xz^2 \\
xv &= z
\end{align*}
\]

So \( \tilde{X} \cap (\mathbb{A}^2 \times \mathbb{A}^1) \) has coordinate ring \( k[x, z, v]/(xv - z, v - x^2 - x^2z^2). \) This shows how to extend the map to \( p. \)

The form is \( \omega = du = -\frac{dv}{y^2}. \) The point \( \tilde{p} \) has \( x = 0, z = 0, v = 0. \) Recall from the computation last time that \( x \) is a local parameter in first factor. So \( B \) around \( p \) is given by \( \frac{d(v \circ \varphi)}{dx} = \frac{dv}{dx}. \)

\[
v = x^2 + x^2v^2 \implies dv = 2xdx + 2xv^2dx + 2x^2vdv \implies \frac{dv}{dx} = \frac{2x + 2xv^2}{1 - 2x^2v}.
\]

This vanishes to order 1.

We conclude that \( B = [1] + [0] + [-1] + [\infty]. \)
Chapter 6

The Hilbert Polynomial

6.1 Construction of the Hilbert Polynomial

**Theorem 6.1.1.** Let $M$ be a finitely generated graded module over $S = k[x_0, \ldots, x_n]$. Then there exists a polynomial $P_M(t) \in \mathbb{Q}[t]$ of degree at most $n$ such that $P_M(d) = \dim_k M_d$ for $d \gg 0$.

**Proof.** We induct on $n$. If $n = -1$, then the result is trivial since $M$ is then a finite dimensional vector space over $k$. Now suppose the theorem holds for $< n$. Let $N' = \{ m \in M \mid x_n m = 0 \}$ and $N'' = M/x_n M$. Then $N', N''$ are finitely generated graded $k[x_0, \ldots, x_{n-1}]$ modules. By induction, there exists $P_{N'}(t) \in \mathbb{Q}[t]$ and $P_{N''}(t) \in \mathbb{Q}[t]$ with $P_{N'}(d) = \dim_k N'_d$ and $P_{N''}(d) = \dim_k N''_d$ for $d \gg 0$. We have an exact sequence

$$0 \to N'_d \to M_d \xrightarrow{x_n} M_{d+1} \to N''_{d+1} \to 0.$$ 

So $\dim M_{d+1} - \dim M_d = \dim N''_{d+1} - \dim N'_d = P_{N''}(d+1) - P_{N'}(d)$. But $R(t) := P_{N''}(t+1) - P_{N'}(t) \in \mathbb{Q}[t]$ with degree at most $n - 1$.

**Lemma 6.1.2.** Let $f(t) \in \mathbb{Q}[t]$ of degree $m$. Then there exists $g(t) \in \mathbb{Q}[t]$ of degree $m + 1$ such that $g(t+1) - g(t) = f(t)$.

**Exercise 6.1.3.** Easy exercise.

So there exists $P(t) \in \mathbb{Q}[t]$ such that $P(t+1) - P(t) = P_{N''}(t+1) - P_{N'}(t) = \dim M_{d+1} - \dim M_d$ for $d \gg 0$. Then $\dim M_{d+1} - P(d+1) = \dim M_d - P(d)$ for $d \gg 0$, implying that $\dim M_d - P(d)$ is constant for all sufficiently large $d$. We then set $P_M(d)$ to be the constant plus $P(d)$.

In particular, let $M = S/I(X)$ where $X$ is an algebraic set in $\mathbb{P}^n$. Then $P_X := P_{S/I(X)}$ is called the Hilbert polynomial of $X$. 

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Example 6.1.4. (i) Let $X = \mathbb{P}^n$. Then $M = S$, and $M_d$ which has dimension $\binom{n+d}{n}$.

$$P_{\mathbb{P}^n}(t) = \binom{t+n}{n} = \frac{1}{n!}t^n + \ldots$$

(ii) Now suppose $X = V(f) \subset \mathbb{P}^n$. Then $S(X) = S/f$. Suppose $\deg f = d$.

$$0 \to S_m \xrightarrow{\times f} S_{m+d} \to S(X)_{m+d} \to 0$$

So $\dim_k S(X)_{m+d} = \dim_k S_{m+d} - \dim_k S_m = \binom{m+d+n}{n} - \binom{m+n}{n}$ for $m \geq d$. So

$$P_X(t) = \binom{t+n}{n} - \binom{t-d+n}{n} = \frac{d}{(n-1)!} t^{n-1} + \ldots$$

(iii) Let $X = \{p_1, \ldots, p_m\} \subset \mathbb{P}^n$. We may assume that $p_i \notin H_0 = V(X_0)$. We have a left exact sequence

$$0 \to I(X)_d \to S_d \to \mathbb{k}^m$$

$$f \mapsto \left(\frac{f}{x_0^d(p_1)}, \ldots, \frac{f}{x_0^d(p_m)}\right).$$

For $d \gg 0$ (e.g. $d > m$), this is also surjective. To see this, we can just consider products of hyperplanes passing through one of the points and no others. So for $d \gg 0$, $S(X)_d \simeq \mathbb{k}^m$, implying that $P_X(t) = m$.

Remarks. Let $X$ be a smooth, projective variety.

(i) Let $L_X(d) = \mathbb{P}(S(X)_d)$, so $\dim L_X(d) = P_X(d) - 1$ for $d \gg 0$. We proved that this linear system is complete for $d \gg 0$, so $\dim |dH| = \dim L_X(d)$ for $d \gg 0$.

In other words,

$$P_X(d) = \ell(dH) \quad d \gg 0.$$  

(ii) We saw that the canonical ring $\bigoplus_{d \geq 0} \mathcal{L}(dK_X)$ was finitely generated. There exists a polynomial $P(t)$ such that

$$P(d) = \ell(dK_X) \quad d \gg 0.$$  

The degree of $P(d)$ is called the Kodaira dimension of $X$, which can be any integer in $[-1,d]$. (Degree $-1$ means the polynomial is the zero polynomial)

6.2 Numerical invariants

Definition 6.2.1. Let $X^r \subset \mathbb{P}^n$ be projective. We define the arithmetic genus of $X$ to be

$$p_a(X) = (-1)^r (P_X(0) - 1).$$
Theorem 6.2.2. (i) If $X \simeq Y$ as varieties, then $p_a(X) = p_a(Y)$. (ii) If $X$ and $Y$ are smooth and birational, then $p_a(X) = p_a(Y)$.

