18.089 TAKE-HOME EXAM 3

DUE: TUE JULY 14 NOON, E18-301W

Name: Solutions

This exam consists of eight problems, not arranged in any particular order. Please solve all problems in the space provided (or attaching additional sheets as necessary), showing all work as neatly and cleanly as possible.

All work must be your own. In particular, you may not seek help from other students in the class or any "live" internet resources. But feel free to use reference works such as your class notes, a textbook, or Wikipedia. If you have any questions about what constitutes an acceptable source, please ask.

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Problem 1 (20 pt) Let \( \mathbf{F} = (2x + ay)i + (2y + 2x)j \).

(a) For what value of \( a \) is this a conservative field? For that value, find a potential function.

(b) Assume the value of \( a \) from the first part, what are the possible values of \( \int_C \mathbf{F} \cdot d\mathbf{r} \), where \( C \) is a curve (not necessarily closed) that starts at the origin and is contained in the unit disk?

(c) With the same value \( a \), find all the possible values of the line integral, where \( C \) is allowed to be any curve which is contained in the unit disk? (Hint: first convert all the curves into ones that pass through the origin.)

\[ (a) \quad N_x = M_y \quad \Rightarrow \quad |2 = a | \]

\[ \mathbf{F} = \langle 2x + 2y, 2y + 2x \rangle \]

\[ f = \int M \, dx + g(y) = x^2 + 2xy + g(y) \]

\[ f_y = 2x + g'(y) = 2x + 2y \quad \Rightarrow \quad g'(y) = 2y \quad \Rightarrow \quad g(y) = y^2 + C \]

\[ f = x^2 + 2xy + y^2 + C \]

\[ (b) \quad C \text{ starts at } (0,0) \text{ and ends at } (x, y) \]

then \( \int_C \mathbf{F} \cdot d\mathbf{r} = f(x, y) - f(0,0) = x^2 + 2xy + y^2 = (x+y)^2 \)

\((x, y) \) can be any point inside \( \{ x^2 + y^2 \leq 1 \} \)

therefore \( x+y \) takes value between \(-\sqrt{2} \) and \( \sqrt{2} \). So \((x+y)^2 \) takes value between \( 0 \) and \( 2 \).

\[ 0 \leq \int_C \mathbf{F} \cdot d\mathbf{r} \leq 2 \]

\( (c) \quad \text{For curve starts at } A \text{ and ends at } B \)

\[ \int_C \mathbf{F} \cdot d\mathbf{r} = (\int_{C_1} + \int_{C_2}) \mathbf{F} \cdot d\mathbf{r} \]

\( C_1 \text{ takes value between } 0 \text{ and } 2 \) \( \text{ direction) } \)

\[ \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = -2 \]

\( C_2 \) between \( 0 \) and \( 2 \) therefore \( -2 \leq \int_{C_2} \mathbf{F} \cdot d\mathbf{r} \leq 2 \).
Problem 2 (10 pt) Show that the value of $\int_C -y^3dx + x^3dy$ around any positively oriented simple closed curve $C$ is always positive.

Using Green's theorem:

$$\int_C -y^3dx + x^3dy$$

$$= \iint_R \left( \frac{\partial(x^3)}{\partial x} - \frac{\partial(-y^3)}{\partial y} \right) \, dxdy$$

$$= \iint_R 3(x^2 + y^3) \, dxdy$$

$x^2 + y^3 \geq 0$ everywhere, and is strictly positive except $(0,0)$.

Therefore, the integration

$$\iint_R 3(x^2 + y^3) \, dxdy > 0.$$
**Problem 3 (10 pt)** Compute the area of the region bounded by the following three curves: the straight line between $(0,0)$ and $(0,1)$, the straight line between $(0,0)$ and $(1,0)$, and the curve given by \( x = (1-t)^3, y = t^2, 0 \leq t \leq 1 \). You can use any method.

