Uniqueness of Circuits and Systems Containing One Nonlinearity

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Abstract—We study systems containing *one* memoryless nonlinearity. We show that two such systems have the same I/O operator only when they are related by simple scaling, delay, and loop transformations. The theory is applied to one-port networks containing one nonlinear element.

I. Introduction

IN [1] the authors considered systems consisting of a memoryless nonlinearity sandwiched between two linear time-invariant (LTI) operators. We showed that if two such systems have the same I/O operator, then one can be got from the other by scaling the LTI operators and memoryless nonlinearity, and possibly redistributing some delay between the LTI operators. Thus, such systems are *essentially unique*, in the sense that the I/O operator determines the nonlinearity and the pre- and post-LTI filters up to scaling and delays.

In this paper we continue our study of systems which are interconnections of LTI and memoryless operators. We consider systems containing *one* nonlinearity, possibly in a feedback loop, and show that these systems too are essentially unique, in this case modulo scaling, delays, and loop transformations (Theorem 3). Using this fact we show that the I/O maps realizable with some common structures for nonlinear systems (we have called these the cascade, Lur'e, and complementary Lur'e structures) are completely disjoint. This raises the possibility of determining internal structure from I/O measurements.

In Section VII we apply the theory to one-port networks containing one nonlinear element and show that two such networks are equivalent, that is, look the same from the external port, only if they are related in a simple way (Theorem 4).

II. Notation and Foundations

In order to easily handle memoryless nonlinearities we extend the usual Volterra series formalism slightly to allow measures as kernels. This will allow memoryless operators as well as operators like

$$Au(t)=\int u(t-\tau)^2h(\tau)\ d\tau$$

which are called Volterra-like by Sandberg [6], [7], and arise in interconnections of memoryless and LTI operators. In fact, the operators we allow are included in even more general formalisms, e.g., that of deFigueiredo [8]. A complete discussion of our formulation can be found in Boyd *et al.* [9].

Let μ be a bounded measure on R^n. $\|\mu\|$ will denote, as usual, $\|\mu\| \triangleq \int d|\mu| = |\mu|(R^n)$.[1] We say μ is symmetric if $\mu(\sigma E) = \mu(E)$ for all permutations $\sigma \in S^n$ and all (measurable) $E \subseteq R^n$, where

$$\sigma E \triangleq \{(x_{\sigma 1}, \cdots, x_{\sigma n})^T | (x_1, \cdots, x_n)^T \in E\}.$$

Let $\langle a\rangle$ be a sequence where the nth term a_n is a symmetric bounded measure supported on R_+^n ($R_+ \triangleq \{\tau|\tau \geq 0\}$). Define

$$\text{Rad } \langle a\rangle \triangleq [\lim_{n\to\infty} \sup \|a_n\|^{1/n}]^{-1}.$$

Then if Rad $\langle a\rangle = \rho > 0$, $\langle a\rangle$ defines an operator A on B_ρ, the open ball of radius ρ in L^∞, into L^∞ given by[2]

$$Au(t) \triangleq \sum_{n=1} \int \cdots \int u(t-\tau_1) \cdots u(t-\tau_n) a_n(d\tau_1, \cdots, d\tau_n). \tag{2.1}$$

We will only consider operators of the form (2.1). We call a_n the nth time domain kernel of A; we will use more often its Laplace transform

$$A_n(s_1, \cdots, s_n) \triangleq \int \cdots \int \cdot \exp -(s_1\tau_1 + \cdots + s_n\tau_n) a_n(d\tau_1, \cdots, d\tau_n)$$

which is analytic and bounded in $C_+^n \triangleq \{s | \text{Re } s_k > 0, 1 \leq k \leq n\}$. A_n will be called the nth kernel of A, and we will use the notational convention that whenever, say, B is an operator of the form (2.1), $B_n(s_1, \cdots, s_n)$ will denote its nth kernel.

A is LTI if $A_n = 0$, $n > 1$ and in this case we write its only nonzero kernel $A_1(s)$ as $A(s)$; that is, we will use the same notation for an LTI operator and its first kernel. For example, e^{-sT} will denote both an analytic function and the T-second delay operator.

Conversely, if $A_1 = 0$, that is, A has zero linear part, then we say A is strictly nonlinear.

The part of (2.1) due to the masses or delta functions at the origin in the a_n will be called the memoryless part of A; formally MP A is the operator defined by

$$(\text{MP } A)_n \triangleq a_n(\{0\})$$

($\{0\}$ is the set whose only element is $0 \in R_n$). We develop some of the properties of MP in the Appendix. If MP $A = A$ then we say A is memoryless, and then we will also use $A(\cdot)$ to denote the associated function: $R \to R$ given by $A(x) \triangleq \sum A_n x^n$ (the A_n are constants here).

I is as usual the identity operator with kernels

$$I_n = \begin{cases} 1 & n=1 \\ 0 & n>1. \end{cases}$$

If A is memoryless and LTI, it has the form αI for some real

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[1] An excellent reference on bounded measures and this norm is Rudin's book [10].

[2] Just as convolution with a bounded measure is a bounded map from L^p into L^p or $C^{(k)}$ into $C^{(k)}$, A also maps B_ρ in $C^{(k)}$ into $C^{(k)}$, if you prefer these signal spaces.

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constant α; we will simply write it as α. For example, $\alpha B\beta$ is the operator defined by

$$(\alpha B\beta)u=\alpha B(\beta u)$$

where α and β are just real numbers on the right-hand side.

In the sequel H will always denote an LTI operator, F a memoryless operator, and N a memoryless strictly nonlinear operator.

Finally, we list a few facts we will use in the paper. If A and B are operators, then the following holds.

Fact 1: $A = B$ if and only if $A_n = B_n$ for all n. Note that $A = B$ asserts equality of operators, whereas $A_n = B_n$ asserts equality of functions analytic in C_+^n. This is sometimes called the uniqueness theorem.