Example 6.2.3. (i) Since we calculated $P_{\mathbb{P}^n}(t) = \binom{t+n}{n}$, we have $p_a(\mathbb{P}^n)$.
(ii) For $X = V(f)$, deg $f = d$, we saw that
\[
P_X(t) = \binom{t+n}{n} - \binom{t+n-d}{n} \implies P_X(0) = 1 - \binom{n-d}{n}.
\]
Therefore,
\[
p_a(X) = (-1)^n \binom{n-d}{n} = \begin{cases} 0 & d \leq n \\ \binom{d-1}{n} & d > n \end{cases}
\]
So $p_a(y_2 - x^3) = p_a(y^2z = x^3 + z^2x) = p_a(y^2z = x^3 + z^2x) = 1$. Note that we really need smoothness in the second part of the theorem, since singular cubic curves are birational to $\mathbb{P}^1$.

Theorem 6.2.4 (Serre duality). If $X$ is a smooth projective curve, then $p_a(X) = p_9(X)$.

Proposition 6.2.5. Let $X^r \subset \mathbb{P}^n$ be projective. Then
\[
P_X(t) = \frac{d}{r!} t^r + \ldots,
\]
where $d \in \mathbb{Z}$ is positive. This $d$ is called the degree of $X$.

Proof. We saw that this is true if $X$ is a hypersurface. Now let $M \subset \mathbb{P}^n$ be a $(n-r-2)$-plane such that $P_M : X \to X' \subset \mathbb{P}^{r+1}$ is birational. We know that (i) $S(X)$ is a finite $S(X')$-module, and (ii) $k(X) \simeq k(X')$. So $\text{Frac}(S(X)) \simeq \text{Frac}(S(X')) \simeq k(X)(x_0)$ where $x_0 \neq 0$ in $S(X')$.

Let $f_1, \ldots, f_s$ be the generators of $S(X)$ over $S(X')$. We can write $f_i := \frac{g_i}{h_i}$, where $g_i, h_i \in S(X')$. Setting $h = \prod h_i$, we have $S(X) \subset \frac{1}{h}S(X')$, i.e. $hS(X) \subset S(X')$. Write $h = \sum_d h_d$, where each $h_d$ is homogeneous of degree $d$. Since $S(X)$ and $S(X')$ are homogeneous, $h_dS(X) \subset S(X')$ for each $d$. So we can assume that there exists $h \in S(X')$ homogeneous of degree $d_0$ such that $hS(X) \subset S(X')$. Then
\[
\dim S(X') \leq \dim S(X) \leq \dim S(X)_{d+d_0}.
\]
So
\[
P_{X'}(d) \leq P_X(d) \leq P_{X'}(d + d_0).
\]
Then $P_X - P_X'$ is a polynomial of degree less than $\deg P_{X'}(t)$. Therefore $P_{X'}(t) = \frac{D}{r!} t^r + \ldots$, so $P_X$ has the same degree with the same leading coefficient. \qed
6.3 Dimension theory of intersections

**Proposition 6.3.1.** Let $Z^n$ be an affine variety and $X^r, Y^s$ closed subvarieties. Let $x \in X^r \cap Y^s$. Assume that $x$ is smooth in $Z^n$, and write

$$X^r \cap Y^s = W_1 \cup \ldots \cup W_r \cup W^s,$$

where the $W_i$ are irreducible components and $x \notin W^s$. Then $\dim W_i \geq r + s - n$.

**Corollary 6.3.2.** Let $X^r, Y^s \subset \mathbb{P}^n$ be projective with $r + s \geq n$. Then $X^r \cap Y^s \neq \emptyset$.

**Proof.** Take the affine cone $C(X^r), C(Y^s) \subset \mathbb{A}^{n+1}$. Then $0 \in C(X^r) \cap C(Y^s)$ contains 0, and $\dim C(X^r) = r + 1, \dim C(Y^s) = s + 1$, we have $\dim C(X^r) \cap C(Y^s) \geq (r + 1) + (s + 1) - (n + 1) \geq 1$.

**Proof of Proposition 6.3.1.** We can consider $X \cap Y = (X \times Y) \cap \Delta$, where $\Delta$ is the image of the diagonal embedding $Z \to Z \times Z$. The image has dimension $r + s$, and we know that the subvariety cut out by one equation has dimension at least 1 less, so it is enough to show that around $(x, x)$, $\Delta$ is cut out by $n$ equations $f_1, \ldots, f_n$.

Let $x_1, \ldots, x_n$ be local parameters at $x \in Z$. By shrinking $Z$ if necessary, we may assume that the $x_i$ are regular functions on $Z$, i.e. $x_i \in \mathcal{O}(Z)$. Consider $f_i = x_i \otimes 1 - 1 \otimes x_i \in \mathcal{O}(Z \times Z)$. In other words, $f_i(p, p') = x_i(p) - x_i(p')$. Then clearly $f_i \mid_{\Delta} = 0$. But $df_1, \ldots, df_n$ are linearly independent in $T_{(x,x)}(Z \times Z) = T_x^* Z \oplus T_x^* Z$ because $m_{(x,x)} = m_x \otimes \mathcal{O}(Z) + \mathcal{O}(Z) \otimes m_x$, so $x_1 \otimes 1, \ldots, x_n \otimes 1, 1 \otimes x_1, \ldots, 1 \otimes x_n$ are local parameters at $(x, x)$.

Then $f = (f_1, \ldots, f_n) : Z \times Z \to \mathbb{A}^n$ is smooth at $(x, x)$, so $f^{-1}(0) = Z^* \cup Z^{**}$ where $(x, x) \in Z^*$ and $(x, x) \notin Z^{**}$, $\dim Z^* = n$ and $Z^*$ is smooth at $(x, x)$. But also $\Delta \subset f^{-1}(0)$ and contains $(x, x)$ and also has the same dimension, so $Z^* = \Delta$.

**Theorem 6.3.3.** Let $X^r \subset \mathbb{P}^n$ have degree $d$. Then there exists an open subset $U \subset G(n-r, \mathbb{P}^n)$ such that for any $L \in U$,

(i) $\#(L \cap X) = \{p_1, \ldots, p_k\}$, and

(ii) For any $p_i \in X^r \cap L$, where $X^r$ and $L$ are smooth at $p_i$, we have $T_{p_i} X^r + T_{p_i} L = T_{p_i} \mathbb{P}^n$ (iff $T_{p_i} X^r \cap T_{p_i} L = 0$). In this case, $k = d$.

**Definition 6.3.4.** Let $X^r, Y^s \subset Z^n$, where $Z^n$ is smooth. Let $X^r \cap Y^s = \bigcup W_i$, where $W_i$ are irreducible components.

(i) We say that $X$ and $Y$ intersect properly at $W_i$ if $\dim W_i = r + s - n$.

