Antenna Array Pattern Synthesis via Convex Optimization

Hervé Lebret and Stephen Boyd

Abstract—**We show that a variety of antenna array pattern synthesis problems can be expressed as convex optimization problems, which can be (numerically) solved with great efficiency by recently developed interior-point methods. The synthesis problems involve arrays with arbitrary geometry and element directivity, constraints on far- and near-field patterns over narrow or broad frequency bandwidth, and some important robustness constraints. We show several numerical simulations for the particular problem of constraining the beampattern level of a simple array for adaptive and broadband arrays.**

I. Introduction

ANTENNA arrays provide an efficient means to detect and process signals arriving from different directions. Compared with a single antenna that is limited in directivity and bandwidth, an array of sensors can have its beampattern modified with an amplitude and phase distribution called the weights of the array. After preprocessing the antenna outputs, signals are weighted and summed to give the antenna array beampattern. The antenna array pattern synthesis problem consists of finding weights that satisfy a set of specifications on the beampattern.

The synthesis problem has been studied quite a lot. From the first analytical approaches by Schelkunoff [1] or Dolph [2] to the more general numerical approaches such as mentioned in the recent paper by Bucci *et al.* [3], it would be impossible to make an exhaustive list. An important comment in [3] is that in many minimization methods, there is no guarantee that we can reach the absolute optimum unless the problem is convex.

In this paper, we emphasize the importance of convex optimization in antenna array design. Of course, not all antenna array design problems are convex. Examples of nonconvex problems include those in which the antenna weights have fixed magnitude (i.e., phase-only weights), problems with lower bound constraints (contoured beam antennas), or problems with a limit on the number of nonzero weights.

Nevertheless, other important synthesis problems are convex and can thus be solved with very efficient algorithms that have been developed recently. Even nonlinear convex optimization problems can be solved with great efficiency using new interior-point methods that generalize Karmarkar's linear programming method (see [4] and [5]). Moreover, by "solve" here we mean a very strong form: *Global* solutions are found with a computation time that is *always* small and grows gracefully with problem size. Of course, the computation time is not as small as that required by an "analytical" method, but the number and variety of problems that can be handled is much larger. At the other extreme, we find nonconvex optimization, which is completely general—essentially, all synthesis problems can be posed as general optimization problems. The disadvantage is that such methods cannot guarantee global optimality, small computing time, and graceful growth of computing time with problem size. In our opinion, convex optimization is an excellent tradeoff in efficiency/generality between the (fast but limited) analytical methods and the (slow but comprehensive) general numerical techniques.

Convexity of problems arising in engineering design has been exploited in several fields, e.g., control systems [6], mechanical engineering [7], signal and image processing [8], [9], circuit design [10], and optimal experiment design [11]. To our knowledge, however, it has not been used very much in antenna array design.

As the main objective of our paper is to illustrate the importance and utility of convex optimization for antenna array pattern synthesis problems, we will limit most of our examples to simple arrays. In Section II, we formulate the gain pattern for two examples, which are then generalized. In Section III, we will briefly describe the basic properties of convex optimization and of the algorithms mentioned above. In Section IV, we introduce some design problems and show how they can be recast or reduced to convex optimization problems. Finally, in Section V, we show some numerical examples.

II. The Antenna Array Pattern Formulation

A. The Linear Array Pattern

Consider a linear array of N isotropic antennas at locations $x_1, \cdots, x_N \in \boldsymbol{R}^2$. A harmonic plane wave with frequency ω and wavelength λ is incident from direction θ and propagates across the array (which, we assume for simplicity does not change the field). The N signal outputs s_i are converted to baseband (complex numbers), weighted by the weights w_i, and summed to give the well-known linear array beampattern

$$G(\theta) = \sum_{i=1}^{N} w_i \exp\left(-j\frac{2\pi}{\lambda} x_i \cos\theta\right) \quad (1)$$

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where $w = [w_1, \cdots, w_N]^T$ is the complex weight vector to be designed. The weights are chosen to give a desired beampattern expressed as specifications on $G(\theta)$.

