Potentially Useful Information

Trigonometric Identities:

\[ \sin(a + b) = \sin(a) \cos(b) + \cos(a) \sin(b) \]
\[ \cos(a + b) = \cos(a) \cos(b) - \sin(a) \sin(b) \]

\[ \cos^2 x + \sin^2 x = 1 \]
\[ \sec^2 x = 1 + \tan^2 x \]
\[ \csc^2 x = 1 + \cot^2 x \]

\[ \cos^2 x = \frac{1 + \cos 2x}{2} \]
\[ \sin^2 x = \frac{1 - \cos 2x}{2} \]

Some trigonometric derivatives and integrals:

\[ \frac{d}{dx}(\tan x) = (\sec x)^2 \]
\[ \frac{d}{dx}(\sec x) = \sec x \tan x \]

\[ \int \sec x dx = \ln|\sec x + \tan x| + C \]
18.089 Take-home Midterm 1

Name:

1. Find the following limits (1pt each). Write $\pm \infty$ if the limit is $\pm \infty$ and write DNE if the limit does not exist.

   a. $\lim_{x \to -2} \frac{x^3 + 8}{x^2 + 3x + 2}$

   b. $\lim_{x \to \pi} \frac{1 - \cos x}{x}$

   c. $\lim_{x \to 0} \frac{e^x - 1}{1 - \cos x}$

   d. $\lim_{x \to \infty} \sqrt{x^2 - 6x + 18} - x$

   e. $\lim_{x \to 0} \frac{\tan x - \sin x}{x - x \cos x}$

2. (5pts) Prove that the derivative of $\cos x$ is $-\sin x$ from the definition of the derivative. You may use the facts that $\lim_{x \to 0} \frac{\sin x}{x} = 1$ and $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$. 
3. Find the derivatives of the following functions (2pts each)

a. $2x^5 - x^3 + x + 4$

b. $\tan^{-1}(\sqrt{x})$

c. $\frac{(\tan x)^2}{x}$

d. $x^{\sin x}$

e. $\frac{\ln x}{x}$

f. $(5x + 1)^{\frac{4}{3}}$

g. $x\sin(e^{\sqrt{2x+1}})$

h. $x^2 e^x$

i. $\ln(\ln(\ln x))$

j. $x^{(x^x)}$
4. (5pts) If you are selling widgets which cost $20 to buy and you expect that you will sell \( \frac{8000}{p^2} \) widgets where \( P \) is the price you set, what price should you sell the widgets for to maximize your profit?

5. (5pts) You are currently at the point \((0, -4)\) and are trying to reach the point \((12, 0)\). If you have speed 5 while on the x-axis and speed 3 while not on the x-axis, what path should you take to reach your destination in the shortest possible time?
6. (5pts each) Sketch the following functions, finding all zeros, critical points, asymptotes, and the behavior near points where the function is undefined.

a. \( f(x) = x^3 + 3x^2 \).

b. \( f(x) = \frac{1}{x^2+2x} \).
7. Find the following integrals (2pts each)

a. \( \int_0^{\pi/2} \frac{\cos x}{\sqrt{\sin x}} \, dx \)

b. \( \int x \cos x \, dx \)

c. \( \int x^2 \tan^{-1} x \, dx \)

d. \( \int_1^2 \frac{x^3 - 1}{x^2 + x} \, dx \)

e. \( \int_0^1 \sqrt{7x + 9} \, dx \)

f. \( \int (\sin x)^2 \frac{\cos x}{(\cos x)^4} \, dx \)

g. \( \int_0^2 \frac{x + 1}{\sqrt{16 - x^2}} \, dx \)
h. \( \int \frac{1}{e^x+1} \, dx \)

i. \( \int (\sin x)^4 \, dx \)

j. \( \int \frac{x}{\sqrt{x^2+2x+2}} \, dx \)

9. Find the following volumes (5pts each)

a. The area between the curves \( y = \sqrt{x} \) and \( y = x \) rotated about the x-axis

b. The area between the curves \( y = \sqrt{x} \) and \( y = x \) rotated about the y-axis
10. (5pts) What is the length of the curve \( f(x) = \frac{x^2}{4} - \frac{\ln x}{2} \) between \( x = 1 \) and \( x = 2 \)?

11. (5pts) What is the surface area of the solid formed by rotating the curve \( y = \sqrt{x} \) from \( x = 1 \) to \( x = 4 \) about the x-axis? Include the areas from the left and right sides of this solid.

13. (2pts) What is \( \sum_{n=1}^{\infty} \frac{2^n}{5^n} \)?

14. (3pts) What is \( \sum_{n=1}^{\infty} \frac{1}{n^2+4n+3} \)?
15. (1pt each) Decide whether or not the following series converge and briefly explain your reasoning.

a. \( \sum_{n=1}^{\infty} \frac{2n+1}{n^3+n} \)

b. \( \sum_{n=1}^{\infty} \frac{1}{5n+100} \)

c. \( \sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)} \)

d. \( \sum_{n=1}^{\infty} \frac{n}{e^n} \)

e. \( \sum_{n=1}^{\infty} \frac{1}{n(\ln(n)+1)^2} \)
Extra Credit:

1. (5pts) Find the first four terms of the Taylor series for \( f(x) = \sqrt{x + 1} \) around \( x = 0 \). Using this, approximate \( \sqrt{1.04} \) to an accuracy of 6 decimal places.

2. (1+4pts)

Consider the series \( \sum_{n=0}^{\infty} (-1)^n 2^{1-\lfloor \log_2(n+2) \rfloor} \) which goes as follows:

\[
1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{4} - \frac{1}{4} + \frac{1}{4} - \frac{1}{4} + \frac{1}{4} - \frac{1}{4} + \frac{1}{8} - \frac{1}{8} + \frac{1}{8} - \frac{1}{8} + \frac{1}{8} - \frac{1}{8} + \frac{1}{8} - \frac{1}{8} + \frac{1}{8} - \frac{1}{8} + \cdots
\]

a. Does this series converge? If so, what does it equal?

b. Describe how the terms of this series can be rearranged to make it sum to any given integer value.