GEOMETRIC CLASS FIELD THEORY I

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1. Classical class field theory

1.1. The Artin map. Let’s start off by reviewing the classical origins of class field theory. The motivating problem is basically to “describe” in some meaningful way the abelian extensions of a number field $F$, or what is essentially the same: the abelianized Galois group $\text{Gal}(\overline{K}/K)^{ab}$ (since its finite quotients are the Galois groups of finite abelian extensions of $K$).

If we’re going to describe the group in an intrinsic way, we had better have a way to write down some intrinsic elements of it. Well, each prime $p$ of $K$ is a point of the form $\text{Spec } \mathbb{F}_q$. This is the arithmetic version of a circle, since its étale fundamental group is $\hat{\mathbb{Z}}$, i.e. it has a unique extension of degree $n$ for each $n$. In particular, there is a canonical generator for $\text{Gal}(\mathbb{F}_q/\mathbb{F}_q)$, the “arithmetic Frobenius” $\text{Frob}_p$ at $p$, which takes $x \mapsto x^q$. Given an extension $L/K$ unramified over $p$, we can lift $\text{Frob}_p$ to an element of $\text{Gal}(L/K)$, which we denote by the same name. Normally this choice of lift is only well-defined up to conjugacy, but since we’re in the case of an abelian extension it is completely well-defined.

This already gives a hint of why the abelian case is so much more tractable. The issue with the “absolute Galois group” $G_K := \text{Gal}(\overline{K}/K)$ is that it is not “canonical”, reflecting the fact that there is no canonical choice of algebraic closure $\overline{K}$. Two such choices will differ by non-canonical choices, leading to isomorphisms of the absolute Galois groups which are not actually unique. This means that $\text{Gal}(\overline{K}/K)$ is really a “group up to conjugation”, and that the right way to access it is through conjugation-invariant means, i.e. representation theory. But in the abelian case, we can hope to have a more direct understanding.

Anyway, the modern formulation for class field theory centers around the “Artin map”

$$\Psi: K^\times \backslash \mathbb{A}_K^\times \to G_K^{ab}.$$  

This exists for both number fields and function fields, and it is almost an isomorphism in either case, but not quite.

- In the number field case $\Psi$ is surjective, but has as kernel the connected component of the identity. That this latter must be in the kernel is evident from the fact that $G_K^{ab}$ is profinite, hence totally disconnected.

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• In the function field case $\Psi$ is injective, but not surjective. It lands in the “Weil group”, a subgroup of the Galois group who image in $G_{\mathbb{F}_q}$ (via restriction) lands in $\mathbb{Z} \subset \hat{\mathbb{Z}}$:

$$
\begin{array}{ccc}
K^\times \backslash A_K^\times & \xrightarrow{\Psi} & G_K \\
\downarrow & & \downarrow \\
\mathbb{Z}^\times & \xrightarrow{\Psi} & \hat{\mathbb{Z}}
\end{array}
$$

One uniform way of stating the “almost isomorphism” property is that $\Psi$ induces an isomorphism of profinite completions. In particular, the image of $\Psi$ is dense, which means that the composition of $\Psi$ with any finite quotient $\text{Gal}(K/K) \to \text{Gal}(L/K)$ is surjective. This fact is a manifestation of Artin reciprocity. One compelling way to state Artin reciprocity is that $\text{Gal}(L/K)$ is generated by Frobenius elements, and the image of a Frobenius element is determined by congruence conditions.

**Example 1.1.** In particular, in the case of $K = \mathbb{Q}$ this says, for instance, that whether or not $p$ splits in an extension $L/\mathbb{Q}$ is determined completely by congruence conditions.