B. Near-Field Broadband Acoustic Array

The next example is from acoustics. An isotropic sinusoidal source at frequency f at position $\boldsymbol{x_S} \in \boldsymbol{R}^3$ creates at the point $\boldsymbol{x} \in \boldsymbol{R}^3$ (complex) acoustic pressure given by

$$p(t, \boldsymbol{x}) = \frac{\exp(-jk\|\boldsymbol{x} - \boldsymbol{x_S}\|_2)}{\|\boldsymbol{x} - \boldsymbol{x_S}\|_2} \exp(j2\pi f t) \tag{2}$$

where $\|\boldsymbol{z}\|_2 = \sqrt{\boldsymbol{z}^T \boldsymbol{z}}, c$ is the wave speed, and $k = 2\pi f / c$ is the wave number. (The actual acoustic pressure is the real part of $p(t, \boldsymbol{x})$ in (2), but it is convenient to use complex notation.)

We assume there are omnidirectional microphones at locations $\boldsymbol{x_i} \in \boldsymbol{R}^3, i = 1, \cdots, N$ (which, as above, we assume for simplicity do not disturb the acoustic field). The microphones convert the acoustic pressure into electrical signals s_i, which are discretized at the sampling frequency f_s. These N discrete-time signals are then filtered by an N-input, single-output finite impulse response (FIR) filter to yield the combined output

$$\begin{aligned} G(\boldsymbol{x_S}, f) &= \sum_{i=1}^{N} \sum_{l=1}^{L} w_{il} s_{il} \\ &= \sum_{i=1}^{N} \sum_{l=1}^{L} w_{il} \frac{1}{\|\boldsymbol{x_i} - \boldsymbol{x_S}\|_2} \\ &\quad \cdot \exp\left(j2\pi f \left(\frac{l}{f_s} - \frac{\|\boldsymbol{x_i} - \boldsymbol{x_S}\|_2}{c} \right) \right). \end{aligned} \tag{3}$$

For each fixed set of filter coefficients w_{il}, this gives a response function that depends on the source location $\boldsymbol{x_S}$ and the frequency. The goal is to choose the weights w_{il} so that this response function has the required properties.

C. Extensions

These two examples have the same general pattern formulation:

$$G(\alpha) = w^T s_\alpha \tag{4}$$

where w is the vector of complex weights for the linear array, whereas it becomes the vector of the coefficients of the FIR filters for the acoustic array. The parameters α include the frequency and the geometric positions of the sources, but they could be extended to include, for example, polarization of electromagnetic signals. Finally, vector s_α includes the individual properties of antenna and/or filter outputs. The formulation of the pattern will be modified if the antenna elements have directional patterns. The description of the environment can furthermore modify the formulation of the array pattern if physical boundaries are included, if the characteristics of the propagation medium are detailed, or if coupling between the array elements is taken into account. Nevertheless, the array pattern will keep the general expression of a linear function of the weights as in (4).

III. Convex Optimization

A. Convexity and Its Properties

A set is convex if for any pair of its points, the line joining these two points lies in the set. A function ϕ is convex on a convex set if $\phi(\lambda x + (1-\lambda)y) \le \lambda\phi(x) + (1-\lambda)\phi(y)$ for λ: $0 \le \lambda \le 1$. A convex optimization problem (or convex program) is the minimization of a convex function over a convex set. It is easy to show that any local minimum of a convex function is a global minimum.

Some important and common convex functions include the following: affine functions $a^T x + b$, where a, x are vectors and b is a scalar; quadratic functions $x^T R x$, provided R is a symmetric positive semi-definite matrix; and norms of vectors $\|x\|$ (which include the Euclidean norm, the absolute value, and the maximum value of a set of elements).

An upper bound on a convex function yields a convex set, i.e., if f is convex and $a \in \boldsymbol{R}$, then $\{x | f(x) \le a\}$ is convex. A lower bound on a convex function, however, is, in general, not a convex constraint.

