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Piecewise-Affine State Feedback Using Convex Optimization

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***Abstract*— This paper shows that Lyapunov-based state feedback controller synthesis for piecewise-affine systems can be cast as an optimization problem subject to a set of LMIs analitically parameterized by a vector. Furthermore, it is shown that continuity of the control inputs at the switchings can be guaranteed by adding equality constraints to the problem without affecting its parameterization structure. Finally, it is shown that piecewise-affine state feedback controller synthesis to maximize the decay rate of a quadratic control Lyapunov function can be cast as a set of quasi-convex optimization problems parameterized by a vector.**

I. Introduction

Piecewise-affine systems are multi-model systems that offer a good modeling framework for complex dynamical systems involving nonlinear phenomena. Piecewise-affine systems are also a class of hybrid systems, i.e, systems with a continuous-time state and a discrete event state. Piecewise-affine systems pose challenging problems because of its switched structure. In fact, the analysis and control of even some simple piecewise-affine systems have been shown to be either an $\mathcal{NP}$ hard problem or undecidable [1]. State and output feedback control of continuous-time piecewise-affine systems has received increasing interest over the last years [2], [3], [4]. Previous work of the authors has concentrated on Lyapunov-based controller synthesis methods for continuous-time piecewise-affine (PWA) systems [4], [2]. In [4], controller synthesis was formulated as a bi-convex optimization problem. The bi-convexity structure arises because of the negativity constraint on the derivative of the piecewise-quadratic Lyapunov function over time. This constraint leads to a bilinear matrix inequality (BMI) [5]. Bi-convex optimization problems are non-convex, NP-hard and, therefore, extremely expensive to solve globally from a computational point of view [5]. Although the general Lyapunov-based controller synthesis problem for piecewise-affine systems using piecewise-quadratic Lyapunov functions is non-convex, reference [2] has shown that for the particular case of piecewise-linear state feedback without affine terms, globally quadratic stabilization could be cast as a convex optimization problem. Unfortunately, if affine terms are included in the controller, as stated in [2], "it does not seem that the condition for stabilizability can be cast as an LMI", which apparently destroys the convex structure of the problem, making it hard to solve globally. The current paper shows that piecewise-affine state feedback using a globally quadratic Lyapunov function can indeed be solved to a point near the global optimum in an efficient way by a set of parameterized LMIs. In this paper four controller synthesis problems are formulated, relaxed to a finite set of optimization programs and solved. The paper starts by presenting the assumptions that are common to all controller design problems, followed by the statements of the four problems. Section IV formulates the controller synthesis problems as optimization programs and discusses its solution. Finally, after a numerical example, the paper presents the conclusions

II. Problem Assumptions

It is assumed that a PWA system and a corresponding partition of the state space with polytopic cells $\mathcal{R}_i$, $i \in \mathcal{I} = \{1, \ldots, M\}$ are given (see [6] for generating such a partition). Following [3], [2], each cell is constructed as the intersection of a finite number (p_i) of half spaces

$$\mathcal{R}_i = \{x \mid H_i^T x - \tilde{g}_i < 0\}, \tag{1}$$

where $H_i = [h_{i1} \;\; h_{i2} \ldots h_{ip_i}]$, $\tilde{g}_i = [\tilde{g}_{i1} \;\; \tilde{g}_{i2} \ldots \tilde{g}_{ip_i}]^T$. Moreover the sets $\mathcal{R}_i$ partition a subset of the state space $\mathcal{X} \subset I\!\!R^n$ such that $\cup_{i=1}^{M} \overline{\mathcal{R}}_i = \mathcal{X}$, $\mathcal{R}_i \cap \mathcal{R}_j = \emptyset$, $i \neq j$, where $\overline{\mathcal{R}}_i$ denotes the closure of $\mathcal{R}_i$. Within each cell the dynamics are affine of the form