(ii) We say that $X$ and $Y$ intersect transversely at $W_i$ if there exists $x \in W_i$ such that (i) $X, Y$ are smooth at $x$ and (ii) $T_x X + T_x Y = T_x Z$.

**Lemma 6.3.5.** If $X$ and $Y$ intersect transversely at $W$, and $x \in W$ as in the definition, then $W$ is smooth at $x$.
Proof. This is similar to Proposition 6.3.1. We have $X \cap Y = (X \times Y) \cap \Delta$. Locally, the diagonal $\Delta$ is cut out by $n$ equations $f_1, \ldots, f_n$, and $X \cap Y = \varphi^{-1}(0)$ where $\varphi : X \times Y \to \mathbb{A}^n$ is the map defined by $f_1, \ldots, f_n$. The condition $T_xX + T_xY = T_xZ$ implies that $\varphi$ is smooth at $(x, x)$, and then we can apply our results on smooth morphisms: there exists a unique component of the pre-image passing through $x$, and this component is smooth at $x$. \hfill \Box

Theorem 6.3.6. Let $X^r, Y^s \subset \mathbb{P}^n$ be projective varieties intersecting transversely. Then $\deg X \deg Y = \sum_{W_i \subset X \cap Y} \deg W_i$.

Proof. For $W_i$, let $W'_i \subset W_i$ be the open subset consisting of $x$ such that $X, Y$ are smooth at $x$ and $T_xX + T_xY = T_xZ$. We have shown that such a point exists on each component, but since it is an open condition there is automatically an open dense subset of such points.

**Step 1.** First, suppose $Y$ is a plane $L^s$, so $X^r \cap Y^s = \bigcup W_i$ with $\dim W_i = r + s - n$. Choose $M^{2n-r-s}$ such that $M \cap W_i$ transversely for any $i$ (this is possible since there is a Zariski open set of choices for each $W_i$, so the intersection is nonempty).

$$\#(X \cap (L \cap M)) = \#((X \cap L) \cap M) = \sum \#(W_i \cap M) = \sum \deg W_i$$

and $L \cap M$ is an $n - r$ plane.

**Step 2.** Suppose $r + s = n$. Consider $X \subset \mathbb{P}^n \subset \mathbb{P}^{2n+1}$ embedded by $(a_0, \ldots, a_n) \mapsto (a_0, \ldots, a_n, 0, \ldots)$ and $Y \subset \mathbb{P}^n \subset \mathbb{P}^{2n+1}$ embedding in the last coordinates. Define the join of $X$ and $Y$:

$$J(X, Y) = \bigcup_{x \in X, y \in Y} x \overline{y} = \{(a_0, \ldots, a_{2n+1}) \mid (a_0, \ldots, a_n, 0, \ldots) \in X \text{ and } (\ldots, 0, a_{n+1}, \ldots, a_{2n+1}) \in Y\}.$$

Note that $S(J(X, Y)) = S(X) \otimes S(Y)$. Therefore,

$$\dim S(J(X, Y))_\ell = \sum_{i=0}^{\ell} \dim S(X)_i \dim S(Y)_{\ell-i}.$$

By definition, we have

$$\dim S(X)_i = d_x \binom{i + r}{r} + o(i^r) + \ldots$$

$$\dim S(Y)_{\ell-i} = d_y \binom{\ell - i + s}{s} + o((\ell - i)^s).$$
So then
\[
\dim S(J(X, Y)) = \sum_{i=0}^{\ell} \left( d_x \left( \frac{i + r}{r} \right) + o(i^r) \right) \left( d_y \left( \frac{\ell - i + s}{s} \right) + o((\ell - i)^s) \right)
\]
\[
= d_x d_y \sum_{i=0}^{\ell} \left( \frac{i + r}{r} \right) \left( \frac{\ell - i + s}{s} \right) + o(\ell^{r+s+1})
\]
\[
= d_x d_y \left( \frac{\ell + r + s + 1}{r + s + 1} \right) + o(\ell^{r+s+1}).
\]
This tells us that
\[
P_{J(X,Y)}(t) = \frac{d_x d_y}{(r + s + 1)!} t^{r+s+1} + \ldots,
\]
i.e. \( J(X, Y) \) has dimension \( r + s + 1 \) and degree \( d_x d_y \).

Now comes the trick. Take \( L = (x_0 - x_{n+1}, x_1 - x_{n+2}, \ldots, x_n - x_{2n+1}) \). Then \( L \cap J(X, Y) = X \cap Y \) and the intersections being transverse in both cases is equivalent. But \( L \) is an \( n \)-plane, so \# \( X \cap Y = \# L \cap J(X, Y) = d_x d_y \).

**Step 3.** We now turn to the general case. Choose a \( M^{2n-r-s} \) plane intersecting each \( W_i \) transversely. Again,
\[
\# (M \cap (X \cap Y)) = \# (M \cap (\bigcup W_i)) = \sum \deg W_i.
\]
On the other hand, \( M \cap (X \cap Y) = X \cap (Y \cap M) \). We claim that \( Y \) intersects \( M \) transversely: let \( x \in Y \cap M \). By assumption, \( T_x W_i + T_x M = T_x \mathbb{P}^n \) so \( T_x Y + T_x M = T_x \mathbb{P}^n \), so \( Y \cap M \) transversely. Let \( Y \cap M = \bigcup Y_i \), and note that \( \dim Y_i = n - r \). Then
\[
\# (X \cap (Y \cap M)) = \# (X \cap (\bigcup Y_i))
\]
\[
= \sum \# (X \cap Y_i)
\]
\[
= \sum \deg X \deg Y_i
\]
\[
= \deg X \sum \deg Y_i,
\]
by Step 2, and \( \sum \deg Y_i = \deg Y \) by Step 1. \( \square \)

Suppose \( X^r, Y^s \subset \mathbb{P}^n \) intersect properly and transversely with irreducible components \( W_i \), then Bezout’s theorem tells us that \( \deg X \cdot \deg Y = \sum \deg W_i \). In particular, if \( X \) and \( Y \) are plane curves in \( \mathbb{P}^2 \) of degree \( d \) and \( e \), then \( \# X \cap Y = de \).