B. Interior Point Methods

In the last 10 years, there has been considerable progress and development of efficient algorithms for solving wide classes of convex optimization problems. One family of methods that has been greatly developed is interior point methods (IPM's), which are always efficient in terms of complexity theory and are also very efficient in practice. These methods gained great popularity when Karmakar [5] showed their polynomial complexity when applied to linear programming. Since then, a considerable amount of work has been done on the subject. One can consult a review by Gonzaga [12] and a very complete work by Nesterov and Nemirovsky [13], who developed a very general framework for solving nonlinear convex optimization problems using IPM's.

IPM's have another important advantage: It is possible to exploit the underlying structure of the problem under consideration to develop very efficient algorithms (see, e.g., Vandenberghe and Boyd [4], [14]). In the following examples, all the problems have been solved with various specific IPM's.

IV. The Array Pattern Specifications

The specifications that we describe here apply mainly to our first example, and we will just mention some differences for the nearfield broadband acoustic array.

A. Convex Synthesis Problems

As an example, let us consider the minimization of the beampattern level over a given zone with the possibility of level constraints in other areas:

$$\begin{array}{lll} \min\limits_{w_i} & \max\limits_{i=1,\cdots,L,} & |G(\theta_i)|, \\ & \text{subject to} & |G(\theta_j)| < U_j, j = 1, \cdots, M, \\ & & G(\theta_0) = 1. \end{array} \tag{5}$$

The way to express it as a standard convex optimization problem is rather natural and very similar to a framework

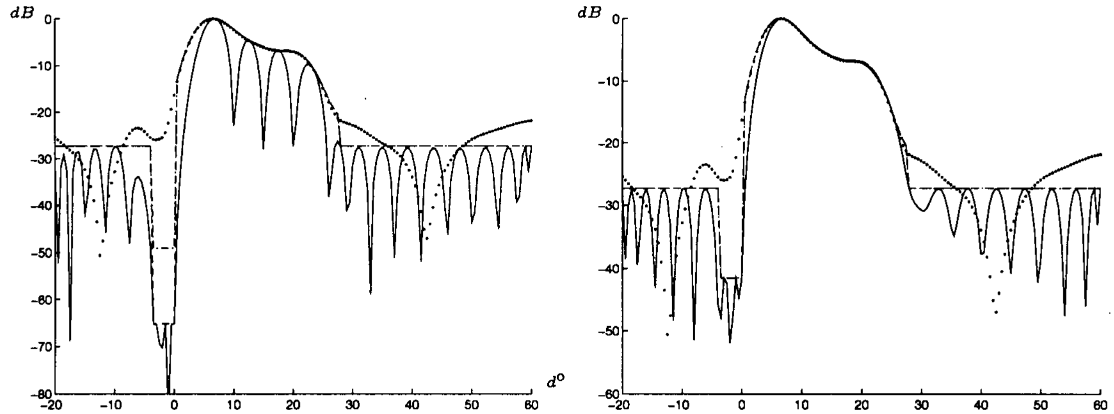

Fig. 1. Optimal beampatterns. The solid lines give the minimized patterns, whereas the dotted lines correspond to the original pattern. The right plot includes four more equality constraints than the right one, giving the desired mainlobe.

