July 2. Green's theorem is not simply connected case.

\[ \oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \]

\[ \text{Closed curve} \]

\[ \text{Path independent} \]

Conservative

\[ \mathbf{F} \text{ defined on simply connected region} \]

\[ N_x = N_y \]

Example: \[ \mathbf{F} = \hat{y} \, i - \frac{x}{y^2} \, j \]

Is it a gradient field?

\[ M = \frac{1}{y} \quad N = -\frac{x}{y^2} \]

\[ N_y = -\frac{1}{y^2} = N_x \]

\[ \text{Defined on } y \neq 0 \]

\[ \text{Simply connected} \]

\[ \mathbf{F} = \nabla f \]

We can find the potential:

\[ f_x = \frac{1}{y} \Rightarrow f = \frac{x}{y} + g(y) \Rightarrow g'(y) = 0 \Rightarrow g(y) = C \]

\[ f_y = -\frac{x}{y^2} \]

Therefore \[ f = \frac{x}{y} + C \]

(Constant can be different on \( y > 0 \) and \( y < 0 \))

Recall Green's theorem on simply connected regions:

\[ \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (N_x - N_y) \, dx \, dy \]

What if \( R \) has holes?
\[ \Phi_1 \cdot \vec{F} \cdot d\vec{r} + \Phi_2 \cdot \vec{F} \cdot d\vec{r} + \Phi_3 \cdot \vec{F} \cdot d\vec{r} + \Phi_4 \cdot \vec{F} \cdot d\vec{r} = \iint_R (N_x - M_y) \, dx \, dy \]

(Note the direction of curves: region stays left side)

Proof: decompose the region into simply connected ones, the extra edges cancel out.

Example: \[ \vec{F} = -\frac{y}{x^2+y^2} \hat{i} + \frac{x}{x^2+y^2} \hat{j} \]

\[ \iint_R \text{curl} \vec{F} \, dA = 0 \]

\[ \Phi_1 \cdot \vec{F} \cdot d\vec{r} = \Phi_2 \cdot \vec{F} \cdot d\vec{r} = \Phi_3 \cdot \vec{F} \cdot d\vec{r} = \Phi_4 \cdot \vec{F} \cdot d\vec{r} = 2\pi \]

(take \( C_1 \) to be unit circle)

This is true for any loop around origin!

Now we talk about the other Green's theorem:

\[ \iint_R \vec{F} \cdot \nabla \, dA = \oint_C \vec{F} \cdot d\vec{s} \] (is about the tangent part of flow)

\[ \int_{(ds, ds)} \vec{F} \cdot \vec{n} \, ds \] gives the flux across the curve

\[ \vec{n} = \left( \frac{dy}{ds}, -\frac{dx}{ds} \right) \quad \vec{n} \perp \vec{T} \quad |\vec{n}| = 1 \]

Now \[ \oint_C \vec{F} \cdot \vec{n} \, ds = \int_C (M, N) \cdot (\frac{dy}{ds}, -\frac{dx}{ds}) \, ds = \int_C M \, dy - N \, dx \]
Green's theorem for flux:
\[ \oint_C \mathbf{F} \cdot d\mathbf{n} = \iint_R (M_x + N_y) \, dA \]

We can get this form from the old Green's theorem:
\[ \oint_C P \, dx + Q \, dy = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA \]

Example: \( \mathbf{F} = -y\mathbf{i} + x\mathbf{j} \) \( C : (0,0) \to (2,4) \) along \( y = x^2 \)
Compute the work of \( \mathbf{F} \) along \( C \) and the flux across \( C \).

Solution: Parameterize \( \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, \quad 0 \leq t \leq 2 \)
\[ \mathbf{F}(\mathbf{r}(t)) = -t^2\mathbf{i} + t\mathbf{j} \]
Work: \( \int_M dx + N dy = \int_{t=0}^{2} (-t^2) \, dt + t \, dt \bigg|_{t=0}^{2} = \frac{23}{3} \]
Flux: \( \iint_R M dx + N dy = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA \)
\[ = \frac{23}{3} \]

Why is flux negative?: Flow out: positive

Exercise: Verify Green's theorem in normal form for the vector field \( \mathbf{F} = x\mathbf{i} + y\mathbf{j} \) and region bounded by \( x \)-axis's and upper half-circle.
\[ \oint_C \mathbf{F} \cdot d\mathbf{n} = \int_{C_1} + \int_{C_2} (M_y - N_x) \, dA \]
For \( C_1 \), \( \int_{C_1} M dy - N dx = 0 \)
For \( C_2 \), \( \int_{C_2} \cos t \, d(smt) - \sin t \, d(\cos t) = \pi \)