\[
\oint_C \vec{F} \cdot d\vec{r} = \iint_R \left( \frac{\partial M}{\partial x} - \frac{\partial N}{\partial y} \right) \, dx \, dy = \iint_R \, dx \, dy = \text{Area}(R)
\]

\[
\oint_C \vec{F} \cdot d\vec{r} = \int_{C_1} + \int_{C_2} + \int_{C_3}
\]

\[
\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_1} x \, dy = 0
\]

\[
\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{t=0}^{1} (1-t)^3 \, d(t^3) = \int_{0}^{1} (1-3t+3t^2-t^3) \, 2t \, dt
\]

\[
= \left[ t^2 - 2t^3 + \frac{3}{2} t^4 - \frac{3}{5} t^5 \right]_0^1 = \frac{1}{10}
\]

\[
\int_{C_3} \vec{F} \cdot d\vec{r} = \int_{C_3} x \, dy = 0
\]

Area = \[
\oint_C \vec{F} \cdot d\vec{r} = 0 + \frac{1}{10} + 0 = 1 \frac{1}{10}
\]
Problem 4 (10 pt) (a) Calculate \( \text{div} \vec{F} \) and \( \text{curl} \vec{F} \) for
\[
\vec{F} = r^n (xi + yj), \quad r = \sqrt{x^2 + y^2}.
\]
Hint: simplify the differentiation by using \( r_x = \frac{x}{r}, r_y = \frac{y}{r} \). You can include \( r \) in your final answer.
(b) For which value(s) of \( n \) is \( \text{div} \vec{F} = 0 \)?
(c) For which value(s) of \( n \) is \( \text{curl} \vec{F} = 0 \)?

(a) \( M = r^n x \quad N = r^n y \quad P > 0 \)
\[
\text{div} \vec{F} = M_x + N_y + P_z = (nr^{n-1} r_x x + r^n) + (nr^{n-1} r_y y + r^n)
\]
\[
r_x = \frac{x}{r}, \quad r_y = \frac{y}{r} = nr^{n-1} \cdot \frac{x^2}{r} + r^n + nr^{n-1} \cdot \frac{y^2}{r} + r^n
\]
\[
= nr^{n-2}(x^2+y^2) + 2r^n
\]
\[
x^2 + y^2 = r^2 = \frac{r^n}{(n+2)}
\]
\[
\text{curl} \vec{F} = (N_x - M_y) \hat{k} = (nr^{n-1} r_y y - nr^{n-1} r_x x) \hat{k}
\]
\[
= (nr^{n-1} \cdot \frac{y}{r} - nr^{n-1} \cdot \frac{x}{r}) \hat{k} = \frac{2}{r}
\]

(b) \( n = -2 \)
\[ \text{div} \vec{F} = 0 \]

(c) \( \text{any } n \)
\[ \text{curl} \vec{F} = 0 \]
Problem 5 (10 pt) Find the point(s) on the surface \(xyz^2 = 1, x, y, z \geq 0\) which is closest to the origin. Write out the \((x, y, z)\) coordinate of the point(s). Then justify your answer.

Hint: it is easier to minimize the square of distance.

\[
\begin{align*}
z^2 &= \frac{1}{xy} \\
\mathbf{r}^2 &= x^2 + y^2 + z^2 = x^2 + y^2 + \frac{1}{xy} \\
\text{minimize} \quad f(x, y) &= x^2 + y^2 + \frac{1}{xy} \quad \text{over } x, y \geq 0 \\
\nabla f &= 0 \quad \Rightarrow \\
f_x &= 2x - \frac{1}{x^2y} = 0 \\
f_y &= 2y - \frac{1}{xy^2} = 0 \\
\begin{aligned}
x &= y &= \sqrt[4]{2} \\
x = y = 2^{-\frac{1}{4}}
\end{aligned}
\end{align*}
\]

The corresponding point is \(\left(2^{-\frac{1}{4}}, 2^{-\frac{1}{4}}, 2^{-\frac{1}{4}}\right)\).

Why this is the minimum:

- When \(xy \rightarrow 0\), \(z^2 \rightarrow \infty \) therefore \(\mathbf{r}^2 \rightarrow \infty\).
- \(x, y \rightarrow \infty\) \(\mathbf{r}^2 \rightarrow \infty\).