$$\dot{x}(t) = A_i x(t) + \tilde{b}_i + B_i u(t), \tag{2}$$

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where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$. For system (2), we adopt the definition of trajectories or solutions presented in [3]. Any two cells sharing a common facet will be called *level-1* neighboring cells. Let $\mathcal{N}_i = \{$*level-1* neighboring cells of $\mathcal{R}_i\}$. It is also assumed that vectors $c_{ij} \in \mathbb{R}^n$ and scalars d_{ij} exist such that the facet boundary between cells $\mathcal{R}_i$ and $\mathcal{R}_j$ is contained in the hyperplane described by $\{x \in \mathbb{R}^n \mid c_{ij}^T x - d_{ij} = 0\}$, for $i = 1, \ldots, M$, $j \in \mathcal{N}_i$. A parametric description of the boundaries can then be obtained as [2]

$$\overline{\mathcal{R}_i} \cap \overline{\mathcal{R}_j} \subseteq \{x = \tilde{l}_{ij} + F_{ij}s \mid s \in \mathbb{R}^{n-1}\} \quad (3)$$

for $i = 1, \ldots, M$, $j \in \mathcal{N}_i$, where $F_{ij} \in \mathbb{R}^{n \times (n-1)}$ (full rank) is the matrix whose columns span the null space of c_{ij}, and $\tilde{l}_{ij} \in \mathbb{R}^n$ is given by $\tilde{l}_{ij} = c_{ij}\left(c_{ij}^T c_{ij}\right)^{-1} d_{ij}$. It is also assumed that each $\mathcal{R}_i$ can be outer approximated by a finite union of (possibly degenerate) ellipsoids ε_{ij} for $j = 1, \ldots, J_i$. To describe the ellipsoidal covering, it is assumed that matrices E_{ij} and $\tilde{f}_{ij}$ exist such that

$$\mathcal{R}_i \subseteq \cup_{j=1}^{J_i} \varepsilon_{ij} \quad (4)$$

where

$$\varepsilon_{ij} = \{x \mid \|E_{ij}x + \tilde{f}_{ij}\| \leq 1\}. \quad (5)$$

This covering is especially useful in the case where $\mathcal{R}_i$ is a slab because in this case the covering has only one degenerate ellipsoid ε_i and it is exact, *i.e.,* $\varepsilon_i \subseteq R_i$ and $R_i \subseteq \varepsilon_i$. More precisely, if $\mathcal{R}_i = \{x \mid d_1 \leq c_i^T x \leq d_2\}$, then the degenerate ellipsoid is described by $E_i = 2c_i^T/(d_2 - d_1)$ and $\tilde{f}_i = -(d_2 + d_1)/(d_2 - d_1)$. Finally, it is assumed that the control objective is to stabilize the system to a given point x_{cl}. Setting $z = x - x_{cl}$ the problem is transformed to the stabilization of the origin of the system

$$\dot{z}(t) = A_i z(t) + b_i + B_i u(t), \quad (6)$$

where $b_i = \tilde{b}_i + A_i x_{cl}$. The parametric description of the boundaries (3) is written as

$$\overline{\mathcal{R}_i} \cap \overline{\mathcal{R}_j} \subseteq \{z = l_{ij} + F_{ij}s \mid s \in \mathbb{R}^{n-1}\} \quad (7)$$

where $l_{ij} = \tilde{l}_{ij} - x_{cl}$ for $i = 1, \ldots, M$, $j \in \mathcal{N}_i$. The description of the polytopic cells is

$$\mathcal{R}_i = \{z \mid H_i^T z - g_i < 0\}, \quad (8)$$

where $g_i = \tilde{g}_i - H_i^T x_{cl}$. With $f_{ij} = \tilde{f}_{ij} + E_{ij}x_{cl}$, the ellipsoidal covering elements ε_{ij} are described by

$$\varepsilon_{ij} = \{z \mid \|E_{ij}z + f_{ij}\| \leq 1\}. \quad (9)$$

III. Problem Statement

There are four Lyapunov-based controller synthesis problems that will be solved in this paper. For the four problems, the piecewise-affine state feedback input signal is parameterized by K_i and m_i in the form

$$u = K_i z + m_i, \quad z \in \mathcal{R}_i \quad (10)$$

with $-l_0 \leq m_i \leq l_0$ where l_0 is a vector of upper bounds for the entries of m_i, $i = 1, \ldots, M$. The globally quadratic candidate control Lyapunov function is parameterized by $P = P^T$ as

$$V(z) = z^T P z. \quad (11)$$

PROBLEM 1: Find a globally quadratic control Lyapunov function and a piecewise-affine state feedback controller that exponentially stabilizes the origin of (6),
PROBLEM 2: From the controllers that exponentially stabilize the origin, find the one that maximizes the decay rate of the control Lyapunov function,
PROBLEM 3: The same as problem 1 with continuous input signals at the switching boundaries,
PROBLEM 4: The same as problem 2 with continuous input signals at the switching boundaries.