Fact 2: $A + B$ and AB (composition of A and B) are operators with kernels $(A+B)_n = A_n + B_n$ and

$$(AB)_n=\mathrm{SYM}\sum_{m=1}^{n}\left[\sum_{\substack{i_1,\cdots,i_m\geq 1\\ i_1+\cdots+i_m=n}}\right.\cdot A_m(s_1+\cdots+s_{i_1},\cdots,s_{n+1-i_m}+\cdots+s_n)\cdot\left. B_{i_1}(s_1,\cdots,s_{i_1})\cdots B_{i_m}(s_{n+1-i_m},\cdots,s_n)\right]$$

where SYM symmetrizes a function on C_+^n

$$\mathrm{SYM}\ f\triangleq(n!)^{-1}\sum_{\sigma\in S^n}f(s_{\sigma 1},\cdots,s_{\sigma n}).$$

The n in SYM can be determined by context; it is the order of the kernel on the left-hand side of the equation. When one of the operators is LTI the composition formula simplifies to

$$(AH)_n(s_1,\cdots,s_n)=A(s_1,\cdots,s_n)H(s_1)\cdots H(s_n)$$

$$(HA)_n(s_1,\cdots,s_n)=H(s_1+\cdots+s_n)A(s_1,\cdots,s_n).$$

Fact 3: If A is strictly nonlinear, $I + A$ has an inverse (near 0) which is an operator in our sense. In particular, Rad $[(I + A)^{-1}] > 0$.

We refer the reader to [9] for proofs of these facts.

III. Problem Setup

We will be concerned with systems which are stable interconnections of various LTI operators $H_k(s)$ and one memoryless nonlinear operator $F(\cdot)$ (see Fig. 1). Specifically, we assume that the linearized system ($F(\cdot)$ replaced by F_1) is *internally stable*.[3] Under this assumption, we may extract N, the strictly nonlinear part of F, collect the rest of the system into a two-input two-output LTI operator H, and redraw Fig. 1 as Fig. 2. Here

$$H=\begin{bmatrix}H_{yu} & H_{yd}\\ H_{xu} & H_{xd}\end{bmatrix}.$$

and the overall I/O operator S is therefore

$$S=H_{yu}+H_{yd}N(I-H_{xd}N)^{-1}H_{xu}. \tag{3.1}$$

Remark 1: Facts 2 and 3 of Section II can be used to show that S is indeed given by a Volterra series, i.e., has the form (2.1). To see this, note that by Fact 2 $-H_{xd}N$ is an operator with first kernel $-H_{xd}(s_1)N_1 = 0$; that is, $-H_{xd}N$ is strictly nonlinear. By Fact 3 then $(I - H_{xd}N)^{-1}$ is an operator of the form (2.1); a few more applications of Fact 2 establish that S has form (2.1).

[3] By internally stable we mean that if we inject a signal u into a summing node placed anywhere in the system, and pick off an output y from anywhere in the system, the resulting map $\Phi: u \to y$ is LTI in our sense (in particular, $\Phi(s) \neq (1-s)^{-1}$, s, etc).

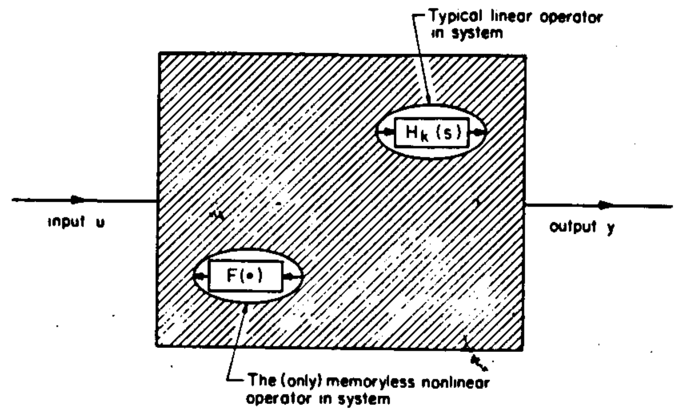

Fig. 1. System which is interconnection of various LTI operators $H_k(s)$ and one memoryless nonlinear operator $F(\cdot)$.

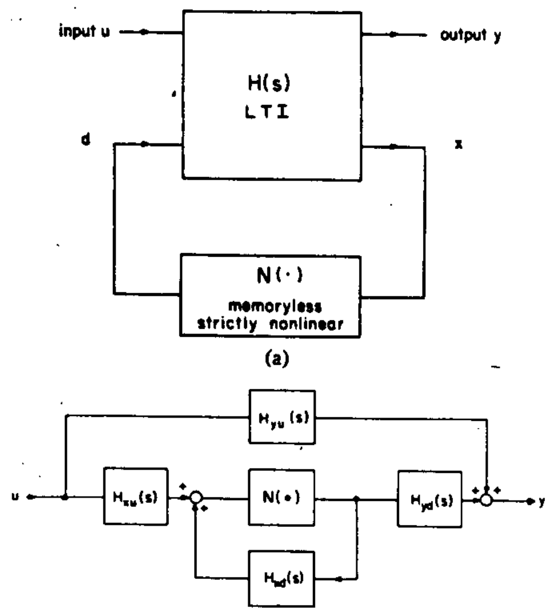

Fig. 2. (a) System redrawn as two-input two-output LTI operator $H(s)$ and strictly nonlinear memoryless operator $N(\cdot)$. (b) Block diagram.

Remark 2: This form is a special case of the class of systems Sandberg considers in [6], [7], and occurs whenever a system is decomposed into two subsystems, one of which is linear.[4]

We now ask the question, under what conditions could two systems of the form (3.1) have the same I/O operator?

IV. System Transformations

We first describe three system transformations which leave the I/O operator S unchanged: scaling, delay, and loop transformations.

Scaling Transformations: Let α and β be nonzero real constants. Consider the system shown in Fig. 3. It clearly has I/O operator S independent of α and β. That is, if

$$\tilde{H}_{yu}=H_{yu}\qquad \tilde{H}_{yd}=\beta H_{yd}$$
$$\tilde{H}_{xu}=\alpha H_{xu}\qquad \tilde{H}_{xd}=\alpha\beta H_{xd}$$

[4] In the notation of [6], [7], we consider the special case where all the operators are SISO, N is memoryless strictly nonlinear, and A, B, C, and D are given by convolution with bounded measures. Not all LTI bounded causal operators $L^\infty \to L^\infty$ are given by convolution with bounded measures, although all the ones of engineering interest are [9].