What if \( X \) and \( Y \) intersect properly but not transversely? If you count the *multiplicity* of each point properly, then we still get \( \# (X \cap Y) = de \). More precisely, assume \( X^r, Y^s \subset Z^n \) where \( Z^n \) is smooth and \( X \) and \( Y \) intersect properly. Let \( X \cap Y = \bigcup W_i \). Then one defines the intersection number \( i(X, Y; W_i) \). We won’t define this in general, but if \( f = V(f), Y = V(g) \) and \( p \in X \cap Y \), then \( i(X, Y; p) \) to be the length of \( \mathcal{O}_{\mathbb{P}^2,p}/(f, g) \) as an \( \mathcal{O}_{\mathbb{P}^2,p} \) module.
Theorem 6.3.7 (Bezout). If $X, Y \subset \mathbb{P}^n$ intersect properly, then

$$\deg X \deg Y = \sum_{W_i \in X \cap Y} i(X, Y; W_i) \deg W_i.$$
Chapter 7

Algebraic Curves

7.1 Curves and function fields

Definition 7.1.1. An algebraic curve is an algebraic variety of dimension 1.

Lemma 7.1.2. Let $X$ be a smooth curve and $Y$ be a projective variety. Then every rational map $\varphi : X \to Y$ is a morphism.

Proof. This follows from our version of Zariski’s main theorem, that a birational map of smooth quasiprojective varieties is defined away from codimension 2. (Technically, we only stated the theorem for $X$ quasiprojective. One can reduce to this case by looking at an affine cover of the graph $\Gamma_{\varphi}$.)

Corollary 7.1.3. Let $X$ and $Y$ be two smooth projective curves. If $X$ and $Y$ are birational, then they are isomorphic.

So as far as smooth curves go, they are completed determined by their function fields $k(X)$, which has transcendence degree 1.

Definition 7.1.4. A function field $K/k$ is a finitely generated field over $k$ of transcendence degree 1.

Question: Given a function field $K$, can we find a smooth, projective curve $X$ with $k(X) \simeq K$? Equivalently, given any curve $Y$, can we find a smooth projective curve birational to it?

The following theorem assures us that the answer is yes.

Theorem 7.1.5. There is an equivalence of categories between smooth projective curves over $k$ and function field over $k$.

7.2 Normality

Before embarking on the proof, we recall the notion of normality.
Definition 7.2.1. A variety is called **normal** if for all \( x \in X \), \( \mathcal{O}_{X,x} \) is integrally closed in \( k(X) \).

**Lemma 7.2.2.** If \( X \) is affine, then \( X \) is normal if and only if \( \mathcal{O}(X) \) is integrally closed in \( k(X) \).

**Proof.** Suppose \( \mathcal{O}(X) \) is integrally closed. More generally, if \( R \) is integrally closed then \( S^{-1}R \) is integrally closed. To see this, suppose \( x \in \text{Frac}(R) \) satisfies

\[
x^n + a_{n-1}x^{n-1} + \ldots + a_0 = 0
\]

where \( a_i \in S^{-1}R \). For some \( a \in S \), \( aa_i \in R \) for all \( i \). Then we multiply the above equation by \( a^n \):

\[
(ax)^n + aa_{n-1}(ax)^{n-1} + \ldots + a_na_0 = 0.
\]

Since \( R \) is integrally closed, \( ax \in R \), so \( x \in S^{-1}R \).

Applying this in the case where \( R = \mathcal{O}(X) \) and \( S = R \setminus m_p \), we see that \( \mathcal{O}_X \) integral \( \implies \mathcal{O}_{X,p} \) integral. Conversely, it is a general fact that

\[
R = \bigcap R_m,
\]

where the intersection ranges over the maximal ideals \( m \) of \( R \). To see this, if \( x \in R \), then the set \( \{ a \in R : ax \in R \} \) is an ideal in \( R \), not contained in any maximal ideal since \( x \in R_m \) implies that there exists \( s \in R \setminus m \) such that \( sx \in R \). Therefore, this set is the whole ring; in particular, \( 1x = x \in R \). Therefore, \( \mathcal{O}_{X,p} \) integral for all \( p \implies \mathcal{O}(X) \) integral by the above remarks and the Nullstellensatz.

**Definition 7.2.3.** Let \( X \) be an algebraic variety. Then the **normalization** of \( X \) is a normal algebraic variety \( \tilde{X} \) together with a morphism \( \pi : \tilde{X} \to X \) such that if \( Y \) is normal and \( f : Y \to X \) is dominant, then there exists a unique \( \tilde{f} \) such that the diagram commutes

\[
\begin{array}{ccc}
Y & \xrightarrow{\tilde{f}} & \tilde{X} \\
\downarrow{f} & & \downarrow{\pi} \\
X & & \\
\end{array}
\]

**Proposition 7.2.4.** The normalization exists and is unique.

**Proof.** If \( X \) is affine, let \( A \) be the integral closure of \( \mathcal{O}(X) \) in \( k(X) \) and \( \tilde{X} = \text{Spec } A \). We have a natural embedding \( \mathcal{O}(X) \to A \), hence a morphism \( \tilde{X} \to X \). A dominant map \( Y \to X \) corresponds to an embedding \( \mathcal{O}(X) \to \mathcal{O}(Y) \). Write \( Y = \bigcup Y_i \) where \( Y_i \) is affine, so the map factors through \( A \).

In the general case, \( X = \bigcup X_i \), where the \( X_i \) are affine. By the above construction, we get an \( \tilde{X}_i \) for each \( X_i \), and we check the agreement on the intersections, so we may construct product, etc. \( \square \)
Theorem 7.2.5. If $X$ is smooth, then $X$ is normal.

Proof. Recall a commutative algebra fact that we stated (but did not prove) earlier: if $X$ is smooth at $x$, then $\mathcal{O}_{X,x}$ is a UFD. Also, UFDs are integrally closed; this is essentially the “rational root theorem.” Suppose we have an integral equation

$$u^n + a_1u^{n-1} + \ldots + a_n = 0, a_i \in \mathcal{O}_{X,x}.$$  

Write $u = \frac{f}{g}$, where $\gcd(f,g) = 1$. Then

$$f^n + a_1gf^{n-1} + \ldots + a_ng^n = 0.$$  

Let $p$ be a prime divisor of $fg$. Then we must have $p | f^n$, so $p | f$. Contradiction.

Theorem 7.2.6. If $X$ is normal, then $X_{\text{sing}}$ has codimension $\geq 2$.

Example 7.2.7. The cone $xy - z^2 \subset \mathbb{P}^3$ is not smooth, but it is normal.

Proposition 7.2.8. Let $X$ be normal and $Y \subset X$ be a closed subvariety of codimension 1. Then there exists $U \subset X$ open such that $U \cap Y = \emptyset$ and $Y = V(f)$ on $U$.

