HW 3 Solution:

1. Take \((x, y, z)\) to be the coordinates for the vertex diagonal to \((0, 0, 0)\)

\[ xyz = 1 \Rightarrow z = \frac{1}{xy} \]

the surface area (without lid) is

\[ A = xy + 2xz + 2yz \]

\[ = xy + 2x \cdot \frac{1}{xy} + 2y \cdot \frac{1}{xy} \]

\[ = xy + \frac{2}{x} + \frac{2}{y} \]

Gradient \( \nabla A = (A_x, A_y) = (y - \frac{2}{x^2}, x - \frac{2}{y^2}) = 0 \)

\[ y - \frac{2}{x^2} = 0 \quad \Rightarrow \quad y = \frac{2}{x^2} \]

\[ x - \frac{2}{y^2} = 0 \quad \Rightarrow \quad x = \frac{2}{y^2} \]

In this case, \((x, y, z) = (\frac{2}{\sqrt{2}}, \frac{2}{\sqrt{2}}, \frac{1}{2^{\frac{3}{2}}})\)

Why this is minimum when \(x\) or \(y\) go to infinity,

\[ A = xy + \frac{2}{y} + \frac{2}{x} \] will go to infinity.

Since \((\frac{2}{\sqrt{2}}, \frac{2}{\sqrt{2}})\) is the only critical point, it is the minimum.

2. (a) \[ \iint_S xe^{xy} \, dx \, dy = \int_{x=-2}^{2} \left( \int_{y=-1}^{1} xe^{xy} \, dy \right) \, dx \]

\[ = \int_{-2}^{2} \left. \left( e^{xy} \right) \right|_{y=-1}^{1} \, dx = \int_{-2}^{2} \left( e^{x} - e^{-x} \right) \, dx \]

\[ = \left( e^{x} + e^{-x} \right) \bigg|_{x=-2}^{2} = 0 \]

(The other way doesn't work)
(b) \[ M = \iint \delta(x, y) \, dx \, dy \]
\[ = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_{r=0}^{r=1} r \, dr \, d\theta \]
\[ = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{3}{2} \, d\theta = \frac{2\pi}{3} \]
\[ \bar{x} = \frac{\iint x \delta(x, y) \, dx \, dy}{M} = \frac{\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_{r=0}^{r=1} r^2 \cos \theta \, dr \, d\theta}{M} \]
\[ = \frac{\frac{25}{3} \cdot 2}{M} = \frac{250}{2\pi} = \frac{20}{3\pi} \approx 2.12 \]

By symmetry, \( \bar{y} = 0 \)

Therefore, center of mass is \( \left( \frac{20}{3\pi}, 0 \right) \)

(3) \[ \text{Two boundaries: } y = 1 \text{ and } y = 2 \]
\[ \Rightarrow r \sin \theta = 1 \text{ and } r \sin \theta = 2 \]
\[ 0 \leq \theta \leq \pi, \quad \frac{1}{\sin \theta} \leq r \leq \frac{2}{\sin \theta} \]

Therefore, the integral is \[ \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_{r=\frac{1}{\sin \theta}}^{r=\frac{2}{\sin \theta}} \frac{1}{r^4} \cdot r \, dr \, d\theta \]
\[ = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{2}{8} \sin^2 \theta \, d\theta \]
\[ = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{3}{16} \, d\theta = \frac{3}{16} \pi. \]
Use \( u = 2x - y \) \( \quad v = 2y - x \)

Therefore \( du \, dv = 3 \, dx \, dy \) or \( dx \, dy = \frac{1}{3} \, du \, dv \)

\[
\int_{x=0}^{1} \int_{y=0}^{1-x-y} 3 \, dx \, dy = \int_{u=0}^{3} \int_{v=0}^{3} (2u + v) \, du \, dv = \frac{27}{2}
\]

(b) \( x, y, z \) are symmetric, just compute \( \bar{x} \).

\[
\bar{x} = \frac{1}{M} \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} x \, dz \, dy \, dx
\]

\[
\bar{x} = \frac{1}{M} \int_{x=0}^{1} \int_{y=0}^{1-x} x(1-x-y) \, dy \, dx = \frac{1}{M} \int_{x=0}^{1} \frac{x(1-x)^2}{2} \, dx = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{4}
\]
(a) The relation between height and radius can be seen from the side:

\[ h = 3 - 3r \]

θ goes from 0 to 2π

\[ V = \int_{r=0}^{1} \int_{\theta=0}^{2\pi} \int_{h=0}^{3-3r} 1 \cdot r \, dh \, d\theta \, dr \]

\[ = \int_{0}^{1} \left( \int_{0}^{2\pi} r (3-3r) \, d\theta \right) \, dr \]

\[ = \int_{0}^{1} 2\pi r (3-3r) \, dr \]

\[ = \frac{\pi}{2} \]

(b) Seen from the side:

\[ V = \int_{r=0}^{1} \int_{\theta=0}^{2\pi} \int_{h=0}^{1-r^2} 1 \cdot r \, dh \, d\theta \, dr \]

(7) \[ 0 \leq r \leq 1 \quad 0 \leq \theta \leq 2\pi \quad 0 \leq \varphi \leq \frac{\pi}{2} \]

\[ \frac{\pi a^4}{4M} = \frac{3a}{8} \]

\[ x = y = 0 \quad \left( 0, 0, \frac{3a}{8} \right) \]
(f(x,y) = 1 - x - y \quad (f_x, f_y) = (-1, -1)

(8) \quad S = \iiint \sqrt{1 + f_x^2 + f_y^2} \, dx \, dy

\quad = \int_{x=0}^{1} \int_{y=0}^{1} \sqrt{1 + 1 + 1} \, dx \, dy

\quad = \sqrt{3}

(9) (a) Vector: magnitude 1:

\[ \mathbf{F} = \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}} \]

(b) There is no x component in the vector.

Recall a circular field like this

\[ \text{magnitude} = \sqrt{y^2 + z^2} \]

Therefore, \( \mathbf{F} = (0, z, -y) \)