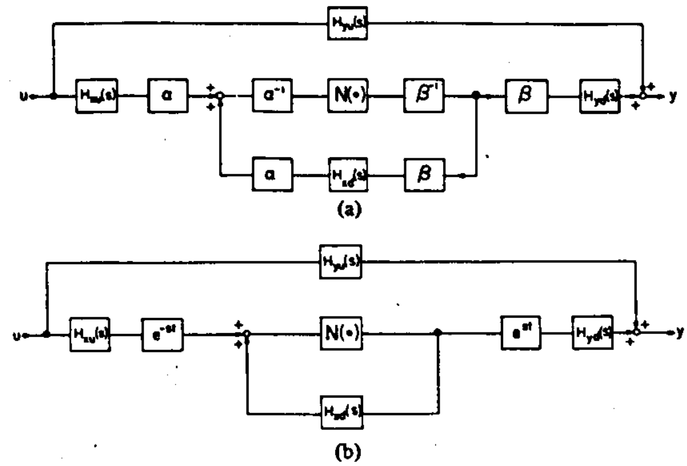

Fig. 3. (a) Scaling transformation of system in Fig. 2. The I/O operator is independent of α and β. (b) Delay transformation: any time delay in H_{xu} and H_{yd} can be distributed arbitrarily between them.

$$\tilde{N}=\beta^{-1}N\alpha^{-1}$$

then $\tilde{S} = S$.

Proof: Obvious from Fig. 3, or more formally

$$\tilde{S}=H_{yu}+\beta H_{yd}\beta^{-1}N\alpha^{-1}(I-\alpha\beta H_{xd}\beta^{-1}N\alpha^{-1})^{-1}\alpha H_{xu}$$
$$=H_{yu}+H_{yd}N((I-\alpha H_{xd}N\alpha^{-1})\alpha)^{-1}\alpha H_{xu}$$

since β commutes with H_{yd} and H_{xd} and $B^{-1}A^{-1} = (AB)^{-1}$ generally. Carefully distributing the α we get

$$=H_{yu}+H_{yd}N(\alpha-\alpha H_{xd}N)^{-1}\alpha H_{xu}$$
$$=H_{yu}+H_{yd}N(I-H_{xd}N)^{-1}H_{xu}=S$$

after extracting the α on the left and using $(AB)^{-1} = B^{-1}A^{-1}$ again.

Delay Transformations: When T is such that

$$\tilde{H}_{xu}=e^{-sT}H_{xu} \qquad \tilde{H}_{yd}=e^{sT}H_{yd}$$

are operators of the form we consider (i.e., still causal), then

$$\tilde{H}_{yd}N(I-H_{xd}N)^{-1}\tilde{H}_{xu}=H_{yd}N(I-H_{xd}N)^{-1}H_{xu}.$$

[See Fig. 3(b).] This follows from the time invariance of $N(I - H_{xd}N)^{-1}$ and is easily verified.

Loop Transformations: Let k be any real constant and consider the feedback subsystem shown in Fig. 4. The I/O operator of the subsystem shown in Fig. 4(b) is independent of k; that is, if

$$\tilde{H}_{xd}=H_{xd}+k \qquad \tilde{N}=N(I+kN)^{-1}$$

then

$$\tilde{N}(I-\tilde{H}_{xd}\tilde{N})^{-1}=N(I-H_{xd}N)^{-1}$$

and thus $\tilde{S} = S$ if $\tilde{H}_{yu} = H_{yu}$, $\tilde{H}_{xu} = H_{xu}$, and $\tilde{H}_{yd} = H_{yd}$. Note that the transformed subsystem has the same structure: a strictly nonlinear memoryless operator with LTI feedback around it. By Facts 2 and 3 of Section II, $\tilde{N}$ has a positive radius of convergence. We leave to the reader the proof that $\tilde{N}$ is strictly nonlinear and that the transformed subsystem has the same I/O operator.

It will be convenient to say that the subsystem in Fig. 4(a) is normalized if MP $H_{xd} = 0$. Since MP $\tilde{H}_{xd} =$ MP $H_{xd} + k$, any subsystem of the form in Fig. 4(a) can be brought to an equivalent normalized by a loop transformation with $k = -\text{MP}H_{xd}$. This normalization has an intuitive interpretation: a normalized H_{xd} has some sort of response delay or smoothness: its step response is continuous at $t = 0$.

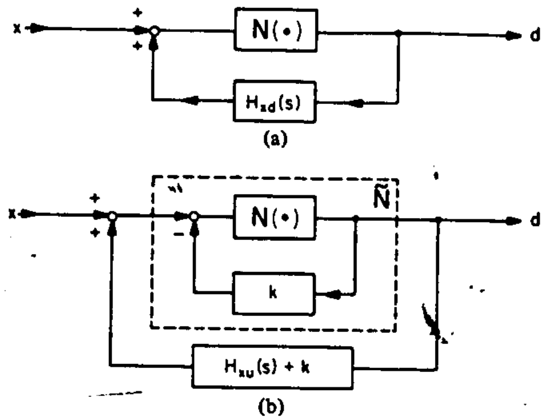

Fig. 4. Loop transformation of the feedback subsystem. The transformed subsystem in (b) has the same form as the original subsystem: a strictly nonlinear memoryless operator with LTI feedback around it.

V. Statement and Proof of Main Theorems

In this section we will show that if two systems as in Fig. 2 have the same I/O operator, then the systems are related by a scaling, delay, and loop transformation. Thus, the transformations described in the last section are the *only* transformations which preserve the I/O operator. We first develop some results concerning the feedback subsystem shown in Fig. 4(a).

Lemma 1: Let $G = N(I - HN)^{-1}$, where H is LTI, MP $H = 0$, and N is memoryless strictly nonlinear. Then MP$G = N$.

Intuitively, there is some "delay" in the feedback loop (the subsystem is normalized), so that only the feedforward path N contributes to the memoryless part of the closed-loop operator G.

Proof: Deferred to the Appendix.