Since we only have one critical point, this is the minimum.
Problem 6 (20 pt) Compute the following flux integrals. You can use whatever method you like:

(a) \( \vec{F} = z \hat{k}, \) S is unit sphere, with normal vector pointing outwards.
(b) \( \vec{F} = x \hat{i} + y \hat{j} + z \hat{k} \) across \( S, \) where \( S \) is the part of the plane \( z + 3x + 3y = 1 \) with \( x \geq 0, y \geq 0, z \geq 0. \)

\[\text{Divergence Theorem}\]

\[(a) \ \iint_{S} \vec{F} \cdot \hat{n} \, dA = \iiint_{R} (\text{div} \, \vec{F}) \, dV \]
\[= \iiint_{R} 1 \, dV = \text{volume (sphere)} = \frac{4\pi}{3}\]

\[(b) \ z = 1 - 3x - 3y = f(x, y) \]
\[\vec{n} \, dA = \langle -f_x, -f_y, 1 \rangle \, dx \, dy = \langle 3, 3, 1 \rangle \, dx \, dy \]
\[\iint_{\Delta} \vec{F} \cdot \vec{n} \, dA = \iint_{\Delta} \langle x, y, z \rangle \cdot \langle 3, 3, 1 \rangle \, dx \, dy \]
\[= \iint_{\Delta} (3x + 3y + 2) \, dx \, dy \]
\[= \text{area (\( \Delta \))} \]
\[= \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{18}\]
Problem 7 (10 pt) Let $\vec{F} = x\hat{i} + xy\hat{j} + z^2\hat{k}$. Let $S$ be a cylinder of radius 3 and height 2, with base in the xy-plane and symmetric around the z-axis, NOT including the top and the bottom. Use Stokes theorem to compute $\iint_S \text{curl} \vec{F} \cdot \hat{n} \, dA$.

\[\iint_S \text{curl} \vec{F} \cdot \hat{n} \, dA = \oint_{C_1} \vec{F} \cdot \hat{r} \, dr + \oint_{C_2} \vec{F} \cdot \hat{r} \, dr\]

with directions as indicated on the picture.

on $C_1$:
\[
\begin{align*}
X(t) &= 3 \cos t \\
y(t) &= 3 \sin t \\
z(t) &= 0
\end{align*}
\]

\[
\oint_{C_1} \vec{F} \cdot \hat{r} \, dr = \int_{0}^{2\pi} \begin{pmatrix} 3 \cos t, 9 \cos t \sin t, 0 \end{pmatrix} \cdot \begin{pmatrix} -3 \sin t, 3 \cos t, 0 \end{pmatrix} \, dt
\]

\[
= \int_{0}^{2\pi} (-9 \cos t \sin t + 27 \cos^2 t \sin t) \, dt
\]

\[
= \left. \left( -\frac{9}{2} \cos^2 t - 9 \cos^3 t \right) \right|_{0}^{2\pi} = 0.
\]

\[\oint_{C_2} \vec{F} \cdot \hat{r} \, dr = \int_{0}^{2\pi} \begin{pmatrix} 3 \cos t, 3 \sin t, 2 \end{pmatrix} \cdot \begin{pmatrix} -3 \sin t, 3 \cos t, 0 \end{pmatrix} \, dt
\]

\[
= \int_{0}^{2\pi} (-9 \sin t \cos t + 27 \sin^2 t \cos t) \, dt
\]

\[
= 0 \quad \text{(the same as $C_1$)}
\]

Therefore \[
\iint_S \text{curl} \vec{F} \cdot \hat{n} \, dA = 0.
\]
Problem 8 (10 pt) Find the volume of the solid region bounded below by the cone \( z^2 = x^2 + y^2 \), and above by the sphere centered at the origin with radius \( \sqrt{2} \).

Use cylindrical coordinates:

For each \( r \), \( h \) goes from \( r \) to \( \sqrt{2-r^2} \)

Volume = \( \int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{2-r^2}} \int_{h=r}^{\sqrt{2-r^2}} r \, dr \, d\theta \, dh \)

\[
= \int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{2-r^2}} \left( \sqrt{2-r^2} - r \right) r \, dr \, d\theta
\]

\[
= \int_{\theta=0}^{2\pi} \left[ \frac{1}{3} \left( 2-r^2 \right)^{3/2} - \frac{1}{3} r^3 \right]_{r=0}^{1} \, d\theta
\]

\[
= \int_{\theta=0}^{2\pi} \frac{1}{3} \left( 2 - \frac{3}{2} \right) \, d\theta
\]

\[
= \frac{2\pi}{3} \left( \frac{3}{2} - 2 \right)
\]