Proof. We will prove the proposition for $\dim X = 1$, so that $Y = \{y\} \subset X$. Let $f \in \mathcal{O}(X)$, where $X$ is affine and $f(y) = 0$. By shrinking $X$, we can assume that the zeros of $f$ are just $y$. By Hilbert’s Nullstellensatz, $\sqrt{(f)} = m_y$. Therefore, there exists $r$ such that $m_y^r \subset (f) \subset m_y$. We may assume that this $r$ is minimal, so there exist $a_1, \ldots, a_{r-1} \in m_y$ such that

$$g = a_1a_2\ldots a_{r-1} \notin (f), \text{ but } gm_y \subset (f).$$  

Let $u = \frac{g}{f}$, so that $u \notin \mathcal{O}_{X,y}$ but $um_y \subset \mathcal{O}_{X,y}$. We claim that $um_y \notin m_y$. Suppose otherwise; then multiplication by $u$ is a $\mathcal{O}_{X,y}$-module homomorphism $m_y \to m_y$, so there exists $F \in \mathcal{O}_{X,y}[t]$ monic such that $F(u) = 0$, implying that $u$ is integral over $\mathcal{O}_{X,y}$. But $\mathcal{O}_{X,y}$ is integrally closed, and $u \notin \mathcal{O}_{X,y}$ by assumption.

Therefore, $m_y$ is strictly contained in $um_y \subset \mathcal{O}_{X,y}$, so $um_y = \mathcal{O}_{X,y}$. Then $m_y = (u^{-1})$, which is what we wanted.

We have shown more generally that if $X$ is a normal curve, $\mathcal{O}_{X,y}$ is a PID. For the general case, recall that if $x \in X$ then

$$\mathcal{O}_{X,x} = \lim_{x \in U} \mathcal{O}(U).$$  

If $Y \subset X$ is a closed subvariety, then

$$\mathcal{O}_{X,Y} = \lim_{U \cap Y \neq \emptyset} \mathcal{O}(U).$$
It is easy to see that $\mathcal{O}_{X,Y}$ is a local ring with residue field $k(Y)$ and maximal ideal

$$m_{X,Y} = \lim_{U \cap Y \neq \emptyset} I(U \cap Y).$$

The proof now follows along similar lines, replacing $\mathcal{O}_{X,Y}$ by $\mathcal{O}_{X,Y}$. Namely, pick an open $U \subset X$ such that $U \cap Y \neq \emptyset$ and $U \cap Y$ is set-theoretically defined by one equation and $\mathcal{O}_{X,Y}$ is integrally closed.

Proof of Theorem 7.2.6. Suppose $\text{codim} X_{\text{sing}} = 1$, i.e. there exists $Y \subset X_{\text{sing}}$ a subvariety of codimension 1. By shrinking $X$, we may assume that $Y = V(f)$. Let $y \in Y$ be a smooth point of $Y$. Let $y_1, \ldots, y_{n-1}$ be local parameters of $Y$ at $y$, i.e. $y_1, \ldots, y_{n-1}$ generate $m_{Y,y}$.

$$0 \to (f) \to \mathcal{O}(X) \to \mathcal{O}(Y) \to 0.$$ 

So the lifts $y_1, \ldots, y_{n-1}, f$ generate $m_{X,y}$, i.e. $dy_1, \ldots, dy_{n-1}, df$ generate $m_{X,y}/m_{X,y}^2$; but $y$ being singular means that the tangent space has dimension greater than $n$. □

Proof of Theorem 7.1.5. Let $K/k$ be a function field. We choose any projective curve $X$ such that $k(X) = K$ (take any $B \subset K$ a finitely generated $k$-algebra having $K$ as its fraction field - e.g. take a finite set of generators for $K$ and the algebra generated by them - $U = \text{Spec} B$ and then take its closure in $\mathbb{P}^n$). Now let $\bar{X}$ be the normalization of $X$, so $\bar{X}$ is smooth. We need to show that it is projective.

Cover $X$ by affine opens $\{U_i\}$ such that any two points of $X$ are contained in some $U_i$. By construction, $\bar{X} = \bigcup \bar{U}_i$, where $\bar{U}_i$ is the normalization of $U_i$. So $\bar{U}_i$ is affine and smooth; we let $\overline{U}_i$ be the closure of $\bar{U}_i$ in some projective space. So we have a rational map $\bar{X} \to U_i$, since $\bar{U}_i$ is a dense open subset of $\bar{X}$; this extends to a morphism $\varphi_i: \bar{X} \to U_i$. Consider $\bar{X} \to \prod U_i$ and let $\varphi: \bar{X} \hookrightarrow X$ be the closure of the image of $\varphi$.

We claim that $\varphi$ is an isomorphism, which would finish off the proof. To see that $\varphi$ is injective, suppose $x, y \in \bar{X}$. By our choice of open cover, there exists $\bar{U}_i$ containing $x$ and $y$. Therefore, $\varphi(x) = \varphi(y) \implies x = y$ under the projection to $\overline{U}_i$, so $x = y$.

Lemma 7.2.9. Let $X$ be a variety and $\bar{X} \to X$ its normalization. Then $\pi$ is surjective.

Proof. By construction, it is enough to show this if $X$ is affine. So assume that $A = \mathcal{O}(X) \subset B = \mathcal{O}(\bar{X}) \subset k(X)$. Let $x \in X$ correspond to the maximal ideal $m_x \subset A$. So it suffices to show that $m_x B$ is a proper ideal of $B$, since in this case
it is contained in a maximal ideal which maps to \( x \). Recall a commutative algebra result from earlier: if \( B \) is integral over \( A \) and finitely generated, then it is finite over \( A \). Then by Nakayama’s lemma, \( m_x B \neq B \).

Let \( \tilde{X} \) be the normalization of \( X \). We next claim that \( \varphi \) is an open embedding, so we may regard \( \tilde{X} \subset \tilde{X} \) as an open subset. So we have a rational map \( \tilde{X} \to X \), which extends to a morphism since \( \tilde{X} \) is normal. By the universal property, this factors through \( \tilde{X} \to X \), giving a map back \( \tilde{X} \to \tilde{X} \).

For \( x \in X \), choose \( x' \in \tilde{X} \) a lift of \( x \); its image in \( \tilde{X} \) will map to \( x \).

\[
\begin{array}{ccc}
\tilde{X} & \longrightarrow & X \subset U_i \\
\uparrow & & \downarrow \\
\tilde{U}_i & \longrightarrow & U_i
\end{array}
\]

At the level of local rings, we have the diagram

\[
\begin{array}{ccc}
\mathcal{O}_{\tilde{X}, x} & \leftarrow & \mathcal{O}_{\tilde{X}, \varphi(x)} \\
\uparrow & & \uparrow \\
\mathcal{O}_{\tilde{U}_i, x} & \leftarrow & \mathcal{O}_{U_i, x}
\end{array}
\]

The map on the left is an isomorphism, as is the map on the bottom. The map on the right is an injection, hence the map on top is an isomorphism. By a homework problem, this is an isomorphism in some neighborhood of any point.