We will need to explicitly compute a few kernels of the subsystem:

Lemma 2: Let $G = N(I - HN)^{-1}$, where H is LTI and N is memoryless with first nonvanishing term N_k, that is, $N_i = 0$, $1 \leq i < k$, $N_k \neq 0$. Then

$$G_1=\cdots=G_{k-1}=0$$
$$G_k=N_k,\ \cdots,\ G_{2k-2}=N_{2k-2}$$
$$G_{2k-1}=N_{2k-1}+kN_k^2 \text{ SYM } H(s_1+\cdots+s_k).$$

Thus, the first $2k - 2$ terms of the closed-loop operator G are simply those of N, as if the feedback were not present. We have to look at the kernel of order $2k - 1$ to even detect the presence of the feedback H.

Proof: Deferred to the Appendix.

We are now ready to state and prove:

Theroem 1: Suppose two normalized systems of the form (3.1) have the same I/O operator. Formally, suppose

$$\tilde{H}_{yu}+\tilde{H}_{yd}\tilde{N}(I-\tilde{H}_{xd}\tilde{N})^{-1}\tilde{H}_{xu}=H_{yu}+H_{yd}N(I-H_{xd}N)^{-1}H_{xu} \tag{5.1}$$

where the H's are LTI, the N's are memoryless strictly nonlinear, MP $\tilde{H}_{xd}$ = MP H_{xd} = 0, and S is not linear.

Then there are real constants T and nonzero α and β such that

$$\tilde{H}_{yu}=H_{yu} \qquad \tilde{H}_{yd}=\beta e^{sT}H_{yd}$$
$$\tilde{H}_{xu}=\alpha e^{-sT}H_{xu} \qquad \tilde{H}_{xd}=\alpha\beta H_{xd}$$
$$\tilde{N}=\beta^{-1}N\alpha^{-1}.$$

Proof of Theorem 1: From Lemma 2 $\tilde{H}_{yd}\tilde{N}(I - \tilde{H}_{xd}\tilde{N})^{-1}\tilde{H}_{xu}$ and $H_{yd}N(I - H_{xd}N)^{-1}H_{xu}$ are strictly nonlinear so the first kernel of (5.1) is

$$\tilde{H}_{yu} = H_{yu}.$$

Subtracting this term from (5.1) yields

$$\tilde{H}_{yd}\tilde{N}(I-\tilde{H}_{xd}\tilde{N})^{-1}\tilde{H}_{xu} = H_{yd}N(I-H_{xd}N)^{-1}H_{xu}. \quad (5.2)$$

N is not zero, for then S would be linear, so suppose N_k is the first nonzero term in N. Then by Lemma 2 the first nonzero kernel in (5.2) is

$$\tilde{H}_{yd}(s_1+\cdots+s_k)\tilde{N}_k\tilde{H}_{xu}(s_1)\cdots\tilde{H}_{xu}(s_k)$$
$$= H_{yd}(s_1+\cdots+s_k)N_kH_{xu}(s_1)\cdots H_{xu}(s_k). \quad (5.3)$$

In particular, $\tilde{N}$ also starts at the kth term. Since S is not linear (5.3) is not identically zero. We claim there are real T and nonzero β, α with

$$\tilde{H}_{yd} = \beta e^{sT}H_{yd} \qquad \tilde{H}_{xu} = \alpha e^{-sT}H_{xu}. \quad (5.4)$$

This is proved in Boyd and Chua [1], so we will give an abbreviated argument here. Find an open ball D in C^n_+ in which (5.3) does not vanish. In D define

$$Q(s_1,\cdots,s_n) \triangleq \ln\left[\frac{\tilde{H}_{yd}}{H_{yd}}(s_1+\cdots+s_n)\right] \quad (5.5)$$

$$= \ln\left[\frac{H_{xu}}{\tilde{H}_{xu}}(s_1)\cdots\frac{H_{xu}}{\tilde{H}_{xu}}(s_n)\frac{N_k}{\tilde{N}_k}\right]. \quad (5.6)$$

From (5.5) and (5.6) we have

$$\frac{\partial^2 Q}{\partial s_1 \partial s_2} = \left[\ln\frac{\tilde{H}_{yd}}{H_{yd}}\right]''(s_1+\cdots+s_n) = 0.$$

Thus, in D and therefore in all of C^n_+

$$\left[\ln\frac{\tilde{H}_{yd}}{H_{yd}}\right](s_1+\cdots+s_n) = \gamma(s_1+\cdots+s_n)+T$$

for some constants γ and T. Hence,

$$\tilde{H}_{yd}(s) = \beta e^{sT}H_{yd}(s)$$

where $\beta = \exp\gamma$. Substituting this back into (5.3) yields the other half of (5.4).

We now claim that (5.2) and (5.4) imply

$$\tilde{N}(I-\tilde{H}_{xd}\tilde{N})^{-1} = \beta^{-1}N(I-H_{xd}N)^{-1}\alpha^{-1} \quad (5.7)$$

which is what we would conclude if we pre- and post-operated on (5.2) with $\tilde{H}_{yd}^{-1}$ and $\tilde{H}_{xu}^{-1}$, respectively. To see that (5.7) is true even when $\tilde{H}_{yd}$ and $\tilde{H}_{xu}$ are not invertible, consider the nth kernel of (5.2). Find an open ball in C^n_+ where $H_{yd}(s_1+\cdots+s_n)$ and $H_{xu}(s_1)\cdots H_{xu}(s_n)$ do not vanish. Then in that ball we have, using (5.4)

$$\{\tilde{N}(I-\tilde{H}_{xd}\tilde{N})^{-1}\}_n(s_1,\cdots,s_n)$$
$$= \beta^{-1}\alpha^{-n}\{N(I-H_{xd}N)^{-1}\}_n(s_1,\cdots,s_n). \quad (5.8)$$

Consequently (5.8) holds in all of C^n_+ and the nth kernels of (5.7) agree. This is true for all n, so (5.7) follows.