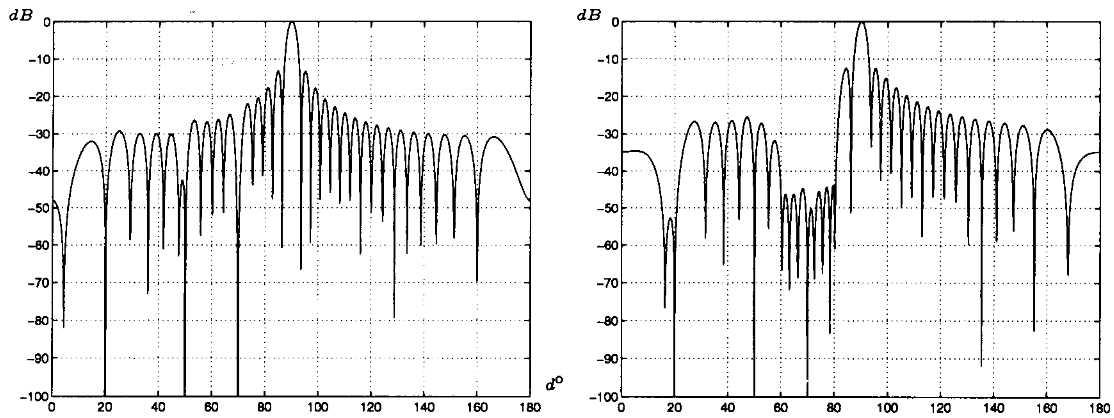

Fig. 2. Optimal properties of an adaptive array: The left plot gives the classical adaptive array and the right one the constrained adaptive array. In both cases, the interferences at $20, 50,$ and $70°$ are rejected, but the right plot includes a -40 dB level constraint between 60 and $80°$.

used in a paper by Lasdon *et al.* [15]. We first eliminate the equality constraint $G(\theta_0) = 1$ by expressing the last weight with the first ones:

$$w_N = e^{j(2\pi/\lambda)x_N \cos\theta_0}\left(1 - \sum_{i=1}^{N-1} w_i e^{-j(2\pi/\lambda)x_i \cos\theta_0}\right). \quad (6)$$

Then, we replace the objective of the problem with a new variable t by adding new (convex) constraints $|G(\theta_i)| < t$ for $i = 1, \cdots, L$. If we now create the vector x with

$$x = [\Re(w_1), \Im(w_1), \cdots, \Re(w_{N-1}), \Im(w_{N-1}), t]^T \quad (7)$$

it is easy to express the original problem as

$$\begin{array}{ll} \text{minimize} & e^T x \\ \text{subject to} & \|A_k x + b_k\| < c_k^T x + d_k, k = 1, \cdots, L + M \end{array} \quad (8)$$

which is a convex problem that is easily solved with IPM's. Thus, the complex number $G(\theta_k)$ is replaced with a 2-D vector $A_k x + b_k$, whose components are its real and imaginary parts. To achieve the explanation, let us finally notice that $e = [0, \cdots, 0, 1]$ and that for each direction $\theta_k, c_k = e$, and $d = 0$ for the first L constraints, and $c_k = 0$ and $d = U_j$ for the last M constraints.

Furthermore, we could just as easily add new convex constraints such as level constraints on the weights $(\|w\| < W)$ while preserving the convexity of the problem. Each new equality constraint is taken into account by elimination of a weight as a variable. The same kind of techniques can be used for convex quadratic constraints such as power constraints or objectives. The signal power is generally expressed as $w^T Rw$, where R is the signal correlation matrix.

B. The Problem of Robustness

Robustness specifications are probably among the most important. When some information about the weights or the direction of arrival is only known approximately, it is essential that the performances of the array are not degraded with slightly different parameter values. We refer now more specifically to papers by Evans [16] and Cantoni [17]. The authors deal with the standard adaptive array processing problem (see also (10)), but they furthermore introduce robustness constraints. The authors have shown that these problems can be expressed as convex problems with *real* weights, but it is possible to generalize their approach and to show that even with *complex* weights, the robustness problem can be replaced by the following convex problem:

$$\min_{|Cw - d| + \Gamma|w| \le \varepsilon} \quad w^T Rw. \quad (9)$$

where Γ and ε include the robustness elements. The important point we want to make here is that (9) is convex and, therefore, can be solved with interior point methods. It is also important

