18.089 Problem Set 1 Solutions

Taylor Series and Power Series

1. (5pts each) Find the first six terms (including the constant term and terms which are 0) of the Taylor series with $a = 0$ for the following functions:

a. $\tan x$

Ans: $x + \frac{x^3}{3} + \frac{2x^5}{15}$

If $f(x) = \tan x$, the first five derivatives of $f$ are as follows:

$f'(x) = (\sec x)^2$

$f''(x) = 2(\sec x)^2 \tan x$

$f'''(x) = 2(\sec x)^4 + 4(\sec x)^2 (\tan x)^2$

$f''''(x) = 16(\sec x)^4 (\tan x) + 8(\sec x)^2 (\tan x)^3$

$f'''''(x) = 16(\sec x)^6 + 88(\sec x)^4 (\tan x)^2 + 16(\sec x)^2 (\tan x)^4$

$f'(0) = 1, f''(0) = 2, f'''(0) = 16, f(0) = f''(0) = f'''(0) = 0$

The first few terms of the Taylor series are $x + \frac{x^3}{3} + \frac{2x^5}{15}$

b. $\sec x$

Ans: $1 + \frac{x^2}{2} + \frac{5x^4}{24}$

If $f(x) = \sec x$, the first five derivatives of $f$ are as follows:

$f'(x) = \tan x \sec x$

$f''(x) = (\sec x)^3 + (\sec x)(\tan x)^2$
\[ f'''(x) = 5 \sec^3 x \tan x + \sec x \tan^3 x \]
\[ f''''(x) = 5 \sec^5 x + 18 \sec^3 x \tan^2 x + \sec x \tan^4 x \]
\[ f'''''(x) = 61 \sec^5 x \tan x + 58 \sec^3 x \tan^3 x + \sec x \tan^5 x \]
\[ f(0) = 1, f''(0) = 1, f'''(0) = 5, f'(0) = f'''(0) = f'''''(0) = 0 \]

The first few terms of the Taylor series are \[ 1 + \frac{x^2}{2} + \frac{5x^4}{24} \]

c. \[ \frac{-1}{\sqrt{1-x}} \]

Ans: \[ -1 - \frac{x}{2} - \frac{3x^2}{8} - \frac{5x^3}{16} - \frac{35x^4}{128} - \frac{63x^5}{256} \]

If \( f(x) = \frac{-1}{\sqrt{1-x}} \), the first five derivatives of \( f \) are as follows:

\[ f'(x) = \frac{-1}{2} (1 - x)^{\frac{3}{2}} \]
\[ f''(x) = \frac{-3}{4} (1 - x)^{\frac{5}{2}} \]
\[ f'''(x) = \frac{-15}{8} (1 - x)^{\frac{7}{2}} \]
\[ f''''(x) = \frac{-105}{16} (1 - x)^{\frac{9}{2}} \]
\[ f'''''(x) = \frac{-945}{32} (1 - x)^{\frac{11}{2}} \]

\[ f(0) = -1, f'(0) = -\frac{1}{2}, f''(0) = -\frac{3}{4}, f'''(0) = -\frac{15}{8}, f''''(0) = -\frac{105}{16}, f'''''(0) = -\frac{945}{32} \]

The first few terms of the Taylor series are

\[ -1 - \frac{x}{2} - \frac{3x^2}{8} - \frac{5x^3}{16} - \frac{35x^4}{128} - \frac{63x^5}{256} \]
d. $\sin^{-1} x$

Ans: $x + \frac{x^3}{6} + \frac{3x^5}{40}$

From the answer to part c, the first few terms of the power series for $\frac{1}{\sqrt{1-u}}$ are $1 + \frac{u}{2} + \frac{3u^2}{8} + \frac{5u^3}{16} + \ldots$

Plugging $u = x^2$ into this, the first few terms of the power series for $\frac{1}{\sqrt{1-x^2}}$ are $1 + \frac{x^2}{2} + \frac{3x^4}{8} + \frac{5x^6}{16} + \ldots$

Integrating this, the first few terms of the Taylor series for $\sin^{-1} x$ are $x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \ldots$

2. (5pts each) Find the power series for the following functions as well as their radius of convergence.

a. $e^{(x^2)}$

Ans: $\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$, $R = \infty$

The Taylor series for $e^u$ is $e^u = \sum_{n=0}^{\infty} \frac{u^{2n}}{n!}$. Plugging in $u = x^2$ gives $e^{(x^2)} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$

Using the ratio test, $\frac{x_{n+1}}{x_n} = \frac{x^2}{n+1}$ so $\lim_{n \to \infty} \frac{x_{n+1}}{x_n} = 0$ for all $x$. Thus, $R = \infty$.