Now we look at the memoryless part of (5.7); by Lemma 1

$$\text{MP }[\tilde{N}(I-\tilde{H}_{xd}\tilde{N})^{-1}]$$
$$= \tilde{N} = \text{MP }[\beta^{-1}N(I-H_{xd}N)^{-1}\alpha^{-1}] = \beta^{-1}N\alpha^{-1}.$$

By the last part of Lemma 2 and (5.7)

$$\tilde{N}_{2k-1} + k\tilde{N}_k^2 \text{ SYM } \tilde{H}_{xd}(s_1+\cdots+s_k)$$
$$= \beta^{-1}\alpha^{1-2k}[N_{2k-1}+kN_k^2 \text{ SYM } H_{xd}(s_1+\cdots+s_k)].$$

Canceling $\tilde{N}_{2k-1} = \beta^{-1}\alpha^{1-2k}N_{2k-1}$ and dividing by $k\tilde{N}_k^2$ yields

$$\text{SYM } \tilde{H}_{xd}(s_1+\cdots+s_k) = \frac{N_k^2}{\beta\alpha^{2k-1}\tilde{N}_k^2}$$
$$\cdot \text{ SYM } H_{xd}(s_1+\cdots s_k) = \alpha\beta \text{ SYM } H_{xd}(s_1+\cdots+s_k).$$

For $s \in C_+$, we evaluate this last equation at $s_1 = \cdots = s_k = s/k$ to get

$$\tilde{H}_{xd}(s) = \alpha\beta H_{xd}(s)$$

which completes the proof of Theorem 1.

In the next section we will need the following.

Remark: Under the hypotheses of Theorem 1, $\tilde{H}_{xd} = \alpha\beta H_{xd}$ and $\det\tilde{H} = \alpha\beta\det H$.

Theorem 2: Suppose two systems of the form in Fig. 2 have the same I/O operator. Then there are real constants α, β, T, and γ such that (using previous notation)

$$\tilde{H}_{yu} = H_{yu} \qquad \tilde{H}_{yd} = \beta e^{sT}H_{yd}$$
$$\tilde{H}_{xu} = \alpha e^{-sT}H_{xu} \qquad \tilde{H}_{xd} = \alpha\beta H_{xd}+\gamma$$
$$\tilde{N} = \beta^{-1}N\alpha^{-1}(I+\gamma\beta^{-1}N\alpha^{-1})^{-1}.$$

That is, the two systems are related by a scaling, loop, and delay transformation.

Proof: We first normalize the systems by loop transformations. Let $k = -\text{MP } H_{xd}$ and $\tilde{k} = -\text{MP } \tilde{H}_{xd}$. Then Theorem 1 applies with H_{xd} replaced by $H_{xd}+k$, N replaced by $N(I+kN)^{-1}$, and similarly for the tilded expressions. Three of the conclusions above pop out immediately from Theorem 1; we also conclude

$$\tilde{H}_{xd}+\tilde{k} = \alpha\beta(H_{xd}+k) \quad (5.9)$$

$$\tilde{N}(I+\tilde{k}\tilde{N})^{-1} = \beta^{-1}N(I+kN)^{-1}\alpha^{-1}. \quad (5.10)$$

Letting $\gamma = \alpha\beta k - \tilde{k}$ in (5.9) yields the fourth conclusion of Theorem 2. To get the last conclusion requires some work. In general, if $B = A(I+A)^{-1}$, then $A = B(I-B)^{-1}$, so from (5.10) we have

$$\tilde{k}\tilde{N} = \tilde{k}\beta^{-1}N(I+kN)^{-1}\alpha^{-1}[I-\tilde{k}\beta^{-1}N(I+kN)^{-1}\alpha^{-1}]^{-1}.$$

Dividing by $\tilde{k}$ and carefully moving the $(I+kN)^{-1}\alpha^{-1}$ into the bracketed expression we get

$$\tilde{N} = \beta^{-1}N[\alpha+\alpha kN-\tilde{k}\beta^{-1}N]^{-1}$$
$$= \beta^{-1}N\alpha^{-1}[I+\alpha kN\alpha^{-1}-\tilde{k}\beta^{-1}N\alpha^{-1}]^{-1}$$
$$= \beta^{-1}N\alpha^{-1}[I+\gamma\beta^{-1}N\alpha^{-1}]^{-1}$$

which is the last conclusion of Theorem 2.

VI. Structural Uniqueness

Theorems 1 and 2 allow us to determine under what conditions two systems (or one-port networks) containing one nonlinearity have the same I/O operator (port (v, i) pairs). These systems are often described, perhaps after simplification such as lumping together cascaded LTI operators, by a simple structure like those in Fig. 5. Of course these systems can be put in the general form considered in the last section, but a structure like those in Fig. 5 is usually a more natural description. Indeed the individual boxes

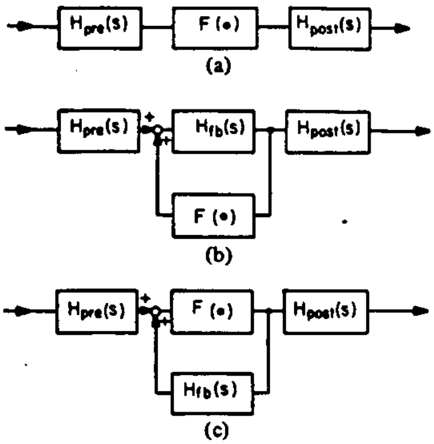

Fig. 5. Three structures for systems with one nonlinearity. (a) Cascade structure, (b) Lur'e structure, and (c) Complementary Lur'e structure. Except for trivial cases, the I/O operators of these structures are completely disjoint. From I/O measurements we could determine which structure such a system has.

often correspond to parts of the actual physical system being modeled. So we now rephrase our original question in terms of these structures: when can two systems as in Fig. 5 have the same I/O operator? We will now show that except for the trivial case when the system is linear, the realizable I/O operators for these different structures are completely disjoint, that is, no system with one structure can have the same I/O operator as a system with a different structure.

In fact we could expand the list of structures in Fig. 5, for example, by taking the output (via H_{post}) from the output of $F(\cdot)$ in (5b) or the input of $F(\cdot)$ in (5c): we only intend the next theorem to illustrate what we call structural uniqueness.

Theorem 3: Consider the three structures shown in Fig. 5, where F is memoryless and the H's are as usual LTI. Suppose F and H_{fb} are not constant, H_{pre} and H_{post} are not identically zero, and H_{fb} is strictly proper; that is $H_{fb}(\infty) \triangleq \lim_{s\to\infty} H_{fb}(s) = 0$.