IV. Problem Solution

This section formulates mathematically the four problems defined in section III and proposes two algorithms to solve them numerically.

A. Stabilization - Problem 1

The candidate control Lyapunov function (11) becomes a Lyapunov function with decay raye α if for fixed $\alpha \geq 0$, $V > 0$ and $\dot{V} < -\alpha V$. Using (6) and (10), sufficient conditions for exponential stability are $P = P^T > 0$,

$$z \in \mathcal{R}_i \Rightarrow 2\left[(A_i + B_i K_i)z + (b_i + B_i m_i)\right]^T P z \quad + \\ +\alpha z^T P z < 0 \quad (12)$$

For $z \in \mathcal{R}_i$, this expression can be recast as

$$\begin{bmatrix} z \\ 1 \end{bmatrix} \begin{bmatrix} \bar{A}_i^T P + P\bar{A}_i + \alpha P & P\bar{b}_i \\ (P\bar{b}_i)^T & 0 \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} < 0, \quad (13)$$

where $\bar{A}_i = A_i + B_i K_i$ and $\bar{b}_i = b_i + B_i m_i$. If we relax the condition $z \in \mathcal{R}_i$ in (13) by $z \in \varepsilon_{ij}$ for $j = 1, \ldots, J_i$ and if we use expression (9) and the $S-procedure$ in a similar way as it was done in [2] yields the following sufficient conditions for quadratic stabilization

$$P = P^T > 0, \ \lambda_{ij} < 0, \ i = 1, \ldots, M, \ j = 1, \ldots, J_i$$

$$\begin{bmatrix} \bar{A}_i^T P + P\bar{A}_i + X & (\cdot) \\ (P\bar{b}_i + \lambda_{ij} E_{ij}^T f_{ij})^T & -\lambda_{ij}\left(1 - f_{ij}^T f_{ij}\right) \end{bmatrix} < 0, \quad (14)$$

where $X = \alpha P + \lambda_{ij} E_{ij}^T E_{ij}$. Using new variables $Q = P^{-1}$, $\mu_{ij} = \lambda_{ij}^{-1}$ and a standard algebraic procedure [7], [2] conditions (14) are equivalent to

$$Q = Q^T > 0,\ \mu_{ij} < 0,\ i = 1, \ldots, M,\ j = 1, \ldots, J_i$$

$$\begin{bmatrix} \bar{A}_i Q + Q\bar{A}_i^T + Y & (\cdot) \\ (\mu_{ij}\bar{b}_i f_{ij}^T + QE_{ij}^T)^T & -\mu_{ij}\left(I - f_{ij} f_{ij}^T\right) \end{bmatrix} < 0, \quad (15)$$

where $Y = \alpha Q + \mu_{ij}\bar{b}_i\bar{b}_i^T$. Setting $\bar{A}_i = A_i + B_i K_i$ and introducing new variables $Y_i = K_i Q$ in (15) yields

$$Q = Q^T > 0, \quad \mu_{ij} < 0, \quad i = 1, \ldots, M, \quad j = 1, \ldots, J_i$$

$$\begin{bmatrix} A_i Q + QA_i^T + W & (\cdot) \\ (\mu_{ij}\bar{b}_i f_{ij}^T + QE_{ij}^T)^T & -\mu_{ij}\left(I - f_{ij} f_{ij}^T\right) \end{bmatrix} < 0, \quad (16)$$

where $W = B_i Y_i + Y_i^T B_i^T + \alpha Q + \mu_{ij}\bar{b}_i\bar{b}_i^T$, $\bar{b}_i = b_i + B_i m_i$.

