Multi-Period Trading via Convex Optimization

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Outline

Setting

- manage a portfolio of assets over multiple periods
- take into account
  - market returns
  - trading cost
  - holding cost
- choose trades
  - using forecasts updated each period
  - respecting constraints on trades and positions
- goal is to achieve high (net) return, low risk

Some trading strategies

- traditional
  - buy and hold
  - hold and rebalance
  - rank assets and long/short
  - stat arb
  - momentum/reversion
- academic
  - stochastic control
  - dynamic programming
- optimization based

Optimization based trading

- solve optimization problem to determine trades
- traces to Markowitz (1952)
- simple versions widely used
- trading policy is shaped by selection of objective terms, constraints, hyper-parameters
- **topic of this talk**

Why now?

- huge advances in computing power
- mature convex optimization technology
- growing availability of data, sophisticated forecasts
- can handle many practical aspects

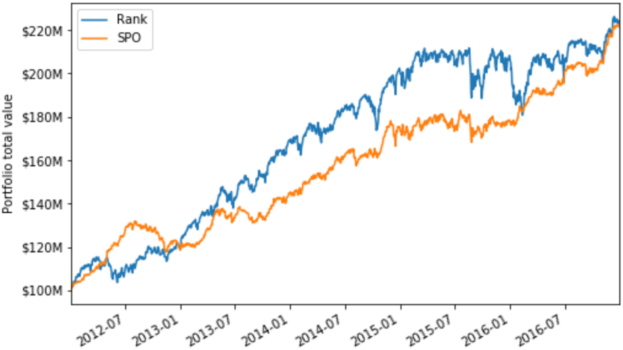
Example: Traditional versus optimization-based

- S&P 500, daily realized returns/volumes, 2012–2016
- initial allocation \$100M uniform on S&P 500
- simulated (noisy) market return forecasts

- rank ('long-short') trading
  - rank assets by return forecast
  - buy top 10, sell bottom 10; 1% daily turnover

- single-period optimization (SPO)
  - empirical factor risk model
  - forecasts of transaction and holding cost
  - hyper-parameters adjusted to match rank trading return

Example: Traditional versus optimization-based



- rank: return 16.78%, risk 13.91%
- SPO: return 16.25%, risk 9.08%

Outline

Portfolio positions and weights

- portfolio of n assets, plus a cash account
- time periods $t = 1, \ldots, T$
- (dollar) holdings or positions at time t: $h_t \in \mathbf{R}^{n+1}$
- net portfolio value is $v_t = \mathbf{1}^T h_t$

- we work with **normalized portfolio** or **weights** $w_t = h_t / v_t$
- $\mathbf{1}^T w_t = 1$
- leverage is $\|(w_t)_{1:n}\|_1$

Trades and post-trade portfolio

- $u_t \in \mathbf{R}^{n+1}$ is (dollar value) trades, including cash
- assumed made at start of period t
- post-trade portfolio is $h_t + u_t$

- we work with **normalized trades** $z_t = u_t / v_t$
- turnover is $\|(z_t)_{1:n}\|_1 / 2$

Transaction and holding cost

- **normalized transaction cost** (dollar cost/v_t) is $\phi_t^{\mathrm{trade}}(z_t)$
- **normalized holding cost** (dollar cost/v_t) is $\phi_t^{\mathrm{hold}}(z_t)$
- these are separable across assets, zero for cash account
- self-financing condition:

$$\mathbf{1}^T z_t + \phi_t^{\mathrm{trade}}(z_t) + \phi_t^{\mathrm{hold}}(w_t + z_t) = 0$$

- this determines cash 'trade' $(z_t)_{n+1}$ in terms of asset holdings and trades $(w_t)_{1:n}$, $(z_t)_{1:n}$

Single asset transaction cost model

- trading dollar amount x in an asset incurs cost

$$a|x| + b\sigma \frac{|x|^{3/2}}{V^{1/2}} + cx$$

  - a, b, c are transaction cost model parameters
  - σ is one-period volatility
  - V is one-period volume
- a standard model used by practitioners
- variations: quadratic term, piecewise-linear, . . .
- same formula for normalized trades, with $V \mapsto V/v_t$

Single asset holding cost model

- holding x costs $s(x)_- = s \max\{-x, 0\}$
- $s > 0$ is shorting cost rate
- variations: quadratic term, piecewise-linear, ...
- same formula for normalized portfolio (weights)

Investment

- hold post-trade portfolio for one period
- $h_{t+1} = (1 + r_t) \circ (h_t + u_t)$
- $r_t \in \mathbf{R}^{n+1}$ are asset (and cash) returns
- $\circ$ is elementwise multiplication
- portfolio return in terms of normalized positions, trades:

$$R_t^{\mathrm{p}} = \frac{v_{t+1} - v_t}{v_t} = r_t^T (w_t + z_t) - \phi_t^{\mathrm{trade}}(z_t) - \phi_t^{\mathrm{hold}}(w_t + z_t)$$

