18.089 Problem Set 2
Due 6/23/15

Polar Coordinates Problems

1. (5pts) What is the equation for the curve $r(\theta) = \sin \theta + \cos \theta$ in Cartesian coordinates?

Ans: $(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 = \frac{1}{2}$

Plugging $\sin \theta = \frac{y}{r}$ and $\cos \theta = \frac{x}{r}$ into our equation gives $r = \frac{x+y}{r}$ so $r^2 = x^2 + y^2 = x + y$. This is sufficient but can be re-expressed as follows:

$x^2 - x + y^2 - y = 0$

$x^2 - x + \frac{1}{4} + y^2 - y + \frac{1}{4} = \frac{1}{2}$

$(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 = \frac{1}{2}$

Thus, this curve is the circle of radius $\frac{1}{\sqrt{2}}$ centered at $(\frac{1}{2}, \frac{1}{2})$.

2. (5pts) What is the equation for the curve $y = x^2 - 2x$ in polar coordinates?

Ans: $r = \frac{\sin \theta + 2 \cos \theta}{(\cos \theta)^2}$

Plugging in $y = r \sin \theta$ and $x = r \cos \theta$ gives

$r \sin \theta = r^2 (\cos \theta)^2 - 2r \cos \theta$

Dividing this by $r$ gives $\sin \theta = r(\cos \theta)^2 - 2 \cos \theta$.
Solving this for $r$ gives $r = \frac{\sin \theta + 2 \cos \theta}{(\cos \theta)^2}$

3. (5pts each) Sketch the following functions

a. $r(\theta) = \theta$ (only sketch the part where $\theta \geq 0$)

Ans:
b. $r(\theta) = 1 + 2 \sin \theta$

Ans:

4. (5pts) What is the area inside the curve $r(\theta) = 2 + \sin \theta$?

Ans: $\frac{9\pi}{2}$

The area inside the curve is

$$A = \int_0^{2\pi} \frac{r^2}{2} \, d\theta = \int_0^{2\pi} \frac{(2 + \sin \theta)^2}{2} \, d\theta = \int_0^{2\pi} 2 + 2 \sin \theta + \frac{(\sin \theta)^2}{2} \, d\theta$$

$\int_0^{2\pi} 2 \, dr = 4\pi$ and $\int_0^{2\pi} \sin \theta \, dr = -\cos(2\pi) + \cos(0) = 0$

$\int_0^{2\pi} \frac{(\sin \theta)^2}{2} \, d\theta = \int_0^{2\pi} \frac{1 - \cos 2\theta}{4} \, d\theta = \frac{\pi}{2} - \int_0^{2\pi} \frac{-\cos 2\theta}{4} \, d\theta = \frac{\pi}{2}$

Putting everything together, $A = \frac{9\pi}{2}$
Level Curves, Partial Derivative, Directional Derivative, and Gradient Problems

1. (5pts) Sketch the level curves for the function $f(x, y) = y^2 - x^2$

Ans: The red lines are the level curve for $f(x, y) = -1$. The blue lines are the level curve for $f(x, y) = 0$. The green lines are the level curve for $f(x, y) = 1$.

2. (5pts) Find the directional derivative $f'_{\hat{v}}(0,0)$ for an arbitrary $\hat{v}$ for the function $f(x, y) = 0$ if and $\frac{x^2y}{x^2+y^2}$ otherwise.

Ans: If $\hat{v} = \langle a, b \rangle$ then $f'_{\hat{v}}(0,0) = \frac{a^2b}{a^2+b^2}$

By definition, if $\hat{v} = \langle a, b \rangle$ then

$$f'_{\hat{v}}(0,0) = \lim_{h \to 0} \frac{f(ah, bh) - f(0,0)}{h} = \frac{(a^2b)h^2}{(a^2+b^2)h^3} = \frac{a^2b}{a^2+b^2}$$
3. (3pts each) Find the gradients for the given functions at the given points

a. \( f(x, y) = \frac{y^2}{x} \) at \((1,1)\)

Ans: \(< -1,2 >\)

\[ \nabla f = <\frac{-y^2}{x^2}, \frac{2y}{x}> = < -1,2 > \text{ at } (1,1) \]

b. \( f(x, y) = \frac{x}{y} + \frac{y}{x} \) at \((2,1)\)

Ans: \(< \frac{3}{4}, -\frac{3}{2}>\)

\[ \nabla f = <\frac{1}{y} - \frac{y}{x^2}, \frac{1}{x} - \frac{x}{y^2}> = < \frac{3}{4}, -\frac{3}{2}> \text{ at } (2,1) \]

c. \( f(x, y) = x^3 + xy + y^3 \) at \((-1,1)\)

Ans: \(< 4,2 >\)

\[ \nabla f = < 3x^2 + y, 3y^2 + x > = < 4,2 > \text{ at } (-1,1) \]

Tangent Plane and Min-Max Problems:

1. (3pts each) Find the tangent plane to the given functions at the given points

a. \( z = \frac{y}{x} \) at \((2,2,1)\)

Ans: \( \frac{1}{2}x - \frac{1}{2}y + z = 1 \)
At \((2,2,1)\), \(\frac{\partial z}{\partial x} = -\frac{y}{x^2} = -\frac{1}{2}\) and \(\frac{\partial z}{\partial y} = \frac{1}{x} = \frac{1}{2}\) so the tangent plane has the form \(-\frac{1}{2} x + \frac{1}{2} y - z = C\). To pass through the point \((2,2,1)\), \(C = -1\) so our plane is \(-\frac{1}{2} x + \frac{1}{2} y - z = -1\)

b. \(z = y^2 - x^3\) at \((1, -1, 0)\)

Ans: \(3x + 2y + z = 1\)

At \((1, -1, 0)\), \(\frac{\partial z}{\partial x} = -3x^2 = -3\) and \(\frac{\partial z}{\partial y} = 2y = -2\) so the tangent plane has the form \(-3x - 2y - z = C\). To pass through the point \((1, -1, 0)\), \(C = -1\) so our plane is \(-3x - 2y - z = -1\)

c. \(z = \frac{x+y}{x^2+y^2}\) at \((1,3,.4)\)

Ans: \(-.02x + .14y + z = .8\)

At \((1,3,.4)\), \(\frac{\partial z}{\partial x} = \frac{1}{x^2+y^2} - \frac{2x(x+y)}{(x^2+y^2)^2} = .02\) and \(\frac{\partial z}{\partial y} = \frac{1}{x^2+y^2} - \frac{2y(x+y)}{(x^2+y^2)^2} = -.14\) so the tangent plane has the form \(.02x - .14y - z = C\). To pass through the point \((1,3,.4)\), \(C = -.8\) so our plane is \(.02x - .14y - z = -.8\)

2. (10pts each) Find the maximum and minimum values of the given function over the specified region

a. \(f(x, y) = x^2 - 3xy + 2y^2\) over the box where \(|x| \leq 1\) and \(|y| \leq 1\)

Ans: The maximum value of this function occurs at the corner points \((-1,1)\) and \((1, -1)\). At these points, \(f(x, y) = 6\). The minimum value
of this function occurs at the points \((-1, -\frac{3}{4})\) and \((1, \frac{3}{4})\). At these points, \(f(x, y) = -\frac{1}{8}\).

\[ \nabla f = <2x - 3y, 4y - 3x> \] and this is only 0 at the origin. Thus, our only critical point is (0,0) and \(f(0,0) = 0\). On the boundary \(x = -1\), \(f(y) = 2y^2 + 3y + 1\). \(f'(y) = 4y + 3 = 0\) when \(y = \frac{-3}{4}\) so \((-1, -\frac{3}{4})\) is a critical point on this boundary. \(f(-1, -\frac{3}{4}) = -\frac{1}{8}\). Since \(f(-x, -y) = f(x, y)\), by symmetry, \((1, \frac{3}{4})\) will be a critical point on the boundary \(x = 1\) and \(f\left(1, \frac{3}{4}\right) = -\frac{1}{8}\). On the boundary \(y = -1\), \(f(x) = x^2 + 3x + 2\). \(f'(x) = 2x + 3 = 0\) when \(x = -\frac{3}{2}\) but this is not inside our region. Thus, there are no critical points on the boundary \(y = -1\). By symmetry, there are no critical points on the boundary \(y = 1\) either. Checking the four corners, \(f(-1, -1) = f(1,1) = 0\) and \(f(-1,1) = f(1, -1) = 6\).

b. \(f(x, y) = \frac{2x+y}{x^2+y^2+1}\) over \(\mathbb{R}^2\).

Ans: The maximum value of this function occurs at the point \(\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)\) and \(f\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) = \frac{\sqrt{5}}{2}\). The minimum value of this function occurs at the point \(\left(-\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right)\) and \(f\left(-\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right) = -\frac{\sqrt{5}}{2}\).

\[ \nabla f = \left< \frac{2}{x^2+y^2+1} - \frac{2x(2x+y)}{(x^2+y^2+1)^2}, \frac{1}{x^2+y^2+1} - \frac{2y(2x+y)}{(x^2+y^2+1)^2} \right> \]

This is 0 if and only if \(2(x^2 + y^2 + 1) - 2x(2x + y) = 0\) and \((x^2 + y^2 + 1) - 2y(2x + y) = 0\). Dividing the first equation by 2 and simplifying we obtain

\[ y^2 - xy - x^2 + 1 = 0 \]
Simplifying the second equation we obtain that

\[ x^2 - 4xy - y^2 + 1 = 0 \]

Adding these two equations together, \(-5xy + 2 = 0\) so \(y = \frac{2}{5x}\).