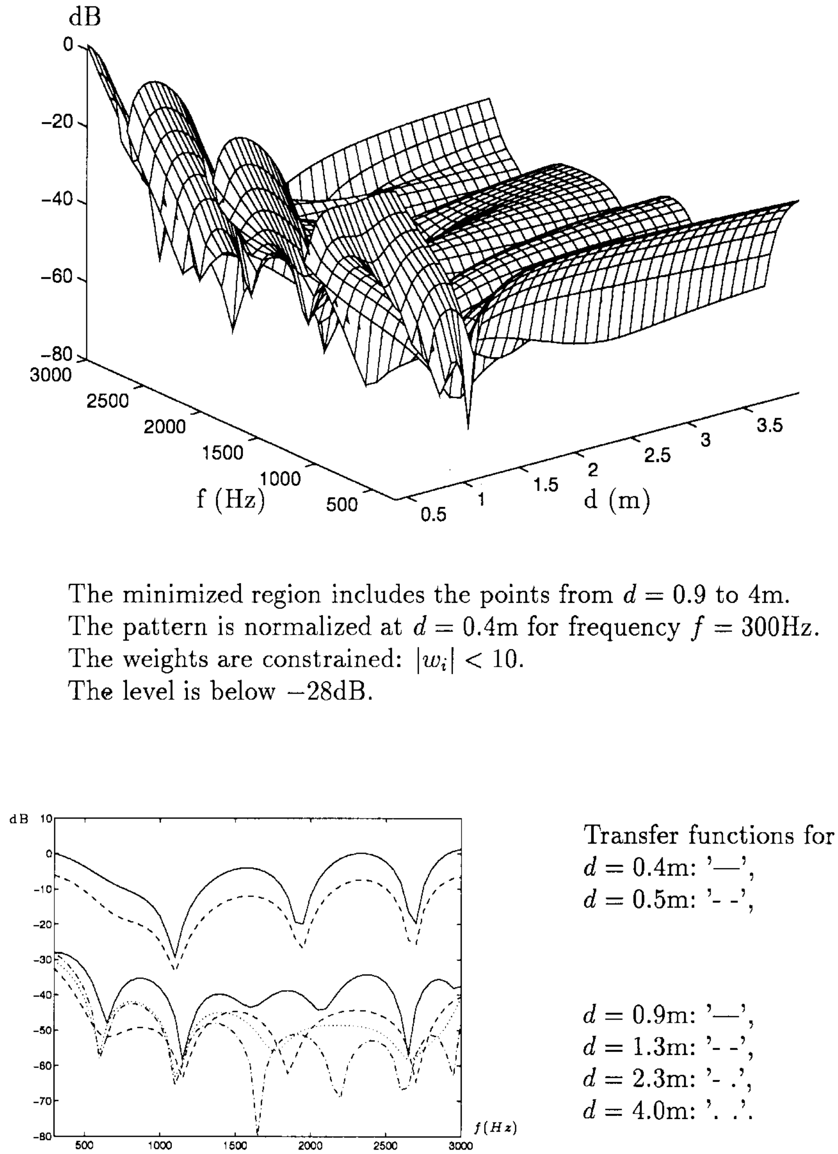

Fig. 3. Optimized pattern for six elements with 10 taps.

to notice that contrary to the previous examples, this was not an obvious convex problem. This means that many more problems might be convex problems, although they do not look convex at first sight. We will not show specific simulations here, but some can be found in [18].

V. Numerical Results

A. The Linear Array: Contrained Synthesis

As a first example, we tried to optimize a cosecant diagram with various constraints. The array has 24 elements with a 0.56λ interelement distance. The mainlobe direction is $\theta_0 = 96.5°$, but for a more convenient description, we assume that $90°$ is now the origin ($\theta_0 = 6.5°$). The original diagram was given and is showed on both plots of Fig. 1 by a dotted line. The left plot gives the optimal pattern for the following constraints: The problem was to minimize the level between -3.5 and $0°$ while constraining the sidelobe level to less than -27.3 dB and keeping a mainlobe shape (between 0 and $28°$) as near as possible to the original one (i.e., never above 1% of the initial diagram). The optimization problem is exactly expressed as in Section IV-A with a discretization step of $0.5°$. Because lower bound constraints are not convex, the result is not satisfying, but the most important result given by convex optimization is the following: It is impossible to find a set of weights that give a level under -65 dB with the given specifications and precision (10^{-4}). Now, the right plot is a new optimization with four new equality constraints at angles $10, 15, 20$, and $26°$ (the pattern is identical to the original one for these four values). With these new constraints, a satisfying new pattern is found.