7.3 The Picard Group of Curve

**Definition 7.3.1.** Let \( \varphi : X \to Y \) be a morphism of algebraic varieties. Then \( \varphi \) is called **finite** if for any affine open \( V \subset Y \), \( U := \varphi^{-1}(V) \) is affine and \( \mathcal{O}(U) \) is a finite \( \mathcal{O}(V) \) module.

**Proposition 7.3.2.** Let \( \varphi : X \to Y \) be a non-constant morphism of smooth projective curves. Then \( \varphi \) is finite.

**Proof.** We first observe that \( \varphi \) is surjective since it is closed and nonconstant (projective varieties are complete). Let \( V \subset Y \) be an affine open. Consider \( A = \mathcal{O}(V) \subset k(Y) \), and let \( B \) be the integral closure of \( A \) in \( k(X) \). Let \( U = \text{Spec } B \); \( U \) is smooth since \( X \) is smooth. So we have a rational map \( U \dashrightarrow X \), which extends to a morphism.

We claim that \( \varphi^{-1}(V) = U \). Once we show this, we are done since we know that \( B \) is a finite \( A \)-algebra.

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To establish the claim, suppose for the sake of contradiction that there exists $y_0 \in V$ and $x_0 \notin U$ such that $\varphi(x_0) = y_0$. Then there exists $f \in k(X)$ with poles at $x_0$, but which is a regular function on $U$. This is because $\varphi^{-1}(y_0) = \{x_0, x_1, \ldots, x_n\}$ and we may choose $f$ to be any function regular on $\{x_1, \ldots, x_n\} \cap U$ but with pole at $x_0$ (working in an affine open).

This means that $f \in B$, and $B$ is integral over $A$, so

$$f^n + a_1 f^{n-1} + \ldots + a_n = 0 \quad a_i \in \mathcal{O}(V) = A.$$

So $f = -a_1 - \frac{a_2}{f} - \ldots - \frac{a_n}{f^{n-1}}$. Since $\frac{1}{f} \in \mathcal{O}_{X,x_0}$, we have $f \in \mathcal{O}_{X,x_0}$, which is a contradiction. 

\[\square\]

**Definition 7.3.3.** Let $X$ be a curve. The *degree map* $\deg : \text{Div} X \to \mathbb{Z}$ is defined by $\sum n_i p_i \mapsto \sum n_i$.

**Theorem 7.3.4.** Let $X$ be a smooth projective curve. If $D = (f)$ is principal, then $\deg D = 0$.

**Corollary 7.3.5.** The degree map factors through the Picard group.

**Definition 7.3.6.** We define $\text{Pic}^0(X) = \ker(\deg : \text{Pic}(X) \to \mathbb{Z})$.

**Proposition 7.3.7.** Let $\varphi : X \to Y$ be a dominant morphism of smooth projective curves. Let $D \in \text{Div}(Y)$, $\varphi^* D \in \text{Div}(X)$. Then $\deg \varphi^* D = \deg D \deg \varphi$, where $\deg \varphi = [k(X) : k(Y)]$.

**Proof.** It suffices to show this for $D = p$. Let $\varphi^{-1}(p) = \{p_1, \ldots, p_\ell\} \subset X$. Let $y$ be a local parameter at $p$. In $\mathcal{O}_{X,p_i}$, $y = u_i x_i^{r_i}$, where $u_i \in \mathcal{O}_{X,x_i}$, and $\varphi^* D = \sum r_i p_i$. We wish to show that $\sum r_i = \deg \varphi$.

Let $V$ be an affine open containing $p$, $A = \mathcal{O}(V)$, $U = \varphi^{-1}(V)$, and $B = \mathcal{O}(U)$. Since $\varphi$ is a finite map, $U$ is affine and $B$ is a finite $A$-module. Let $m_p \subset A$ be the maximal ideal corresponding to $p$. So $\mathcal{O}_{Y,p} \simeq A_{m_p} \to B_{m_p}$, and $B_{m_p}$ is a finite $A_{m_p}$-module. In fact, it is also a free $A_{m_p}$-module, say of rank $n$, since $A_{m_p}$ is a PID and $B$ is an integral domain which is a finite module over it (we are invoking the structure theorem for finitely generated modules over a PID). Furthermore, $n = \deg \varphi$, as is apparent from the fact that $B_{m_p} \otimes k(Y) = k(X)$. So $B_{m_p}/m_p B_{m_p}$ is an $n$-dimensional $k$-vector space.

We claim that $B_{m_p} = \bigcap \mathcal{O}_{X,p_i}$.

$$B_{m_p} = \{ \frac{f}{g} \mid f \in B, \ g \in \mathcal{O}(V), g(p) \neq 0 \}.$$

Now the inclusion $\subset$ is obvious, since the lift of a regular function at $p$ will be a regular function at all of the $p_i$. On the other hand, suppose $h \in \bigcap \mathcal{O}_{X,p_i}$. So
$h$ is regular away from $q_1, \ldots, q_m$, and $p \in W := V - \{ \varphi(q_j) \} \subset V$. So $h \in \mathcal{O}(U) \otimes \mathcal{O}(V) \mathcal{O}(W) \subset B_{mp}$.

Therefore,

$$B_{mp} / (y) \simeq \prod \mathcal{O}_{X, p_i} / (x_i^{r_i}).$$

This implies that

$$n = \dim_k B_{mp} / (y) = \sum \dim_k \mathcal{O}_{X, p_i} / (x_i^{r_i}) = \sum r_i.$$

\[ \square \]

**Proof of 7.3.4.** Let $f$ be any nonconstant rational function on $k(X)$, which is equivalent to a dominant rational map $X \dashrightarrow \mathbb{P}^1$. So $(f) = \varphi^*([0] - [\infty])$, pretty much by definition. So the proposition implies that $\deg(f) = \deg \varphi \deg([0] - [\infty]) = 0$. \[ \square \]

**Corollary 7.3.8.** Let $X$ be a smooth projective curve. If there exist $x, y \in X$ such that $x - y \in \text{Div}^0(X)$, then $X \simeq \mathbb{P}^1$.

**Proof.** If $x - y = (f)$, then $f$ defines a rational map which extends to a morphism $\varphi : X \to \mathbb{P}^1$. So $x = \varphi^*([0])$, so $\deg \varphi = 1$, implying that $\varphi$ is an isomorphism. \[ \square \]

**Example 7.3.9.** (i) $\text{Pic}^0(\mathbb{P}^1) = 0$; we already established that the degree induces an isomorphism with $\text{Pic}(\mathbb{P}^1)$ and $\mathbb{Z}$.