Definition 4.1: The piecewise-affine state feedback synthesis problem (problem 1) is: for fixed $\alpha \geq 0$

$$\begin{aligned} \text{find} \quad & Q,\ Y_i,\ m_i,\ \mu_{ij} \\ s.t. \quad & Q = Q^T > 0, \quad \mu_{ij} < 0, \quad (16) \\ & -l_1 \prec Y_i \prec l_1, \ -l_0 \prec m_i \prec l_0, \\ & i = 1, \ldots, M,\ j = 1, \ldots, J_i \end{aligned}$$

where $\succ$, $\prec$ mean component-wise inequalities and l_0, l_1 are given vector bounds. □

Notice that it is clear from (16) that we cannot formulate this synthesis problem as one convex problem because (16) is not an LMI if the parameters m_i, $i = 1, \ldots, M$ are unknown. However, for fixed m_i, $i = 1, \ldots, M$, expression (16) is indeed an LMI and the problem is convex. Therefore, although the problem formulated in (16) cannot be cast as one convex program, it is an infinite set of convex problems involving an LMI or, equivalently, an infinite number of LMIs analitically parameterized by the vector $\gamma = [m_1^T\ m_2^T\ \ldots\ m_M^T]^T$. The following algorithm is suggested to solve the state-feedback problem:

Algorithm # 1 – Sampling Method:

1) Define a grid for the domain of the vector γ to sample it at N points,
2) For fixed $\alpha \geq 0$, solve the corresponding feasibility problem 4.1 for each of the points in the grid until a feasible point is found.
3) If step 2 is successful or if the maximum number of iterations was reached, stop. Otherwise, increase the grid density and go back to Step 2.

Remark 1: *The feasibility problem 4.1 can be transformed into an optimization problem if the Q with minimum condition number is sought. In that case, for fixed $\epsilon > 0$, the constraints $\eta > 0$, $\epsilon I < Q < \eta\epsilon I$ should be added to the problem (usually ϵ is selected to be unitary) and η should be minimized [7]. Algorithm # 1 can then be changed to store for all grid points the one that yields the minimum value of η. The algorithm can be further improved if the derivative of the solution with respect to γ at each point is computed. In that case, for each selected sample point, the next sample point should be chosen in the direction opposite to the vector derivative. This will reduce the number of points from the grid that need to be used, thus reducing the computational burden.* □

B. Stabilization - Problem 3

To solve problem 3, similarly to what was done in [4], the boundary description (7) is used to yield the following constraints for continuity of the control signals

$$\begin{aligned} (K_i - K_j)F_{ij} &= 0, & (17) \\ (K_i - K_j)l_{ij} + (m_i - m_j) &= 0,\ \forall j \in \mathcal{N}_i. & (18) \end{aligned}$$

These constraints for continuity cannot be directly used in the problem from definition 4.1 because K_i, $i = 1, \ldots, M$, are not variables in that problem. To be able to express contraints (17)–(18) on the variables Y_i, $i = 1, \ldots, M$, define the matrix $X_{ij} = [F_{ij}\ \ l_{ij}]$. Note that X_{ij} is invertible because F_{ij} is full rank and l_{ij} does not belong to the column space of F_{ij} by construction. Using X_{ij}, (17)–(18) can be written as

$$(K_i - K_j)X_{ij} = [0_{m\times(n-1)}\ \ m_j - m_i]. \quad (19)$$

Then, using (19), the change of variables $Y_i = K_i Q$ and inverting X_{ij}, we can write

$$Y_i = Y_j + [0_{m\times(n-1)}\ \ m_j - m_i]X_{ij}^{-1}Q. \quad (20)$$

Definition 4.2: The stabilization problem 3 is: for fixed $\alpha \geq 0$

$$\begin{aligned} \text{find} \quad & Q,\ Y_i,\ m_i,\ \mu_{ij} \\ s.t. \quad & Q = Q^T > 0,\ \mu_{ij} < 0,\ (16),\ (20) \\ & -l_1 \prec Y_i \prec l_1, \ -l_0 \prec m_i \prec l_0, \\ & i = 1, \ldots, M,\ j = 1, \ldots, J_i \end{aligned}$$

where $\succ$, $\prec$ mean component-wise inequalities and l_0, l_1 are vector bounds. □

Constraints (20) must be included in the optimization problem 4.1 to guarantee that the control signals are continuous at the switching boundaries. Note that for fixed m_i and m_j, the problem can still be formulated as an infinite set of convex optimization programs.