B. A Comparison to Adaptive Array Processing

A well-known technique applied to reject interferences relatively to a desired signal in a direction θ_0 is to minimize the total received power $w^T R w$, where R is the signal covariance

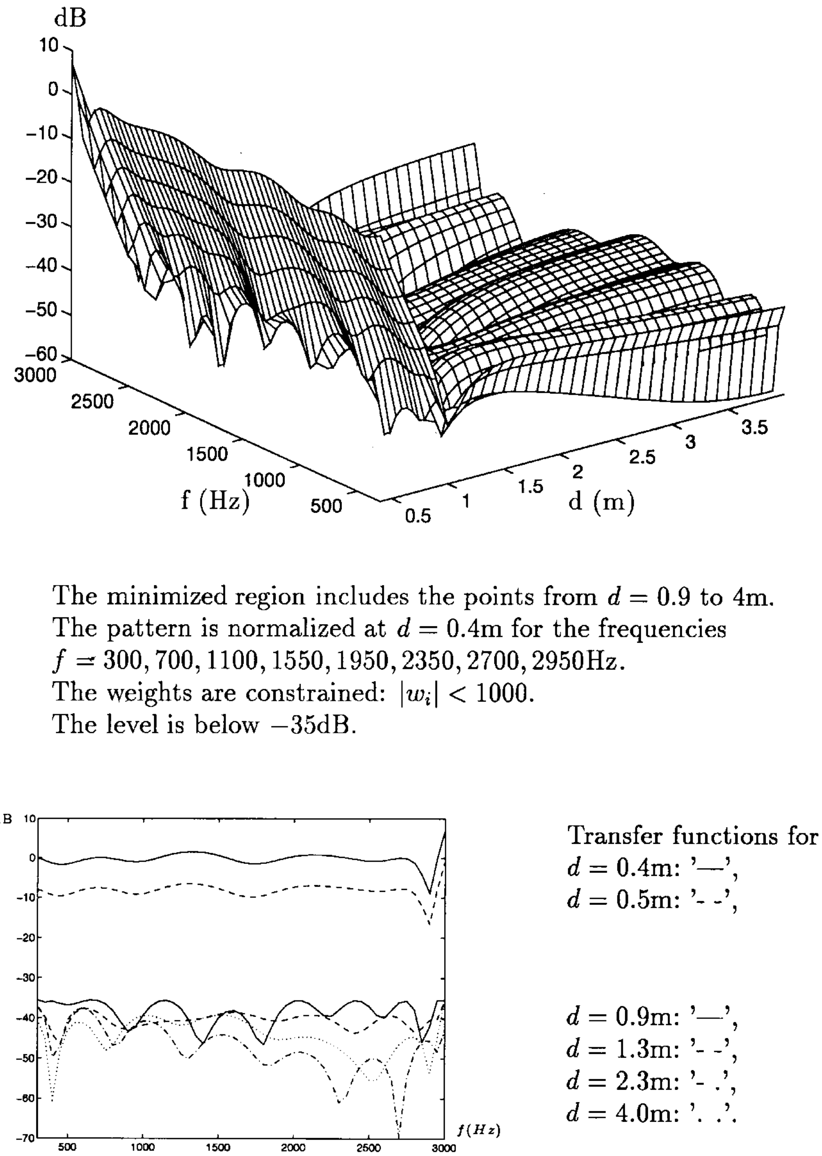

Fig. 4. Normalized and optimized pattern for six elements with 10 taps.

matrix with a normalization gain in the direction θ_0 [19]. When more equality constraints are used, the problem is

$$\begin{array}{ll} \text{minimize} & w^T R w \\ \text{subject to} & Cw = d. \end{array} \tag{10}$$

where C and d are a matrix and a vector with adapted sizes. The solution to the problem is given by the optimal weights vector $w = R^{-1}C^T(CR^{-1}C^T)^{-1}d$.

