June 30: Find potential function, Green's theorem

Find potential function. \( \vec{F} \rightarrow \text{find } \vec{F} \text{ s.t. } \nabla \cdot \vec{F} = 0 \) See notes yesterday.

**Simply connected regions:**
Recall \( \vec{F} = -\frac{y}{x^2+y^2} \hat{i} + \frac{x}{x^2+y^2} \hat{j} \) \( \text{My} = N_x \)

but \( \oint \vec{F} \cdot d\vec{r} = 0 \).

**Problem:** Not defined at \((0,0)\).

Or more specifically: the region it's defined in is not "Simply Connected."

Region with no holes: simply connected.

![Images of regions with different connectivity]

Rigorous definition: any loop can shrink into a point

**Fact:** If \( R \) is simply connected, \( \text{My} = N_x \).

then \( \vec{F} = \nabla f \) for some \( f \).

**Example:** Which of these vector fields are defined on simply connected regions?

(a) \( \vec{F} = \frac{x}{x^2+y^2} \hat{i} + \frac{y}{x^2+y^2} \hat{j} \)

(b) \( \vec{F} = \frac{1}{\sqrt{1-x^2y^2}} \hat{i} + \frac{x}{\sqrt{1-x^2y^2}} \hat{j} \)

(a) \( \nabla \cdot \vec{F} \neq 0 \) No.

(b) \( \nabla \cdot \vec{F} = 0 \) Yes.
What if \( My + N x \). Can we still compute line integrals fast?

For "good loops": yes

\[ \mathbf{F} = \frac{\hat{M}}{x^2} \hat{i} + \frac{\hat{N}}{y^2} \hat{j} \]

\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dx \, dy \quad \text{"Green's theorem"} \]

Example: \( \mathbf{F}(x,y) = xy \hat{i} + x^2 y^3 \hat{j} \)

Line integral: \[ \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} + \int_{C_2} + \int_{C_3} \]

\[ = \int_0^0 - M dx + N dy + \int_{C_2} - M dx + N dy + \int_{C_3} M dx + N dy \]

\[ = 0 + \int_0^1 t^3 \, dt + \int_0^1 (t^2 + t^5) \, dt \]

\[ = -\frac{1}{4} \]

Double integral: \[ \iint_{x=0}^{x=1} (N_x - M_y) \, dx \, dy = \int_0^1 \int_0^x (2y^3 - x) \, dy \, dx = -\frac{1}{4} \]
Physics intuition: \( \oint_C \mathbf{F} \cdot d\mathbf{r} \) : work done by \( \mathbf{F} \) around outside

\[ \mathbf{F} = (M, N) \]

\[ \int_R (N_x - M_y) \, dx \, dy : \text{total "whirlpoolness" inside} \]

\[ \text{difference of force on two sides} \]

Curl \( \mathbf{F} \)

Proof for a rectangle:

\[ \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} \]

\[ = \int_a^b M(x, c) \, dx + \int_c^d N(b, y) \, dy + \int_b^a M(x, d) \, dx \]

\[ \text{attention: integration direction} + \int_d^c N(a, y) \, dy \]

\[ = - \int_a^d (N(b, y) - N(a, y)) \, dy + \int_a^b (M(x, d) - M(x, c)) \, dx \]

\[ = \int_a^b \int_c^d \frac{\partial N}{\partial x} \, dx \, dy - \int_a^b \int_c^d \frac{\partial M}{\partial y} \, dy \, dx \]

\[ = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dx \, dy \]

For a general region \( R \) : divide into 'rectangles.'

The sides inside all cancel out, only the outside boundary left in \( \oint_C \mathbf{F} \cdot d\mathbf{r} \)

What is a good Region? What \( C \) to be simple closed curve.

\( R \) is a simply connected region.

\[ \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dx \, dy \]

\[ \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \nabla \times \mathbf{F} \, dx \, dy \]
Relation to conservative fields:
\[ \mathbf{F} \cdot d\mathbf{r} = \iint_R (N_x - M_y) \, dx \, dy = 0 \] always

Green's theorem is useful in both directions
ugly line integral \(\rightarrow\) nice double integral
double \(\rightarrow\) line

Example: (1) \[ \oint_C y^2 \, dx + x^2 \, dy \]
\[ = \iint_R \left( \frac{\partial (y^2)}{\partial x} - \frac{\partial (x^2)}{\partial y} \right) \, dx \, dy \]
\[ = \int_0^1 \int_0^1 (2x - 2y) \, dx \, dy = \int_{x=0}^1 (2x - 1) \, dx = 0. \]

(2) \( \mathbf{F} = \mathbf{r} \)
\( N_x - M_y = 1 \)

\[ x = \cos \theta \]
\[ y = \sin \theta \]
\[ \iint_R 1 \, dx \, dy = \oint_C \mathbf{F} \cdot d\mathbf{r} \]
\[ = \int_{C_1} + \int_{C_2} + \int_{C_3} \]
\[ = \int_0^1 0 \, dx + \int_0^\theta y^2 \, dy + \int_0^{\pi/2} 0 \, dx + \cos^2 \theta (3 \sin^2 \theta \cos \theta) \, d\theta \]
\[ + \int_0^{\pi/2} M \, dx + 0 \, dy \]
\[ = \int_0^{\pi/2} 0 \, dx \]
\[ = \int_0^\theta 3 \cos^2 \theta \sin^2 \theta \, d\theta = \int_0^\pi 3 \cos^2 \theta \, 5 \sin^2 \theta \, d\sin \theta \]
\[ = \frac{2}{5} \]