Plugging this into the first equation yields

\[ \frac{4}{25x^2} - x^2 + .6 = 0 \]

so \(x^4 - .6x^2 - .16 = 0\)

Using the quadratic equation, \(x^2 = \frac{.6 + \sqrt{.36 + .64}}{2} = \frac{1.6}{2} = \frac{4}{5}\).

\(x = \pm \frac{2}{\sqrt{5}}\) and \(y = \frac{2}{5x} = \pm \frac{1}{\sqrt{5}}\). This gives us our two critical points,

\[ \left(\frac{-2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right) \] and \[ \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) \].

\(f\left(\frac{-2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right) = \frac{-\sqrt{5}}{2}\) and \(f\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) = \frac{\sqrt{5}}{2}\)

No matter how \(x \to \pm \infty\) and/or \(y \to \pm \infty\), \(f(x, y) \to 0\) so these critical points are the only possible maxima and minima.

Jacobian and Chain Rule Problems:

1. (3pts each) Find \(\frac{dF}{dt}\) for each of the following cases using the chain rule.

   a. \(F(x, y, z) = \frac{xy}{z}, x(t) = \sqrt{t}, y(t) = \sin t, z(t) = e^t\)

   Ans: \(\frac{dF}{dt} = \frac{\sin t}{2e^t \sqrt{t}} + \frac{\sqrt{t} \cos t}{e^t} - \frac{\sqrt{t} \sin t}{e^t}\)

   b. \(F(x, y, z) = (x + y + z)^3, x(t) = \cos t, y(t) = \sin t, z(t) = t\)

   Ans: \(3(-\sin t + \cos t + 1)(\cos t + \sin t + t)^2\)

2. (6pts) What is the Jacobian for the function \(F: R^2 \to R^3\) where
\[ F(s, t) = \langle st, \sqrt{s - t}, \sin s + \cos t \rangle \]

Ans: \[
\begin{pmatrix}
\frac{t}{2\sqrt{s-t}} & \frac{s}{2\sqrt{s-t}} \\
\frac{-1}{2\sqrt{s-t}} & \cos s - \sin t
\end{pmatrix}
\]

3. (15pts) If \( F(s, t) = \langle s + t, e^s - e^t, \sqrt{st} \rangle \) and
\( G(x, y, z) = \langle xyz, x^2z + y \rangle \), use the chain rule to find the Jacobian of \( G(F(s, t)) \) at the point (1,1). Then use the chain rule to find the Jacobian of \( F(G(x, y, z)) \) at the point (2,1,2).

Ans: The Jacobian of \( G(F(s, t)) \) at the point (1,1) is
The Jacobian of \( F(G(x, y, z)) \) at the point (2,1,2) is

The Jacobian of \( F(s, t) \) is
\[
\begin{pmatrix}
\frac{1}{e^s} & \frac{1}{-e^t} \\
\frac{\sqrt{t}}{2\sqrt{s}} & \frac{\sqrt{s}}{2\sqrt{t}}
\end{pmatrix}
\]

The Jacobian of \( G(x, y, z) \) is
\[
\begin{pmatrix}
yz & xz & xy \\
2xz & 1 & x^2
\end{pmatrix}
\]

\( F(1,1) = (2,0,1) \) and the Jacobian of \( F(s, t) \) at (1,1) is \[
\begin{pmatrix}
e & -e \\
\frac{1}{2} & \frac{1}{2}
\end{pmatrix}
\]

The Jacobian of \( G(x, y, z) \) at (2,0,1) is \[
\begin{pmatrix}
0 & 2 & 0 \\
4 & 1 & 4
\end{pmatrix}
\] so the Jacobian of \( G(F(s, t)) \) at (1,1) is \[
\begin{pmatrix}
\frac{1}{e} & \frac{1}{-e} \\
\frac{1}{2} & \frac{1}{2}
\end{pmatrix}
\] \[
\begin{pmatrix}
2e & -2e \\
6 + e & 6 - e
\end{pmatrix}
\]

\( G(2,1,2) = (4,9) \) and the Jacobian of \( F(x, y, z) \) at (2,1,2) is \[
\begin{pmatrix}
2 & 4 & 2 \\
8 & 1 & 4
\end{pmatrix}
\]
The Jacobian of $F(s, t)$ at $(4,9)$ is \[ \begin{pmatrix} 1 & 1 \\ \frac{e^4}{3} & -\frac{e^9}{3} \\ \frac{4}{4} & \frac{1}{3} \end{pmatrix} \] so the Jacobian of $F(G(x, y, z))$ at $(2,1,2)$ is

\[
\begin{pmatrix} 1 & 1 \\ e^4 & -e^9 \\ \frac{3}{4} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 2 & 4 & 2 \\ 8 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 10 & 5 & 6 \\ 2e^4 - 8e^9 & 4e^4 - e^9 & 2e^4 - 4e^9 \\ \frac{25}{6} & \frac{10}{3} & \frac{17}{6} \end{pmatrix}
\]