As a simple example, the left plot of Fig. 2 gives the adaptive array processing for a 32-element regular linear with a half-wavelength interelement distance. The covariance matrix is built with a signal of interest in direction $90°$ with level of 0 dB, three interferences in directions $20, 50$, and $70°$ with level 40 dB and a uniform noise density per element of -40 dB. The right plot of Fig. 2 shows the result obtained for the minimization of the power $w^T R w$ with the following new constraints: The level is less than -40 dB between 60 and $80°$ and -12 dB elsewhere in the sidelobes. The discretization step is $1°$, and the precision on the objective is 10^{-5}. Compared with the expression of Section IV-A, we have a difference in the objective that, here, is the signal power, replaced as before with a convex constraint of the type $w^T R w$. Other similar results including interference in the mainlobe with a signal to noise ratio analysis can be found in [20].

C. Broadband Acoustics Array

For the nearfield broadband array, we consider an array of six elements, each with 10 taps. The beam pattern is given here by (3). These elements are microphones receiving the human voice in a frequency band going from 300 to 3000 Hz, with a sampling frequency f_s used in telephone, 8000 Hz. The sound velocity is taken to be equal to $c_s = 330$ m/s.

The array is once again regular and linear with a distance between elements of 4 cm. We limit the geometric zone of reception as a line orthogonal to the array, intersecting it in its center. Furthermore, we only consider the levels of reception at distances from 0.4 to 4 m every 0.1 m. For the frequencies, the step is 50 Hz so that we have a grid of discretization of 37 positions and 55 frequencies, and the objective is to minimize the received level from the distance 0.9 to 4 m at all frequencies. As usual, we add a normalization constraint that is taken here for $d = 0.4$ m and $f = 300$ Hz. We have also added constraints on the weights modules, more precisely, we have $|w_{il}| < 10$. The problem is then precisely

$$\begin{array}{ccc} \min & \max & |G(d,f)|. \\ |w_{il}| < 10, & 0.9 \le d \le 4.0, & \\ i = 1, \cdots, 6, & 300 \le f \le 3000, & \\ l = 1, \cdots, 10, & G(0.4, 300) = 1. & \end{array} \quad (11)$$

As in Section IV-A, a transformation on the problem including the weight level constraints is made to arrive at a formulation similar to (8). The results of the optimization are shown in Fig. 3. The precision was required to be 0.01 on the maximum amplitude $|G(d,f)|$ in the minimized region. On the upper part of the figure, the gain pattern is represented as a function of both the distance and frequency. Although it is not possible to use it for precise data, it shows that the minimized level is very low. We can also see that the frequency transfer function at $d = 0.4$ m has a nonuniform shape that will give a unacceptable reception.

On the lower part of the figure, we have extracted six transfer functions for various distances. We can read the minimum level, whose value is -28 dB. We also notice the gaps in the transfer function at $d = 0.4$ m, which has to be improved.

This can naturally be improved by adding new normalization constraints at frequencies corresponding to the gaps. Fig. 4 shows the result of an optimization with eight normalization constraints. The weight level has been relaxed to $|w_i| < 1000$, bringing it to an acceptable level. A small gap remains at 2900 Hz, but the transfer function at $d = 0.4$ m has been enormously improved. The relaxation of the weights has even enabled a better minimum level of -35 dB.

VI. Conclusion

We have shown how convex optimization can be used to design the optimal pattern of arbitrary antenna arrays. Although the methods used do not give analytical solutions, we think that the enormous advances in available computing power, together with interior-point methods, and the large number of problems that can be handled, make this approach attractive. The method finds global optimum values with a desired precision; the algorithms either find a feasible point or unambiguously determine that the problem is infeasible. In summary, we think that convex optimization is an excellent tool for pattern synthesis. Of course, we can neither solve all synthesis problems nor more general methods because in many cases, only local optima are found, and the choice of good initial points is often crucial. The general methods work when the problems are convex, but then, we state that it is a better choice to use convex optimization algorithms. Our goal was mainly to illustrate the power of convex optimization, and we have chosen simple numerical examples. Nevertheless, the simulations of the near-field broadband array are sufficiently striking in our case to show the power and efficiency of convex optimization for antenna array pattern synthesis.

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