b. $\ln(x^2 + 1)$
Ans: $\ln(x^2 + 1) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n+1}, \quad R = 1$

The Taylor series for $\frac{1}{1-u}$ is $\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$. Integrating this gives $-\ln(1-u) = \sum_{n=0}^{\infty} \frac{u^{n+1}}{n+1}$. Plugging in $u = -x^2$ gives

$$\ln(x^2 + 1) = -\sum_{n=0}^{\infty} \frac{u^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n+1}$$

Using the ratio test, $\frac{x_{n+1}}{x_n} = \frac{-(n+1)x^2}{n+2}$ so $\lim_{n \to \infty} \frac{x_{n+1}}{x_n} = -x^2$. Using the ratio test, this converges if $|x| < 1$ and diverges if $|x| > 1$ so $R = 1$.

Vector Problems

1. (2pts each) Find the following dot products

   a. $< 1,3 > \cdot < 3, -1 >$

      Ans: 0

   b. $< 1, -3, 5 > \cdot < 1, 2, 3 >$

      Ans: 10

   c. $< 1, 4, -3 > \cdot < -4, 2, 1 >$

      Ans: 1

2. (2pts each) Find the following cross products

   a. $< 0, 1, 0 > \times < 1, 0, 0 >$
Ans: $<0,0,-1>$

b. $<1,2,3> \times <1,1,1>$
Ans: $<-1,2,-1>$

c. $<-2,2,2> \times <1,-1,-1>$
Ans: $<0,0,0>$

3. (3pts each) Find the angle $\theta$ between the diagonal of a cube and
a. an adjacent edge
Ans: $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx .955$

One diagonal of the cube is the vector $<1,1,1>$ and one adjacent edge is the vector $<1,0,0>$
The angle between them is

$$
\cos^{-1}\left(\frac{<1,1,1> \cdot <1,0,0>}{|<1,1,1>| \cdot |<1,0,0>|}\right) = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx .955
$$

b. the diagonal of an adjacent face
Ans: $\cos^{-1}\left(\frac{\sqrt{2}}{\sqrt{3}}\right) \approx .615$

One diagonal of the cube is the vector $<1,1,1>$ and one adjacent edge is the vector $<1,1,0>$
The angle between them is

$$
\cos^{-1}\left(\frac{<1,1,1> \cdot <1,1,0>}{|<1,1,1>| \cdot |<1,1,0>|}\right) = \cos^{-1}\left(\frac{\sqrt{2}}{\sqrt{3}}\right) \approx .615
$$
4. (5pts) What is the area of the triangle with vertices (1,1), (3,2), (2,5)?

Ans: $\frac{7}{2}$

The triangle has sides $<2,1>$ and $<1,4>$ and its area is

$$\frac{1}{2} | <2,1> \times <1,4> | = \frac{7}{2}$$

Matrix Problems:

1. (2pts) What is $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$?

Ans: $\begin{pmatrix} 5 \\ -1 \end{pmatrix}$

2. (5pts) If $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$, what are $x$ and $y$?

Ans: $x = 5, y = -1$

Subtracting two times the first row from the second row gives

$$\begin{pmatrix} 1 & 4 \\ 0 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

This implies that $y = -1$. Solving for $x$ now gives $x = 5$.

3. (5pts) What is $\begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 1 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$?

Ans: $\begin{pmatrix} 1 & 6 & 7 \\ 1 & 0 & 1 \\ 0 & 6 & 10 \end{pmatrix}$
4. (5pts each) Find the determinants of the following matrices and their inverses (if they exist)

a. $\begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix}$

Ans: The determinant is $-8$ and the inverse is $\begin{pmatrix} -\frac{7}{8} & \frac{3}{8} \\ \frac{5}{8} & -\frac{1}{8} \end{pmatrix}$

The determinant is $1 \cdot 7 - 3 \cdot 5 = -8$

Now start with

$\begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Subtracting 5 times the first row from the second row gives

$\begin{pmatrix} 1 & 3 \\ 0 & -8 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix}$

Dividing the second row by $-8$ gives

$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \frac{5}{8} & -\frac{1}{8} \end{pmatrix}$

Subtracting three times the first row from the second row gives

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -\frac{7}{8} & \frac{3}{8} \\ \frac{5}{8} & -\frac{1}{8} \end{pmatrix}$ so the inverse is $\begin{pmatrix} -\frac{7}{8} & \frac{3}{8} \\ \frac{5}{8} & -\frac{1}{8} \end{pmatrix}$

b. $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$
Ans: The determinant is 0 so the matrix is not invertible