### 7.4 Jacobians

**Example 7.4.1.** (ii) Let $X$ be the elliptic curve $y^2z = x^3 - xz^2$. Let $p_0 = (0, 1, 0)$ and $L_X(1)$ the linear system of hyperplanes. The hyperplane $z = 0$ is a divisor in $L_X(1)$. We can see that $H_z \cap X = p_0$. Around $p_0$, the curve is given by $z = x^3 - xz^2$ and $x$ is a local parameter. Therefore, the order of $z$ at $p_0$ is at least 3. Then $\text{ord}_{p_0}(xz^2) \geq 7$, implying that $\text{ord}(x^3 - xz^2) = 3$. The conclusion is that $3p_0$ is in the hyperplane class.

Now define a map $AJ : X \to \text{Pic}^0(X)$ sending $p \mapsto p - p_0$. We claim that this is injective:

$$p - p_0 \sim q - p_0 \implies p \sim q \implies p = q$$

since $X \not\simeq \mathbb{P}^1$.

We next claim that $AJ$ is surjective. Let $D \in \text{Div}(X)$, $\deg D = 0$. Write $D = \sum n_i p_i$ with $\sum n_i = 0$; then we may also write $D = \sum n_i (p - p_0)$.

Now we claim that any such divisor is linearly equivalent to a divisor of this form where $n_i \geq 0$. Given $x \in X$, $x p_0 \cap X = \{ x, p_0, y \}$. So $x + p_0 + y \sim 3p_0$, so $x - p_0 \sim p_0 - y$. While $\sum n_i > 1$, $D$ is of the form

$$D = (p_1 - p_0) + (p_2 - p_0) + \ldots$$
Writing $p_1p_2\cap X = \{p_1, p_2, q\}$, so we can replace $(p_1-p_0)+(p_2-p_0) \sim p_0-q \sim q'-p_0$.

The conclusion is that $X$ is bijective with $\text{Pic}^0(X)$; this endows $X$ with a group structure where $p_0$ is the identity element and the group law is defined by the hyperplane class is 0.

**Theorem 7.4.2.** Let $X$ be an elliptic curve. The multiplication map $m : X \times X \to X$ and the inverse $s : X \to X$ are morphisms of algebraic varieties.

**Proof.** Geometrically, the inverse is obtained by intersecting with the vertical line through that point. The group law can also be expressed geometrically by intersecting with lines. This can all be defined by rational functions in terms of the coordinates. \qed

**Example 7.4.3.** Let’s compute the inverse map. Let $(a, b) \in X$. The line through $x$ and $p_0$ is given by $az - x$. The third intersection point is given by intersecting with $y^2z = x^3 + axz^2$, which has the solutions $(0, 1, 0), (a, b, 1), (a, -b, 1)$. So the inverse map sends $(a, b) \mapsto (a, -b)$.

**Definition 7.4.4.** A group variety $G$ is an algebraic variety together an identity $e \in G$ and morphisms $m : G \times G \to G$ and $s : G \to G$, such that $(G, e, m, s)$ form a group in the usual sense. A projective group variety is called an **abelian variety**.

**Corollary 7.4.5.** The variety $X$ defined by $y^2z = x^3 - xz^2$ is an abelian variety.

**Remark 7.4.6.** Assume char $k \neq 2, 3$. Then in fact, every genus one smooth projective curve is isomorphic to a cubic curve in $\mathbb{P}^2$ of the form $y^2z = x^3 + axz^2 + bz^3, -4a^2 - 27b^3 \neq 0$. By the same argument, $X$ is an abelian variety.

Suppose $k = \mathbb{C}$. The regular differential on $C$ is $\omega = \frac{dx}{y} = \frac{dx}{\sqrt{x^3 + ax + b}}$. People wanted to understand this integral; they realized that one needed to pass to a Riemann surface, which was the elliptic curve. Integrating this form on an elliptic curve $X$ from some initial point gives a map from $X$ to $\mathbb{C}$, which is not well-defined because $X$ is not simply-connected. So we need to mod out by the integral over generators for the first homology of the elliptic curve.

**Example 7.4.7.** The following are affine group varieties, called **linear algebraic groups**.

- $\text{GL}_n = \{A \in M_{n \times n} \mid \det A \neq 0\}$.
- $\text{PGL}_n = \text{GL}_n / k^\times I_n$.
- $\mathbb{G}_m = \text{GL}_1 = (\mathbb{A}^1 - \{0\}, \times)$.
- $\mathbb{G}_a \simeq (\mathbb{A}^1, 0)$.

Mathematicians may not always choose to best terminology, but the following theorem shows that they are at least somewhat reasonable.

**Theorem 7.4.8.** Every abelian variety is abelian.
Theorem 7.4.9. If \( X \) is a smooth projective curve of genus \( g \), then \( \text{Pic}^0(X) \) is naturally an abelian variety of dimension \( g \).

Remark 7.4.10. In fact, every algebraic group is built from an abelian variety and an algebraic group. That is, for any algebraic group \( G \) there is an exact sequence

\[ 1 \to N \to G \to A \to 1, \]

where \( N \) is a linear algebraic group and \( A \) is an abelian variety.

7.5 The Riemann-Roch Theorem

Theorem 7.5.1 (Riemann-Roch). Let \( X \) be a smooth projective curve of genus \( g \) over \( k \), \( D \) a divisor on \( X \). Then

\[ \ell(D) - \ell(K_X - D) = \deg D + 1 - g. \]

Corollary 7.5.2. \( \deg K_X = 2g - 2 \).

Proof. Let \( D = K_X \) in the theorem, and use \( \ell(0) = 1, \ell(K_X) = g. \)

Corollary 7.5.3. If \( \deg D \geq 2g - 1 \), then

\[ \ell(D) = 1 - g + \deg D. \]

Proof. \( \deg(K_X - D) < 0 \). If \( \ell(K_X - D) > 0 \), then \( K_X - D \) is linearly equivalent to an effective divisor, but it has negative degree, which is impossible.

Corollary 7.5.4. If \( \deg D \geq 2g \), then \( \ell(D - p) = \ell(D) - 1 \).

Corollary 7.5.5. If \( \deg D \geq 2g \), then \( |D| \) has no base points, hence the rational map \( X \dashrightarrow |D|^* \) extends to a morphism.

Theorem 7.5.6. Let \( D \) be a divisor on \( X \). If for all \( p, q \in X \), \( \ell(D - p - q) = \ell(D) - 2 \), then the rational map \( Z_{|D|} : X \to |D|^* \) is a closed embedding.