C. Decay Rate Maximization

In the problems of sections IV-A and IV-B the parameter α was fixed. Let now α, the desired decay rate for the globally quadratic control Lyapunov function, be

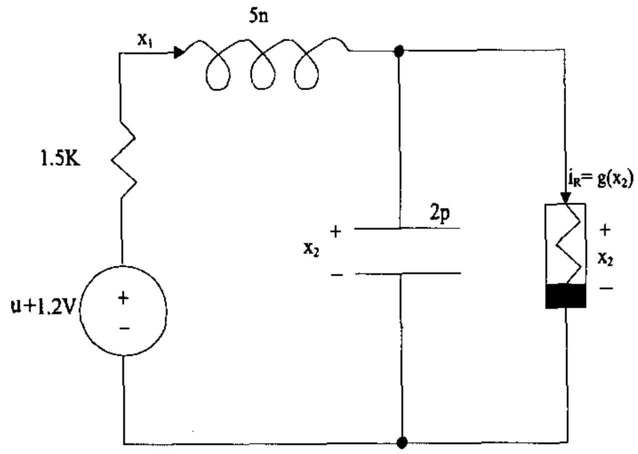

Fig. 1. Circuit with nonlinear resistor

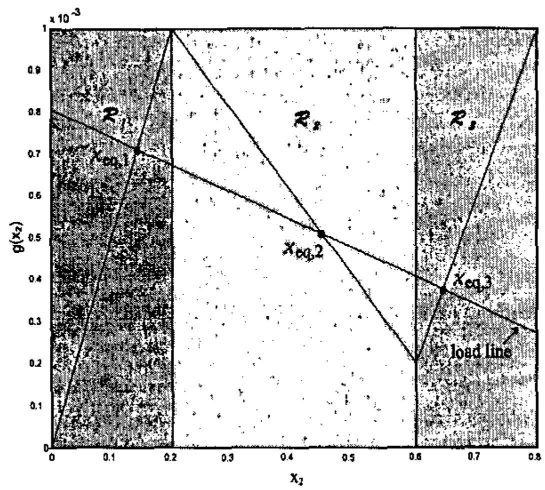

Fig. 2. Nonlinear resistor characteristic.

a variable. Then, we define the performance criterion $\mathcal{J} = \alpha$. The controller design problem is now to find from the class of control signals parameterized in the form $u = K_i z + m_i$ in each region $\mathcal{R}_i$, the one that maximizes $\mathcal{J}$.

Definition 4.3: The decay rate optimization problem 2 is:

$$\begin{array}{ll} \max & \alpha \\ s.t. & Q = Q^T > 0,\ \mu_{ij} < 0,\ \alpha > 0,\ (16) \\ & -l_1 \prec Y_i \prec l_1,\ \ -l_0 \prec m_i \prec l_0, \\ & i = 1, \ldots, M,\ j = 1, \ldots, J_i \end{array}$$

where $\succ, \prec$ mean component-wise inequalities and $l_0,\ l_1$ are vector bounds. □

If there is only one region in the partition of the state space, then $M = 1$, $m_1 = 0$, the system is linear and the decay rate maximization problem is a quasi-convex problem because of the product of variables αQ (see [7] for details). Following the same reasoning as the one used in section IV-A, for the general case of piecewise-affine systems, the decay rate maximization problem is an infinite set of quasi-convex programs analitically parameterized by the vector γ. To formulate problem 4, it suffices to include the continuity constraints (20) in the optimization 4.3 yielding a new optimization problem.

Definition 4.4: The decay rate optimization problem 4 is:

$$\begin{array}{ll} \max & \alpha \\ s.t. & Q = Q^T > 0,\ \mu_{ij} < 0,\ \alpha > 0,\ (16),\ (20) \\ & -l_1 \prec Y_i \prec l_1,\ \ -l_0 \prec m_i \prec l_0, \\ & i = 1, \ldots, M,\ j = 1, \ldots, J_i \end{array}$$

where $\succ, \prec$ mean component-wise inequalities and $l_0,\ l_1$ are vector bounds. □

To solve problems 4.3 and 4.4, note that if γ is again sampled, for each fixed value of γ there is one quasi-convex optimization problem to be solved. For each optimization, a lower bound to the corresponding maximum value of α can then be found, as tight as desired, using the familiar bisection algorithm.