Then two such systems each with structure (a), (b), or (c) have the same I/O operator if and only if:

1) they have the same structure; and furthermore

2) the corresponding operators are related by scaling, and possibly shuttling some delay between H_{pre} and H_{post}.

Proof: We transform the systems into the form considered in the previous section and apply Theorem 1. Let $N = F - F_1$, the strictly nonlinear part of F, and let $K(s) = (1 - F_1H_{fb})^{-1}$. Then in the notation of Section III the systems of Fig. 5 have nonlinearity N and H-matrices

$$H_{(a)} = \begin{bmatrix} H_{pre}F_1H_{post} & H_{post} \\ H_{pre} & 0 \end{bmatrix}$$

$$H_{(b)} = \begin{bmatrix} H_{pre}H_{fb}KH_{post} & KH_{fb}H_{post} \\ H_{pre}KH_{post} & KH_{fb} \end{bmatrix}$$

$$H_{(c)} = \begin{bmatrix} H_{pre}F_1KH_{post} & KH_{post} \\ H_{pre}K & KH_{fb} \end{bmatrix}.$$

Note that the strict properness of the H_{fb} guarantees that these systems are normalized, so by the remark after Theorem 1 we have

A) any system with the same I/O operator as (a) has $H_{xd} = 0$, and

B) any system with the same I/O operator as (b) has $\det H = 0$.

Thus, a system with structure (b) or (c) could have the same I/O operator as (a) only if H_{pre} or H_{post} were zero, a contradiction. If a system with structure (c) has the same I/O operator as (b), then by B) $\det H_{(c)} = H_{pre}H_{post}K = 0$, again a contradiction. This establishes conclusion 1).

Conclusion 2) for the structure (a) is the main theorem of Boyd and Chua [1] and follows immediately from Theorem 1 applied to $H_{(a)}$, so we omit the proof. The proofs for the other two structures are similar, so we will just give the proof of 2) for (c). Assume two systems with structure (c) have the same I/O operator. Then from Theorem 1 there are α, β, and T such that

$$\begin{bmatrix} \tilde{H}_{post}F_1K\tilde{H}_{pre} & \tilde{H}_{post}\tilde{K} \\ \tilde{H}_{pre}\tilde{K} & \tilde{H}_{fb}\tilde{K} \end{bmatrix} = \begin{bmatrix} H_{post}F_1KH_{pre} & \beta e^{sT}H_{post}K \\ \alpha e^{-sT}H_{pre}K & \alpha\beta H_{fb}K \end{bmatrix}. \tag{6.1}$$

$$\tilde{N} = \beta^{-1}N\alpha^{-1}.$$

Thus, $H_{yu}H_{xd}(H_{yd}H_{xu})^{-1}$ is

$$\tilde{H}_{fb}\tilde{F}_1 = H_{fb}F_1$$

so $K = \tilde{K}$. Canceling K from (6.1) yields

$$\begin{bmatrix} \tilde{H}_{post}\tilde{F}_1\tilde{H}_{pre} & \tilde{H}_{post} \\ \tilde{H}_{pre} & \tilde{H}_{fb} \end{bmatrix} = \begin{bmatrix} H_{post}F_1H_{pre} & \beta e^{sT}H_{post} \\ \alpha e^{-sT}H_{pre} & \alpha\beta H_{fb} \end{bmatrix}.$$

So $\tilde{F}_1 = \alpha^{-1}\beta^{-1}F_1$. Coupled with $\tilde{N} = \beta^{-1}N\alpha^{-1}$ this implies

$$\tilde{F} = \beta^{-1}F\alpha^{-1}$$

and we have shown the systems differ only by scaling and shuttling delay between H_{pre} and H_{post}.

Theorem 3 has implications for black box modeling of systems having a structure like those in Fig. 5. It implies that from I/O measurements alone it is possible, in principle, to determine which internal structure such a system has. Furthermore, we can determine the internal blocks H_{pre}, $N(\cdot)$, etc., up to scaling and possibly delay factors. From Lemma 2 and the proof of Theorem 3 we could construct explicit probing signals which distinguish the structures.

Of course, the differences in the I/O maps of the different structures may be subtle, or in some cases unmeasurable. For example, if a system is very nearly second order, that is, its third and higher order kernels are very small, then it may as well be modeled by the cascade structure of Fig. 5(a), since we need to measure the kernel of order three to observe the effects of the feedback (Lemma 2). A similar statement holds for odd systems with unmeasurable fifth and higher order kernels.

VII. Application to Circuit Theory

In this section we present a simple application of the preceding theory to circuit theory. Suppose we have a one-port network N which contains one nonlinear element, say a voltage controlled nonlinear resistor R with characteristic $i_R = \hat{i}(v_R)$, as in Fig. 6(a). We extract the incremental conductance g at 0 of R and partition N into a linear two-port N_{lin} and a strictly nonlinear resistor R_{snl}, as in Fig. 6(b). The network equations are then

$$v_1 = Z_{11}i_1 + Z_{12}i_2$$

$$v_2 = Z_{21}i_1 + Z_{22}i_2$$

$$i_2 = -G(v_2)$$

where $[Z_{ij}]$ is the impedance matrix of N_{lin} and $i = G(v) = \hat{i}_R(v) - gv$ is the constitutive relation of R_{snl}. These equations have the same form as those describing the system we have already studied: the I/O operator S corresponds to the (nonlinear) impedance operator Φ of our network N, and the matrix H corresponds to the impedance matrix of the linear two-port N_{lin}.

If Z is an operator in our sense, Theorem 2 applies and we have the following.