Simulation

- simulation: for $t = 1, \ldots, T$,
  - (arbitrary) trading policy chooses asset trades $(z_t)_{1:n}$
  - determine cash trade $(z_t)_{n+1}$ from self-financing condition
  - update portfolio weights and value

- **backtest**
  - use realized past returns, volumes
  - evaluate candidate trading policies
- **stress test**
  - use challenging (but plausible) data
- **model calibration**
  - adjust model parameters so simulation tracks real portfolio

Outline

Estimated portfolio return

$$\hat{R}_t^{\mathrm{p}} = \hat{r}_t^T(w_t + z_t) - \hat{\phi}_t^{\mathrm{trade}}(z_t) - \hat{\phi}_t^{\mathrm{hold}}(w_t + z_t)$$

- quantities with ^ are estimates or forecasts (based on data available at time t)
- asset return forecast $\hat{r}_t$ is most important
- transaction cost estimates depend on estimates of bid-ask spread, volume, volatility
- holding cost is typically known

Single-period optimization problem

$$
\begin{array}{ll}
\text{maximize} & \hat{R}_t^{\text{p}} - \gamma^{\text{risk}} \psi_t(w_t + z_t) \\
\text{subject to} & z_t \in \mathcal{Z}_t, \quad w_t + z_t \in \mathcal{W}_t, \\
& \mathbf{1}^T z_t + \hat{\phi}_t^{\text{trade}}(z_t) + \hat{\phi}_t^{\text{hold}}(w_t + z_t) = 0
\end{array}
$$

- z_t is variable; w_t is known
- ψ_t is risk measure, $\gamma^{\text{risk}} > 0$ risk aversion parameter
- objective is risk-adjusted estimated net return
- $\mathcal{Z}_t$ are trade constraints, $\mathcal{W}_t$ hold constraints

Single-period optimization problem

- self-financing constraint can be approximated as $\mathbf{1}^T z_t = 0$ (slightly over-estimates updated cash balance)

$$
\begin{array}{ll}
\text{maximize} & \hat{r}_t^T (w_t + z_t) \\
& -\gamma^{\text{risk}} \psi_t (w_t + z_t) \\
& -\hat{\phi}_t^{\text{trade}} (z_t) \\
& -\hat{\phi}_t^{\text{hold}} (w_t + z_t) \\
\text{subject to} & \mathbf{1}^T z_t = 0, \quad z_t \in \mathcal{Z}_t, \quad w_t + z_t \in \mathcal{W}_t
\end{array}
$$

- a **convex optimization problem** provided risk, trade, and hold functions/constraints are

Traditional quadratic risk measure

- $\psi_t(x) = x^T \Sigma_t x$
- Σ_t is an estimate of return covariance
- factor model risk $\Sigma_t = F_t \Sigma_t^{\mathrm{f}} F_t^T + D_t$
  - $F_t \in \mathbf{R}^{n \times k}$ is factor exposure matrix
  - $F_t^T w_t$ are factor exposures
  - Σ_t^{f} is factor covariance
  - D_t is diagonal ('idiosyncratic') asset returns
- variation: $\psi_t(x) = \left(x^T \Sigma_t x - (\sigma^{\mathrm{tar}})^2\right)_+$
  - $(\sigma^{\mathrm{tar}})^2$ is target risk

Robust risk measures

- worst case quadratic risk: $\psi_t(x) = \max_{i=1,\ldots,M} \; x^T \Sigma_t^{(i)} x$
  - $\Sigma^{(i)}$ are scenario or market regime covariances

- worst case over correlation changes:

$$\psi_t(x) = \max_{\Delta} \; x^T(\Sigma + \Delta)x, \qquad |\Delta_{ij}| \leq \kappa \left(\Sigma_{ii}\Sigma_{jj}\right)^{1/2}$$

  $\kappa \in [0, 1)$ is a parameter, say $\kappa = 0.05$

- can express as

$$\psi_t(x) = x^T \Sigma x + \kappa \left(\Sigma_{11}^{1/2}|x_1| + \cdots + \Sigma_{nn}^{1/2}|x_n| \right)^2$$

Return forecast risk

- forecast uncertainty: any return forecast of form

$$\hat{r} + \delta, \quad |\delta| \leq \rho \in \mathbf{R}^{n+1}$$

  is plausible; ρ_i is forecast return spread for asset i
- worst case return forecast is

$$\min_{|\delta| \leq \rho} \; (\hat{r}_t + \delta)^T (w_t + z_t) = \hat{r}_t^T (w_t + z_t) - \rho^T |w_t + z_t|$$