Algorithm # 2 – Bisection:

1) Set $\alpha = 0$, and solve the corresponding convex stabilization problem 1 (or problem 3). If the problem is infeasible stop; there is no piecewise-affine state feedback controller that can quadratically stabilize the system. If the problem is feasible, set $\alpha_{lower} = 0$, $\alpha = \delta$ for small δ and go to step 2.
2) Solve stabilization problem 1 (or problem 3) with $\alpha \leftarrow 10\alpha$ until an infeasible solution is reported.
3) Set $\alpha_{upper} = \alpha$, where α is the one that made problem 1 (or problem 3) infeasible in step 2. Given the desired degree of ϵ tightness of the lower bound, choose the tolerance $tol = \epsilon$.
4) While $\alpha_{upper} - \alpha_{lower} < tol$ solve the convex stabilization problem 1 (or problem 3) with $\alpha \leftarrow 0.5\alpha_{lower} + 0.5\alpha_{upper}$. If the problem is feasible set $\alpha_{lower} = \alpha$, otherwise set $\alpha_{upper} = \alpha$
5) The ϵ-tight lower bound is α_{lower} and the ϵ-optimal controller and control Lyapunov function parameters are the ones that are provided as the solution to problem 1 (or problem 3) using $\alpha = \alpha_{lower}$.

V. Example

This example considers a circuit with a nonlinear resistor taken from [2] and shown in figure 1. With

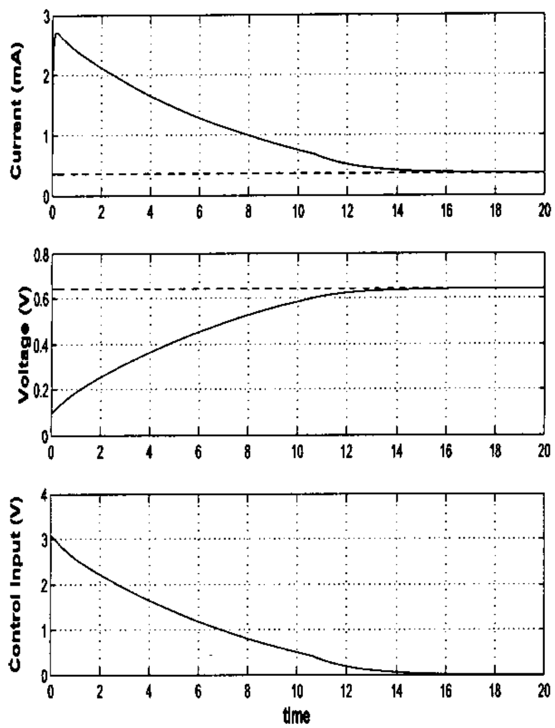

Fig. 3. Piecewise-affine controller for $\alpha = 10^{-9}$

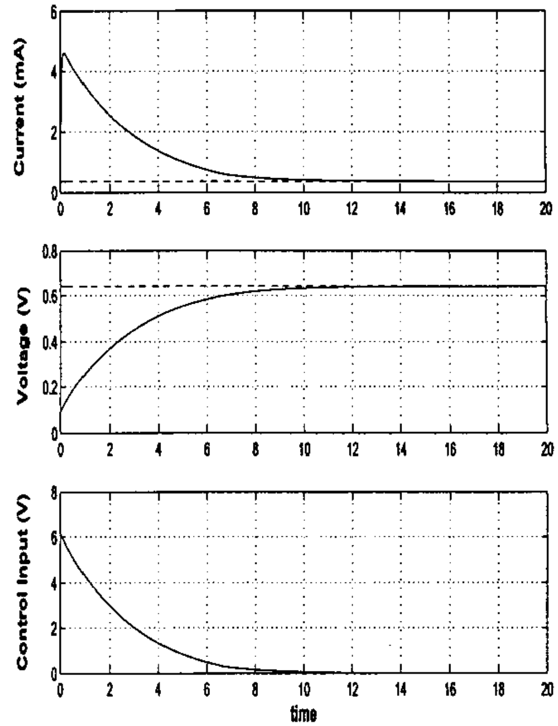

Fig. 4. Piecewise-affine controller for optimal decay rate ($\alpha = 1.01$) using a mesh of 25 points

time in 10^{-10} seconds, the inductor current in mA and the capacitor voltage in Volts, the dynamics are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -30 & -20 \\ 0.05 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 24 \\ -50g(x_2) \end{bmatrix} + \begin{bmatrix} 20 \\ 0 \end{bmatrix} u.$$