We can compute the determinant directly as follows:

\[ 1 \cdot 5 \cdot 9 + 2 \cdot 6 \cdot 7 + 3 \cdot 4 \cdot 8 - 1 \cdot 6 \cdot 8 - 2 \cdot 4 \cdot 9 - 3 \cdot 5 \cdot 7 = 45 + 84 + 96 - 48 - 72 - 105 = 0 \]

Alternatively, expanding by the first row, the determinant is

\[ 1(5 \cdot 9 - 6 \cdot 8) - 2(4 \cdot 9 - 6 \cdot 7) + 3(4 \cdot 8 - 5 \cdot 7) = -3 + 12 - 9 = 0 \]

\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 2 & 3 \\
1 & 3 & 6
\end{pmatrix}
\]

Ans: The determinant is 1 and the inverse is

\[
\begin{pmatrix}
3 & -3 & 1 \\
-3 & 5 & -2 \\
1 & -2 & 1
\end{pmatrix}
\]

We can compute the determinant directly as follows:

\[ 1 \cdot 2 \cdot 6 + 1 \cdot 3 \cdot 1 + 1 \cdot 1 \cdot 3 - 1 \cdot 3 \cdot 3 - 1 \cdot 1 \cdot 6 - 1 \cdot 2 \cdot 1 = 12 + 3 + 3 - 9 - 6 - 2 = 1 \]

Alternatively, expanding by the first row, the determinant is

\[ 1(2 \cdot 6 - 3 \cdot 3) - 1(1 \cdot 6 - 1 \cdot 3) + 1(1 \cdot 3 - 2 \cdot 1) = 3 - 3 + 1 = 1 \]

Now start with

\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 2 & 3 \\
1 & 3 & 6
\end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

Subtracting the first row from the second and third rows gives

\[
\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 2 \\
0 & 2 & 5
\end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}
\]
Subtracting the second row from the first row and two times the second row from the third row gives

\[
\begin{pmatrix}
1 & 0 & -1 \\
0 & 1 & 2 \\
0 & 0 & 1 
\end{pmatrix}, \begin{pmatrix}
2 & -1 & 0 \\
-1 & 1 & 0 \\
1 & -2 & 1 
\end{pmatrix}
\]

Adding the third row to the first row and subtracting two times the third row from the second row gives

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 
\end{pmatrix}, \begin{pmatrix}
3 & -3 & 1 \\
-3 & 5 & -2 \\
1 & -2 & 1 
\end{pmatrix}
\]

Thus, the inverse is

\[
\begin{pmatrix}
3 & -3 & 1 \\
-3 & 5 & -2 \\
1 & -2 & 1 
\end{pmatrix}
\]

Parametric equation problems:

1. (5pts) If the position vector of a particle is \( <t, t^2, t^3> \), what are its velocity and acceleration vectors?

Ans: \( \vec{v} = <1, 2t, 3t^2>, \vec{a} = <0, 2t> \)

2. (5pts) What is the curve described by the parametric equations \( x(t) = t^2, y(t) = t^4 - 1 \)? What portion of the curve is actually described by these equations?

Ans: This corresponds to the curve \( y = x^2 - 1 \) but only the part where \( x \geq 0 \) is actually covered by these parametric equations.
3. (5pts) What is the total distance travelled by a particle whose position vector is \( <t, t^2> \) from \( t = 0 \) to \( t = 4 \)?

Ans: \( \frac{8}{27}(10\sqrt{10} - 1) \)

The total distance is \( L = \int_0^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \int_0^4 \sqrt{1 + \frac{9}{4}t} dt \)

Taking \( u = \frac{9}{4}t + 1, L = \frac{9}{4} \int_1^{10} \sqrt{u} du = \frac{8}{27}(10\sqrt{10} - 1) \)

4. (5pts) What are the parametric equations for a projectile launched with initial velocity \( <v_x, v_y> \) if there is a wind which causes it to accelerate horizontally with acceleration \( a \)?

Ans: \( x(t) = v_x t + \frac{1}{2}at^2, y(t) = v_y t - \frac{1}{2}gt^2 \)

\( \vec{a} = <a, -g> \)

Integrating this gives \( \vec{v} = <at + C_1, -gt + C_2> \) for some \( C_1, C_2 \).

\( \vec{v}(0) = <v_x, v_y> \) so we must have \( \vec{v} = <at + v_x, -gt + v_y> \).

Integrating this again we have that

\( \vec{s} = <v_x t + \frac{1}{2}at^2 + C_3, v_y t - \frac{1}{2}gt^2 + C_4> \) Assuming we start at the origin, \( C_3 = C_4 = 0 \) and \( \vec{s} = <v_x t + \frac{1}{2}at^2, v_y t - \frac{1}{2}gt^2> \)