- same as using nominal return forecast, with a return forecast risk term $\psi_t(x) = \rho^T |x|$

Holding constraints

long only	$w_t + z_t \geq 0$
leverage limit	$\|(w_t + z_t)_{1:n}\|_1 \leq L^{\max}$
capitalization limit	$(w_t + z_t) \leq \delta C_t / v_t$
weight limits	$w^{\min} \leq w_t + z_t \leq w^{\max}$
minimum cash balance	$(w_t + z_t)_{n+1} \geq c_{\min} / v_t$
factor/sector neutrality	$(F_t)_i^T (w_t + z_t) = 0$
liquidation loss limit	$T^{\text{liq}} \hat{\phi}_t^{\text{trade}}((w_t + z_t)/T^{\text{liq}}) \leq \delta$
concentration limit	$\sum_{i=1}^K (w_t + z_t)_{[i]} \leq \omega$

Trading constraints

turnover limit	$\|(z_t)_{1:n}\|_1/2 \leq \delta$
limit to trading volume	$|(z_t)_{1:n}| \leq \delta(\hat{V}_T/v_t)$
transaction cost limit	$\hat{\phi}^{\mathrm{trade}}(z_t) \leq \delta$

Convexity

- objective terms and constraints above are convex, as are many others
- consequences of convexity: we can
  - (globally) solve, reliably and fast
  - add many objective terms and constraints
  - rapidly develop using domain-specific languages
- nonconvexities are not needed or easily handled, *e.g.*,
  - quantized positions
  - minimum trade sizes
  - target leverage (*e.g.*, $\|(x_t + w_t)_{1:n}\|_1 = L^{\mathrm{tar}}$)

Using single-period optimization

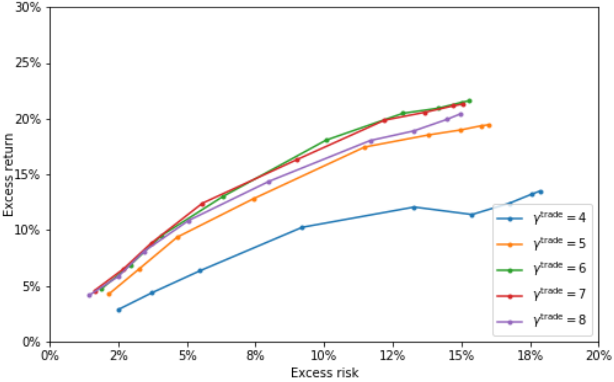
- constraints and objective terms are inspired by estimates of the real values, *e.g.*, of transaction or hold costs
- we add positive (hyper) parameters that scale the terms, *e.g.*, γ^{trade}, γ^{hold}
- these are **knobs** we turn to get what we want
  - absolute value term in $\hat{\phi}^{\text{trade}}$ discourages small trades
  - 3/2-power term in $\hat{\phi}^{\text{trade}}$ discourages large trades
  - shorting cost discourages holding short positions
  - liquidation cost discourages holding illiquid positions
- we simulate/back-test to choose hyper-parameter values

- exact same (meta-) story in control, machine learning, ...

Example

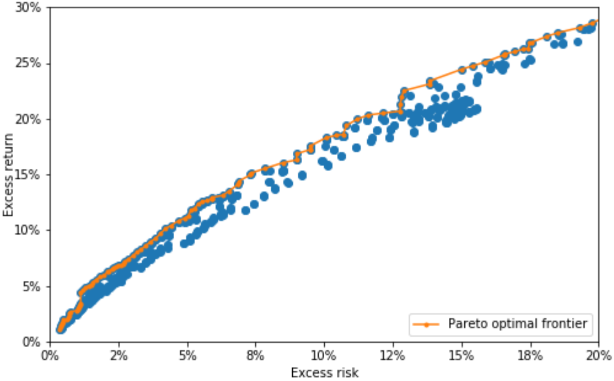
- S&P 500, daily realized returns, volumes, 2012–2016
- initial allocation \$100M uniform on S&P 500
- simulated (noisy) market return forecasts
- risk model: empirical factor model with 15 factors
- volume, volatility estimated as average of last 10 values
- vary hyper-parameters γ^{risk}, γ^{trade}, γ^{hold} over ranges