A famous and familiar special case of this result is quadratic reciprocity:

$$
\left( \frac{q}{p} \right) = \left( \frac{p}{q} \right)^* := (-1)^{\frac{p-1}{2} \cdot \frac{q-1}{2}} \left( \frac{p}{q} \right).
$$

To explain, $p$ splits in the extension $\mathbb{Q}(\sqrt{-q})/\mathbb{Q}$ if $x^2 - q$ has a root modulo $p$, i.e. when $\left( \frac{q}{p} \right) = 1$. Quadratic reciprocity tells us that this is dictated by the class of $p$ modulo $4q$.

In fact, the converse holds: an extension is abelian if and only if the splitting behavior is determined by congruence conditions. That is part of the content of Artin reciprocity. (And it explains why there is not such a clean statement for “cubic reciprocity”, “quartic reciprocity”, etc. - not all field extensions generated by cubic or higher degree polynomials are abelian.) Indeed, if $L/K$ is finite abelian, then let $U \subset K^\times \backslash A_K^\times$ be the kernel of the composition

$$
\Psi: K^\times \backslash A_K^\times \to G_K^\text{ab} \to \text{Gal}(L/K).
$$

A basis of open subsets of $K^\times_v$ are $1 + \pi_v^n \mathcal{O}_v$, so the open subsets are basically “congruence conditions at $v$”. Similarly, a finite-index open subset of $K^\times \backslash A_K^\times$ will be a product of such things, hence a juxtaposition of congruence conditions (almost all trivial since almost all factors will be $\mathcal{O}_{K_v}^\times$, so we get something finite in the end).

1.2. **Local-global compatibility.** We would like to explain how to “compute” the Artin map in practice. A key input is a more explicit description of a local Artin map. That is, local class field theory describes a local Artin map

$$
\Psi_v: K_v^\times \to G_{K_v}^\text{ab}.
$$
Here, a uniformizer $\pi_v$ maps to “Frobenius” and $O_{K_v}^\times$ maps to the ramification group. More precisely, we have

$$
0 \to O_{K_v}^\times \to K^\times \to \mathbb{Z} \to 0
$$

$$
0 \to \text{Gal}(K_{ab}/K_{unr}) \to \text{Gal}(K_{ab}/K) \to \text{Gal}(K_{unr}/K) \to 0
$$

This informs the global Artin map via “local-global compatibility”

$$
K^\times \xrightarrow{\psi_v} G_{K_v}^{ab}
$$

$$
K^\times \backslash A_K^\times \xrightarrow{\psi} G_K^{ab}
$$

Consequently, in terms of the global Artin map the group $O_{K_v}^\times$ maps onto the inertia group over $v$ in any abelian extension $L/K$, hence is killed if $L/K$ is unramified over $v$. Also, the elements $(\ldots,1,\pi_v,1,\ldots)$ maps to $\text{Frob}_v^{-1}$ (beware - sometimes the opposite convention is used).

**Example 1.2.** What extension corresponds to $\mathbb{Q}(\sqrt{p})$? Suppose $p \equiv 1 \pmod{4}$ for simplicity. Then $\mathbb{Q}(\sqrt{p})/\mathbb{Q}$ is unramified except over $p$, so the kernel $U$ of the Artin map contains $\mathbb{Z}^\times$ for each $\ell \neq p$. This immediately implies that $U$ also contains the subgroup $\mathbb{Z} \subset \mathbb{Q}_p^\times$ generated by $p$, since in the idele class group

$$(1,\ldots,1,p,1,\ldots) = (p^{-1},\ldots,p^{-1},1,p^{-1},\ldots) \in \prod_{\ell \neq p} \mathbb{Z}_\ell^\times.$$