Theorem 4: Suppose two one-ports N and $\tilde{N}$ as in Fig. 6 have the same (v, i) pairs, and are not linear. Then there are α, β, T,

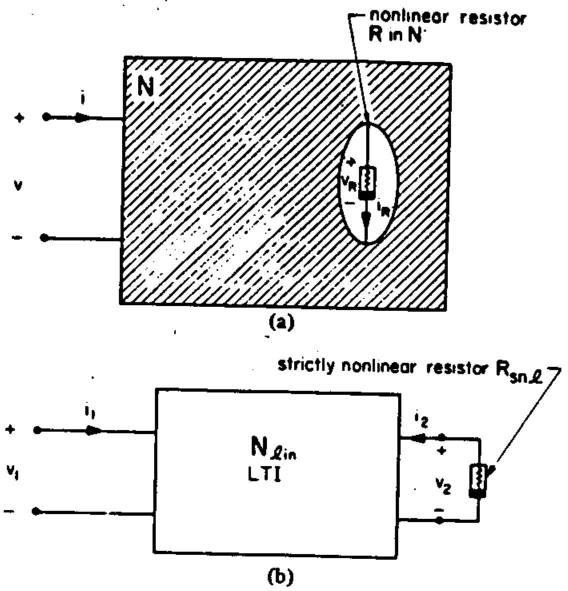

Fig. 6. (a) One-port network N containing one nonlinear element, a resistor R in this case. (b) N partitioned into a LTI 2-port N_{lin} and a strictly nonlinear resistor R_{snl}.

and r_0 such that

$$\tilde{Z}=\begin{bmatrix}1 & 0\\ 0 & \alpha e^{-sT}\end{bmatrix}Z\begin{bmatrix}1 & 0\\ 0 & \beta e^{sT}\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & -r_0\end{bmatrix} \tag{7.1}$$

and the strictly nonlinear resistors are related by

$$\tilde{G}=\beta^{-1}G\alpha^{-1}(I+r_0\beta^{-1}G\alpha^{-1})^{-1}. \tag{7.2}$$

For the case $T = 0$ this has the interpretation shown in Fig. 7.

If in addition N_{lin} and $\tilde{N}_{lin}$ are reciprocal (for example, if they contain only two terminal elements and transformers) then $T = 0$ and $\alpha = \beta$ in (7.1). In Fig. 7 the scalers are then transformers and the networks are related as in Fig. 8.

Proof: If N and $\tilde{N}$ have the same (v, i) pairs, they have the same impedance operator: (7.1) and (7.2) are the conclusions of Theorem 2. Suppose the two-ports are reciprocal. Then (7.1), $Z = Z^T$ and $\tilde{Z} = \tilde{Z}^T$ imply

$$\alpha e^{-sT}Z_{12}(s)=\beta e^{sT}Z_{12}(s).$$

Since Z_{12} is not identically zero, $\alpha\beta^{-1} = \exp(2sT)$, hence, $T = 0$ and $\alpha = \beta$.

Of course, by using another representation (say, admittance) for N_{lin} we can handle current controlled resistors. Similarly, if the original resistor R had been a flux-controlled inductor with $i = \hat{i}_L(\phi)$ we could rewrite the network equations as[5]

$$v_1=Z_{11}i_1+Z_{12}i_2$$
$$\phi_2=s^{-1}Z_{21}i_1+s^{-1}Z_{22}i_2$$

where $S(\cdot)$ is the strictly nonlinear part of $\hat{i}_L$. The conclusions of Theorem 4 then hold with G and $\tilde{G}$ replaced by S and $\tilde{S}$.

We will continue our study of uniqueness in nonlinear circuits in a future paper.

Appendix

The Memoryless Part of an Operator

The main purpose of this section is to prove Lemma 1. While a direct proof is possible we think the approach here is more interesting. We start with a theorem which gives an intuitive interpretation to MP A.

Theorem A1: Suppose $u(t) = 0$, $t < 0$, $\lim_{t\to 0^+} u(t)$ exists [we will call this limit $u(0^+)$], and $\|u\| <$ Rad A. Then $(Au)(0^+)$ exists and $(Au)(0^+) = (\text{MP } A)(u(0^+))$.

Thus MP A is the part of A which reacts instantaneously.

Proof: Let $y(t) = (Au)(t)$. Then $y(t) = \sum_{n=1} y_n(t)$ where

$$y_n(t)=\int\cdots\int u(t-\tau_1)\cdots u(t-\tau_n)\;da_n(\tau_1,\cdots,\tau_n)$$
$$=a_n(\{0\})u(t)^n$$
$$+\int_{R^{+n}-\{0\}}\cdots\int u(t-\tau_1)\cdots u(t-\tau_n)\;da_n(\tau_1,\cdots,\tau_n).$$

Hence,

$$y(t)=\text{MP }A(u(t))$$
$$+\sum_{n=1}\int_{R^{+n}-\{0\}}\cdots\int u(t-\tau_1)\cdots u(t-\tau_n)a_n(d\tau_1,\cdots,d\tau_n). \tag{A1.1}$$

Now the sum in (A1.1) is bounded by

$$\sum_n \|u\|^n|a_n|((0,t]^n). \tag{A1.2}$$

Since the summand in (A1.2) is summable and decreases as $t \to 0^+$, monotone convergence tells us that (A1.2) tends to zero as $t \to 0^+$, and hence the sum in (A1.1) also converges to zero as $t \to 0^+$. Since MP A is analytic near 0,

$$\lim_{t\to 0^+} y(t)=\text{MP }A(u(0^+))$$

which establishes Theorem A1.

Example: Consider the dynamical system

$$\dot{x}=f(x,u)$$
$$y=g(x,u)$$

where f and g are analytic near zero, and the linearized system is exponentially stable near zero, with initial condition $x(0) = 0$. The I/O operator $A: u \to y$ then has a Volterra series ([6], [7]) and Theorem A1 tells us

$$\text{MP }A(\alpha)=g(0,\alpha).$$

Theorem A2: MP $(A + B)$ = MP A + MP B and MP (AB) = MP AMP B.

[5] $s^{-1}Z_{21}$ is sometimes not an operator in our sense, and in fact the same can be said for Z_{21} itself. But the previous theorems still hold with relaxed assumptions on H_{yu}, H_{uu}, and H_{yd}: they can be, e.g., s or s^{-1}.

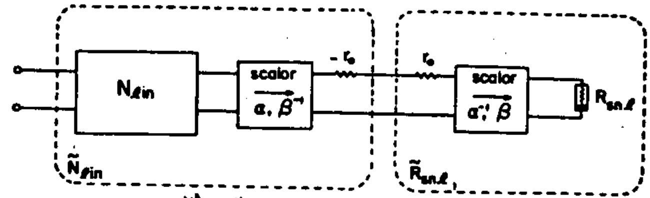

Fig. 7. Relation between one-ports as in Fig. 6 which are port-equivalent. $A(\gamma, \delta)$ scaler is defined by $v_{out} = \gamma v_i$ and $i_{out} = -\delta i_i$ (see [11]).