Example: Risk-return trade-off



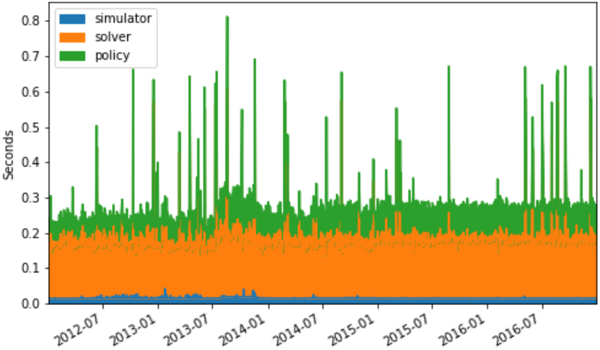
Example: Pareto optimal frontier

- grid search over 410 hyper-parameter combinations



Example: Timing

- execution time, generic CVXPY, single-thread ECOS solver



Outline

Introduction

Model

Single-period optimization

Multi-period optimization

Idea

- at period t, optimize over sequence of portfolio weights

$$w_{t+1}, \ldots, w_{t+H-1}$$

  subject to $\mathbf{1}^T w_\tau = 1$, $\tau = t+1, \ldots, t+H-1$
- H is the (planning) horizon
- execute trades $z_t = w_{t+1} - w_t$
- need forecasts over the horizon, *e.g.*,

$$\hat{r}_{\tau|t}, \quad \tau = t, \ldots, t+H-1$$

  forecast of market return in period τ made at period t
- can exploit differing short- and long-term forecasts

Multi-period optimization

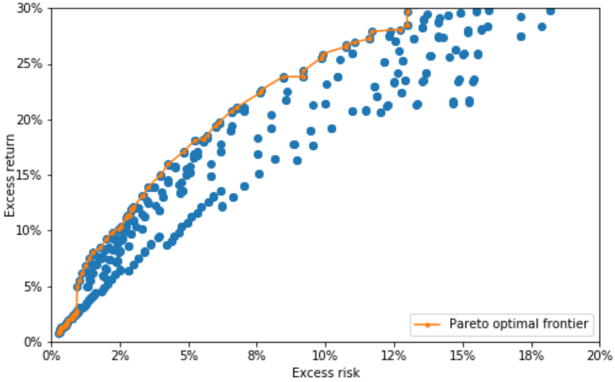
$$
\begin{array}{ll}
\text{maximize} & \sum_{\tau=t+1}^{t+H} \Big(\hat{r}_{\tau|t}^T w_\tau - \gamma^{\text{risk}} \psi_\tau(w_\tau) \\
& \qquad - \gamma^{\text{hold}} \hat{\phi}_\tau^{\text{hold}}(w_\tau) \\
& \qquad - \gamma^{\text{trade}} \hat{\phi}_\tau^{\text{trade}}(w_\tau - w_{\tau-1}) \Big) \\
\text{subject to} & \mathbf{1}^T w_\tau = 1, \quad w_\tau - w_{\tau-1} \in \mathcal{Z}_\tau, \quad w_\tau \in \mathcal{W}_\tau, \\
& \qquad \tau = t+1, \ldots, t+H
\end{array}
$$

- reduces to single-period optimization for $H = 1$
- computational cost scales linearly in horizon H
- same idea widely used in **model predictive control**

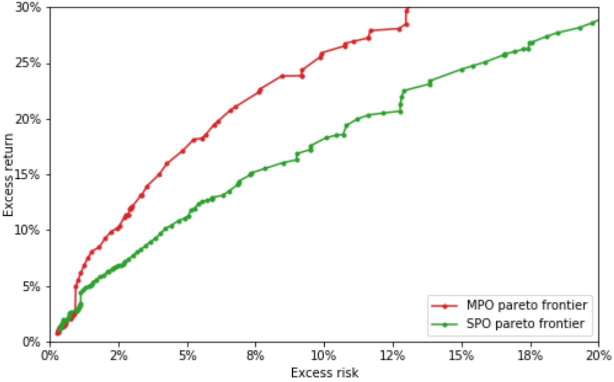
Example

- same data as single-period example
- $H = 2$, so we have forecasts for current and next periods
- grid search over 390 hyper-parameter combinations

Example: Pareto frontier



Example: Multi- and single-period comparison



Conclusions

convex optimization to choose trades

- idea traces to Markowitz (1952), model predictive control
- gives an organized way to parametrize good trading strategies
- works with any forecasts
- handles a wide variety of practical constraints and costs

Is it optimal?

- if we assume (say) $\log(\mathbf{1} + r_t) \sim \mathcal{N}(\mu, \Sigma)$ are independent, the multi-period trading problem is a convex stochastic control problem
- multi-period optimization is almost an optimal strategy (Boyd, Mueller, O'Donoghue, Wang, 2014)
- but real returns are not log-normal, or independent, or stationary, or even a stochastic process

References

- *Active Portfolio Management: A Quantitative Approach*, Grinold & Kahn
- *Convex Optimization*, Boyd & Vandenberghe
- *Multi-Period Trading via Convex Optimization*, Boyd et al., Foundations & Trends in Optimization

- `github.com/cvxgrp/cvxportfolio`