But now $\mathbb{Z}_p^\times$ has a unique subgroup of index 2 which must be killed under the map

$$
\mathbb{Q}^\times \backslash A_{\mathbb{Q}}^\times \xrightarrow{\psi_K} \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{Gal}(\mathbb{Q}(\sqrt{p})/\mathbb{Q}) \cong \mathbb{Z}/2
$$

namely the group of units which reduce mod $p$ to the unique subgroup of index two in the cyclic group $\mathbb{F}_p^\times$, which is the squares. So we have deduced that

$$
U = \prod_{\ell \neq p} \mathbb{Z}_\ell^\times \times U_p
$$

where $U_p$ is the group of index 2 in $\mathbb{Z}_p^\times$, consisting of elements reducing to a square in $\mathbb{F}_p^\times$. The image of $\text{Frob}_q$ is

$$(\ldots,1,q,1,\ldots) = (\ldots,q^{-1},1,q^{-1},\ldots) \mapsto q.$$

Therefore, $q$ spits in $\mathbb{Q}(\sqrt{p})/\mathbb{Q}$ if and only if the image of $\text{Frob}_q$ is trivial, if and only if $q$ is a square modulo $p$. This is quadratic reciprocity!
In particular, the maximal unramified abelian extension is
\[ \mathbb{A}_K^\times / \prod_v \mathcal{O}_v^\times. \]
Notice that this can be identified with the ideal class group of \( K \).

1.3. **Geometric reformulation.** We now focus on the geometric case (\( K = \mathbb{F}_q(C) \) a global function field). We can interpret \( \mathbb{A}_K^\times / \prod_v \mathcal{O}_v^\times \) as the “set” of line bundles on \( C \), which should be the rational points of the Picard scheme \( \text{Pic}_C \). Also, we think of the maximal abelian unramified extension of \( K \) as the étale fundamental group of \( C \). So we might first guess that the geometric reformulation of unramified class field theory is
\[ \pi_{1,\text{ét}}(C)^{\text{ab}} \cong \text{Pic}(\mathbb{F}_q). \]
However, this isn’t quite right. There is a subtle issue of degrees here. Any line bundle in \( \text{Pic}_C(\mathbb{F}_q) \) has a degree, which may be familiar as the degree of the corresponding divisor. On the other hand, an element of \( \pi_{1,\text{ét}}(C) \) has a notion of degree which measures the degree of extension of the constant field \( \mathbb{F}_q \). In other words, there is a map \( \varphi : \pi_{1,\text{ét}}(C)^{\text{ab}} \to \widehat{\mathbb{Z}} \). The actual statement is that there is a commutative diagram
\[
\begin{array}{cccccc}
0 & \to & \text{Pic}^0(C) & \to & \text{Pic}_C & \to & \mathbb{Z} & \to & 0 \\
 & \downarrow & \cong & & \downarrow & & \downarrow & & \\
0 & \to & \ker \varphi & \to & \pi_{1,\text{ét}}(C)^{\text{ab}} & \varphi & \to & \widehat{\mathbb{Z}} & \to & 0
\end{array}
\]
The theorem is then that there is a map \( \text{Pic}(C) \to \pi_{1,\text{ét}}(C)^{\text{ab}} \) inducing an isomorphism
\[ \text{Pic}^0(C) \cong \ker \varphi. \]

2. **Overview of Deligne’s argument**

We begin by reviewing some geometric notions which go into the proof.

2.1. **Picard groups.** For \( C/\mathbb{F}_q \) a smooth curve, \( \text{Pic}_C \) is the “moduli space of line bundles on \( C \)”. We will ignore subtleties concerning the existence of this scheme, and attempt to handle it using our intuition. Its connected components are parametrized by the degree of the line bundle, and the identity component \( \text{Pic}^0_C \) is an abelian variety of dimension equal to the genus of \( C \).

2.2. **The étale fundamental group.** We will also avoid defining the étale fundamental group carefully. Finite étale covers are the algebro-geometric analogues of finite covering spaces, and the étale fundamental group satisfies the same “universal” property with respect to connected finite étale covers as the usual fundamental group with respect to covering spaces: quotients
\[ \pi_{1,\text{ét}}(X) \to G \]
correspond to Galois étale covers with Galois group \( G \).
We have suppressed a detail, which is that we need to choose as basepoint a geometric point.