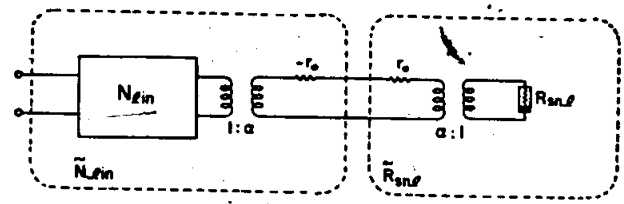

Fig. 8. If N_{lin} and $\tilde{N}_{lin}$ are reciprocal, the relation of Fig. 7 simplifies to that shown here.

Thus, MP maps dynamic operators into memoryless ones, preserving addition and composition. This generalizes the fact that $\mu \rightarrow \mu(\{0\})$ is an algebra homomorphism of the bounded measures on R^+ with convolution into R. We should mention that causality is crucial here, and also that the analogous theorem for discrete time operators is obvious.

Proof: For $|\alpha|$ small ($<\min$ (Rad A, Rad B)) let $u(t) = \alpha 1(t)$, a step of height α. Then from $((A+B)u)(0^+) = Au(0^+) + Bu(0^+)$ and Theorem A1

$$\mathrm{MP}\ (A+B)(\alpha) = \mathrm{MP}\ A(\alpha) + \mathrm{MP}\ B(\alpha)$$

which proves the first assertion; similarly, $Bu(0^+) = \mathrm{MP}\ B(\alpha)$ so $ABu(0^+) = \mathrm{MP}\ A(\mathrm{MP}\ B(\alpha))$. By Theorem A1 $ABu(0^+) = \mathrm{MP}\ (AB)(\alpha)$, hence,

$$\mathrm{MP}\ (AB)(\alpha) = \mathrm{MP}\ A\ \mathrm{MP}\ B(\alpha)$$

establishing Theorem A2.

Theorem A3: If A in invertible, then $\mathrm{MP}\ (A^{-1}) = (\mathrm{MP}\ A)^{-1}$.

Proof: $I = \mathrm{MP}\ I = \mathrm{MP}\ (AA^{-1}) = (\mathrm{MP}\ A)(\mathrm{MP}\ (A^{-1}))$, hence $\mathrm{MP}\ (A^{-1}) = \mathrm{MP}\ A)^{-1}$.

Now we can give the following.

Proof of Lemma 1: In Lemma 1 we have $G = N(I - HN)^{-1}$, where H is LTI, MP $H = 0$, and N is memoryless. By Theorems A1 and A2 MP $(I - HN) = I$; now using Theorems A3 and A2 we have MP $[N(I - HN)^{-1}] = \mathrm{MP}\ N = N$.

Proof of Lemma 2

Recall that $G = N(I - HN)^{-1}$, where H is LTI and N is memoryless strictly nonlinear with first nonvanishing kernel N_k. We first derive a recursive expression for G_n. Since HN is strictly nonlinear, $I - HN$ is invertible (Rad $[(I - HN)^{-1}] > 0$), hence, so is $G = N(I - HN)^{-1}$. Taking the nth kernel of $G(I - HN) = N$ yields

$$[G(I-HN)]_n = N_n.$$

Expanding the left expression using the composition formula:

$$N_n = \mathrm{SYM} \sum_{m=1}^{n} \left[\sum_{\substack{i_1,\cdots,i_m \geq 1 \\ i_1+\cdots+i_m=n}} \right. \cdot G_m(s_1+\cdots+s_{i_1}, \cdots, s_{n+1-i_m}+\cdots+s_n) \cdot \left. (I-HN)_{i_1}(s_1,\cdots,s_{i_1}) \cdots (I-HN)_{i_m}(s_{n+1-i_m},\cdots,s_n) \right].$$

For $n = 1$ this gives $G_1 = 0$, hence, the $m = 1$ term does not contribute. The $m = n$ term is simply $G_n(s_1, \cdots, s_n)$; rearranging the equation above we get a recursive formula for G_n given by

$$G_n(s_1,\cdots,s_n) = N_n - \mathrm{SYM} \sum_{m=2}^{n-1} \left[\sum_{\substack{i_1,\cdots,i_m \geq 1 \\ i_1+\cdots+i_m=n}} \right. \cdot G_m(s_1+\cdots+s_{i_1}, \cdots, s_{n+1-i_m}+\cdots+s_n) \cdot \left. (I-HN)_{i_1}(s_1,\cdots,s_{i_1}) \cdots (I-HN)_{i_m}(s_{n+1-i_m},\cdots,s_n) \right].$$

We can now prove Lemma 2.

Proof of Lemma 2: From the recursive formula for G_n we see that if $G_i = 0$, $i < n$, and $N_n = 0$, then $G_n = 0$. Thus, $G_n = 0$, $n = 1 \cdots k-1$. The outer sum can therefore start at $m = k$. Now we claim that the smallest n for which sum does not vanish is $n = 2k - 1$. By hypothesis,

$$(I-HN)_i = \begin{cases} 1 & i=1 \\ 0 & 1<i<k. \end{cases}$$

The product $(I - HN)_{i_1} \cdots (I - HN)_{i_m}$ will vanish unless each i_j is one or $\geq k$. Since at least one $i_j > 1$, the smallest $n = \sum_{j=1}^{m} i_j$ for which the sum can contribute occurs with $m = k$, one i_j is k, and the others are 1. Thus, $n = m - 1 + k = 2k - 1$. The sum then contains only the k derangements of $(k, 1, \cdots 1)$, so $G_i = N_i$, $i < 2 - 1$ and

$$G_{2k-1} = N_{2k-1} + kN_k^2\ \mathrm{SYM}\ H(s_1+\cdots+s_k)$$

using $G_k = N_k$ and $(I - HN)_k = -H(s_1 + \cdots + s_k)N_k$. So Lemma 2 is proved.

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