Derivative Problems

Differentiate the following functions

1. $e^{(x^2)}$
   
   Answer: $2xe^{(x^2)}$

2. $x10^x$
   
   Answer: $(1 + x(ln 10))10^x$. Letting $y = x10^x$, $ln y = ln x + x ln 10$
   
   $$\frac{1}{y} \frac{dy}{dx} = \frac{1}{x} + ln 10$$
   
   so $\frac{dy}{dx} = \left(\frac{1}{x} + ln 10\right)y = (1 + x(ln 10))10^x$

3. $sec^{-1} x$
   
   Answer: $\frac{1}{x\sqrt{x^2-1}}$. Letting $y = sec^{-1} x$, $sec y = x$ so $sec y tan y \frac{dy}{dx} = 1$

   $sec y = x$ and setting up a triangle, $tan y = \sqrt{x^2 - 1}$ so $\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}$

4. $\frac{ln x}{x}$
   
   Answer: $\frac{1-ln x}{x^2}$

5. $sin^{-1} \sqrt{x}$
   
   Answer: $\frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}$

6. Looking at the curve defined by $y + cos y = x + sin x$, what is $\frac{dy}{dx}$?
   
   Answer: $\frac{1+cos x}{1-sin y}$. 


Curve Sketching Problems

Sketch the function $f(x) = \frac{4x}{x^2+1}$. Answer:

Zero: (0,0)

Critical points:

$(-1, -2), (1,2)$

Asymptote:

$y = 0$

Sketch the function $f(x) = \frac{x^3}{x^2-1}$. Answer:

Zero: (0,0)

Critical points:

$(-\sqrt{3}, -\frac{3\sqrt{3}}{2})$, 

(0,0),

$(-\sqrt{3}, -\frac{3\sqrt{3}}{2})$

Asymptotes:

$x = \pm 1, y = x$

Note: Red lines are asymptotes
Min/Max Problems

1. If you are a widget manufacturer who can make widgets for $50 and you know that you will sell \(200 - 2P\) widgets where \(P\) is the price you set, what price \(P\) should you set to maximize your profit?

Answer: $75. Your total profit is \((200 - 2P)(P - 50)\). Taking the derivative of this and setting it to zero yields

\[-2(P - 50) + 200 - 2P = 300 - 4P = 0\]

which gives \(P = 75\)

\[
\lim_{P \to \pm \infty} (200 - 2P)(P - 50) = -\infty
\]

so this is the maximum profit.

2. If you are building a rectangular garden right next to your house (so one side of the garden will be the house wall) and you have 60ft of fence, what is the largest area that you can enclose for your garden?

Answer: 450. The total length of fence needed is \(2w + l\) so

\[2w + l = 60\]

and thus \(l = 60 - 2w\). The area of the garden is

\[A = lw = 60w - 2w^2\]

Taking the derivative of this and setting it to zero yields \(60 - 4w = 0\) which gives \(w = 15\) and \(l = 30\).

\[
\lim_{w \to 0+} 60w - 2w^2 = \lim_{w \to \infty} 60w - 2w^2 = 0
\]

so this is the maximum area.

3. What is the minimal distance between a point on the curve \(y = x^2\) and the point \((0,1)\)?

Answer: \(\frac{\sqrt{3}}{2}\). Letting \(d^2\) be the square of the distance between a point on the curve \(y = x^2\) and the point \((0,1)\), \(d^2 = x^2 + (1 - x^2)^2\)

Taking the derivative of this and setting it to 0 gives

\[2x - 4x(1 - x^2) = 4x^3 - 2x = 0\]

which gives \(x = 0\) or \(x = \pm \frac{1}{\sqrt{2}}\)
At $x = 0$, $d^2 = 1$. At $x = \pm \frac{1}{\sqrt{2}}$, $d^2 = \frac{3}{4}$. $\lim_{x \to \pm \infty} d^2 = \infty$

Thus, $d^2$ is minimized at $x = \pm \frac{1}{\sqrt{2}}$ so the minimal distance is $\frac{\sqrt{3}}{2}$. 