If $X$ is geometrically connected, then we have a short exact sequence

$$0 \to \pi_1(X, \overline{x}) \to \pi_1(X, x) \to \text{Gal}(k^s/k) \to 1.$$ 

This corresponds to the fact that there is a class of “constant” étale covers of $X$ coming from base-changing with an étale cover over $\text{Spec } k$, which is simply a separable field extension. However, there is another useful way to think of this formula, as part of the “long exact sequence of homotopy groups” associated to the “fibration” $X \to \text{Spec } k$, whose “fiber” is computed by the base-change diagram

\[
\begin{array}{ccc}
X & \longrightarrow & X \\
\downarrow & & \downarrow \\
\text{Spec } k^s & \longrightarrow & \text{Spec } k
\end{array}
\]

Why is this exact at the left and right? Since $X$ is geometrically connected we have $\pi_0(X, x) = 0$. Also, we said in the beginning that you should think of $\text{Spec } k$ as being an algebro-geometric analogue of a circle, which is an Eilenberg-Maclane space $K(\pi_1, 1)$. Therefore, its $\pi_2$ should vanish.

This intuition easily extends to a relative version: if $X \to S$ is a family of geometrically connected varieties, then we have an exact sequence

$$\pi_1(X_s, \overline{x}) \to \pi_1(X, \overline{x}) \to \pi_1(S, \overline{s}) \to 1.$$ 

2.3. **Local systems.** Local systems are locally constant sheaves for the étale topology satisfying finiteness conditions. In particular, we’ll be interested in $\overline{\mathbb{Q}}_\ell$-local systems, and we’ll demand that their stalks be finite-dimensional. As is familiar from topology, there is an equivalence of categories

$$\{\overline{\mathbb{Q}}_\ell\text{-local systems/}{X}\} \iff \{\text{finite } \pi_1,\text{ét}(X)\text{-representations}\}.$$ 

2.4. **Deligne’s argument.** We can now explain Deligne’s idea. Given a $\overline{\mathbb{Q}}_\ell$-local system $\mathcal{L}$ on $C$, we want to produce a $\overline{\mathbb{Q}}_\ell$-local system on on $\text{Pic}(C)$. How can we do this? The idea is to first pull $\mathcal{L}$ back to a local system on the symmetric powers of $C$. We have $d$ projection maps $p_i : C^d \to C$ for $i = 1, \ldots, d$. We can form the local system

$$\mathcal{L}^{\boxtimes d} = p_1^* \mathcal{L} \boxtimes p_2^* \mathcal{L} \boxtimes \ldots \boxtimes p_d^* \mathcal{L}$$

on $C$. This is visibly preserved by the symmetric group $S_d$, so we can hope that it descends to the $d$th symmetric power of $C$,

$$C^{(d)} = C^d / S_d.$$ 

This turns out to be a smooth projective variety.

The upshot is that at this point, we’ve gone from a $\overline{\mathbb{Q}}_\ell$-local system on $C$ to one on $C^{(d)}$. In turn, the symmetric power admits an *Abel-Jacobi map*

$$C^{(d)} \to \text{Pic}^d(C),$$
sending a divisor to the corresponding line bundle, which is surjective. Moreover, if $d$ is large enough then Riemann-Roch tells us that it is fibered in projective spaces. So we have an exact sequence of fundamental groups

$$\pi_1(\mathbb{P}^N) \to \pi_1(C^{(d)}) \to \pi_1(\text{Pic}^d C) \to 1.$$ 

But since projective space is “simply-connected” this is telling us that

$$\pi_1(C^{(d)}) \cong \pi_1(\text{Pic}^d C),$$

so a $\overline{\mathbb{Q}}_\ell$-local system on $C^{(d)}$, which is the same as a representation of $\pi_1(C^{(d)}) \cong \pi_1(\text{Pic}^d C)$, corresponds to a unique $\overline{\mathbb{Q}}_\ell$-local system on $C^{(d)}$!

**References**