Proof. Note that this condition certainly implies that \( \ell(D - p) = \ell(D) - 1 \), since \( \ell(D) - \ell(D - p) \leq 1 \). So \( |D| \) has no base points: the rational map extends to a morphism.

Next, note that \( \varphi \) is injective, since

\[ \mathcal{L}(D - p - q) \subset \mathcal{L}(D - p) \subset \mathcal{L}(D) \]

are all strict inclusions. So there exists \( f \in \mathcal{L}(D - p) \setminus \ell(D - p - q) \), i.e. \( p \in \text{Supp}(f + D) \) but \( q \notin \text{Supp}(f + D) \). This is a hyperplane section passing through the image of \( p \) but not \( q \), so their images are distinct.
Let $X' = \varphi(X) \subset |D|^*$. We need to show that $\varphi : X \to X'$ is an isomorphism. $X \to X'$ is a finite morphism, and $X' \hookrightarrow |D|^*$ is a closed embedding, hence a finite morphism. So for an affine open $V \subset |D|^*$, the pullback $U$ is an affine open, and $B = \mathcal{O}(U)$ is finite over $A = \mathcal{O}(V)$.

$$V \cap X' = \{x \in V \mid f(x) = 0 \text{ if } \varphi^*f = 0 \text{ in } B\}.$$ 

So $\mathcal{O}(V \cap X') = \text{Im}(A \to B)$ and $I(V \cap X') = \ker(A \to B)$. Let $p \in X$; then $\mathcal{O}_{X,p}$ is finite over $\mathcal{O}_{X',\varphi(p)}$.

**Key claim:** $m_{X',\varphi(p)} \to m_{X,p}/m^2_{X,p}$ is surjective. To establish the claim, it suffices to show that 

$$T_{\varphi(p)}^*|D|^* \to T_{\varphi(p)}^*X' \to T_p^*X$$

is surjective. Recall how this map is defined: we choose $(f_0, \ldots, f_n)$ a basis of $\mathcal{L}(D)$. Then $\varphi(p) = (f_0(p), \ldots, f_n(p))$. Let $x_0, \ldots, x_n$ be coordinates on $|D|^*$, $\varphi(p) = (1, 0, \ldots, 0)$. Then

$$\varphi^* \left( \frac{x_1}{x_0} \right) = \frac{f_1}{f_0} \in m_{X,p}.$$ 

Surjectivity means that there exists $i$ such that $\frac{f_i}{f_0} \notin m^2_{X,p}$. Because $\mathcal{L}(D - 2p) \neq \mathcal{L}(D - p)$, there exists $f \in \mathcal{L}(D - p) \setminus \mathcal{L}(D - 2p)$.

Let $n = m_{X',\varphi(p)} \mathcal{O}_{X,p}$, so $n \subset m_{X,p}$. But from the claim, $n$ contains a local parameter of $X$ at $p$, hence $n = m_{X,p}$. Now consider the map of $\mathcal{O}_{X',\varphi(p)}$-modules

$$\mathcal{O}_{X',\varphi(p)} \to \mathcal{O}_{X,p} \to N 	o 0.$$ 

Tensoring by $\mathcal{O}_{X',\varphi(p)}/m_{X',\varphi(p)}$,

$$\begin{array}{ccc}
\mathcal{O}_{X',\varphi(p)}/m_{X',\varphi(p)} & \to & \mathcal{O}_{X,p}/m_{X,p} \mathcal{O}_{X,p} \\
\approx & & \approx \\
k & \approx & 0
\end{array}$$ 

So by Nakayama’s lemma $N = 0$, i.e. $\mathcal{O}_{X',\varphi(p)} \to \mathcal{O}_{X,p}$ is a surjection of $\mathcal{O}_{X',\varphi(p)}$-modules, hence an isomorphism. \(\Box\)

**Corollary 7.5.7.** If $\deg D \geq 2g + 1$, then $Z_{|D|} : X \to |D|^* \simeq \mathbb{P}^\deg D - g$ is a closed embedding.

**Corollary 7.5.8.** Let $g \geq 2$. Then $Z_{|2K_X|}$ is a closed embedding (to $\mathbb{P}^{2g - 4}$).

What about the linear system $Z_{|2K_X|} : X \to \mathbb{P}^G$? (Assume char $k = 0$.)

**Example 7.5.9.** $g = 2$. This gives a map $\varphi : X \to \mathbb{P}^1$ which is separable. We know that $K_X - \varphi^*K_{\mathbb{P}^1}$ is the branch divisor, and $\varphi^*([0]) = K_X$. Hence $\deg \varphi = \deg K_x = 2g - 2$, and $\deg B = 2 - 2(-2) = 6$. So $[k(X) : k(\mathbb{P}^1)] = 2$, implying that $k(X) = k(x)[y]/(y^2 - f(x))$. 

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**Exercise 7.5.10.** Show that the support of the branch divisor $B$ are the points over the zeros of $f(x)$ of odd order, and the point over $\infty$ if $\deg f$ is odd.

**Definition 7.5.11.** A curve $X$ is called *hyperelliptic* if there exists $\varphi : X \to \mathbb{P}^1$ of degree 2.

**Corollary 7.5.12.** Every genus 2 curve is hyperelliptic.

**Example 7.5.13.** Let $g \geq 3$. If $X$ is not hyperelliptic, then $Z_{[K_x]} : X \dashrightarrow \mathbb{P}^{g-1}$ is a closed embedding.

**Example 7.5.14.** Let $\varphi : X \to Y$ be separable of degree $n$. The branch divisor is $B := K_X - \varphi^* K_Y$, and taking degrees we have

$$2g_X - 2 = n(2g_Y - 2) + \deg B.$$

Let $x \in X$, $y = \varphi(x) \in Y$. Let $t_x, t_y$ be the local parameters, $\varphi^* t_y = t_x^e u$, for some $u \in \mathcal{O}^*_{X,x}$.

$$\frac{d \varphi^* t_y}{dt_x} = et_x^{e-1} u + t_x^e \frac{du}{dt_x}.$$ 

**Definition 7.5.15.** $\varphi$ is tame at $x$ if $\text{char } k = p \nmid e$.

So if $\varphi$ is tame at $x$, then $\text{ord}_x \left( \frac{d \varphi^* t_y}{dt_x} \right) = e - 1$. If $\varphi$ is tame everywhere, (e.g. in characteristic 0 or if $\deg n < p$), then the formula above tells us that

$$B = \sum_{x \in X} (e_x - 1)x \implies \deg B = \sum_{x \in X} (e_x - 1).$$

This is called the **Riemann-Hurwitz formula**.