Following [2], the characteristic of the nonlinear resistor $g(x_2)$ is defined to be the piecewise-affine function shown in figure 2 which generates the polytopic regions

$$\begin{aligned} \mathcal{R}_1 &= \{x \in \mathbb{R}^2 \mid -L < x_2 < 0.2\}, \\ \mathcal{R}_2 &= \{x \in \mathbb{R}^2 \mid 0.2 < x_2 < 0.6\}, \\ \mathcal{R}_3 &= \{x \in \mathbb{R}^2 \mid 0.6 < x_2 < L\}, \end{aligned}$$

where $L = 2 \times 10^4$. The (exact) ellipsoidal covering is

$$E_1 = \frac{2}{0.2 + L} e_1, \; E_2 = \frac{2}{0.6 - 0.2} e_2, \; E_3 = \frac{2}{L - 0.6} e_3$$

$$\tilde{f}_1 = \frac{L - 0.2}{L + 0.2} \quad \tilde{f}_2 = -\frac{0.6 + 0.2}{0.6 - 0.2} \quad \tilde{f}_3 = -\frac{L + 0.6}{L - 0.6},$$

where $e_1 = e_2 = e_3 = [0 \ 1]$. Assume that the affine terms of the control law have magnitude bounded by 0.2 so that $l_0 = [0.2 \ \ 0.2 \ \ 0.2]^T$. The objective is to design a piecewise-affine state feedback controller to stabilize the open-loop equilibrium point of $\mathcal{R}_3$

$$x_{cl} = x_{ol}^3 = \begin{bmatrix} 0.3714 \\ 0.6429 \end{bmatrix}.$$

For region $\mathcal{R}_3$ we then must have $m_3 = 0$. We start by fixing $m_1 = 0$ and $m_2 = 0.2$. With these values for m_1 and m_2, Algorithm #1 was then used (with only one point in the grid) enforcing continuity of the control signals and using $\alpha = 1 \times 10^{-9}$, $l_1 = 10^{-13}[8 \ \ 8]^T$ to yield

$$\begin{aligned} K_1 &= \begin{bmatrix} -0.21 \times 10^{-4} & -5.66 \end{bmatrix}, & m_1 &= 0.00, \\ K_2 &= \begin{bmatrix} -0.21 \times 10^{-4} & -5.21 \end{bmatrix}, & m_2 &= 0.20 \\ K_3 &= \begin{bmatrix} -0.21 \times 10^{-4} & -9.88 \end{bmatrix}, & m_3 &= 0.00, \end{aligned}$$

The simulation results are shown in figure 3 for the initial condition $x_1^0 = 0.5$, $x_2^0 = 0.1$ (inside region $\mathcal{R}_1$). If each of the affine terms m_1 and m_2 are now sampled in the interval $[-0.2, 0.2]$ with increments of 0.1, a mesh is obtained for the domain of $\gamma = [m_1 \ m_2]^T$ with 25 points. The optimal controller obtained as the solution to problem 4 using a loop with Algorithm #2 inside Algorithm # 1 is described by (see figure 4)

$$\begin{aligned} K_1 &= \begin{bmatrix} -0.19 \times 10^{-4} & -11.21 \end{bmatrix}, & m_1 &= 0.00, \\ K_2 &= \begin{bmatrix} -0.19 \times 10^{-4} & -11.66 \end{bmatrix}, & m_2 &= -0.20 \\ K_3 &= \begin{bmatrix} -0.19 \times 10^{-4} & -7.00 \end{bmatrix}, & m_3 &= 0.00, \end{aligned}$$

$\alpha = 1.01$

It is clear from figure 4 that maximizing the decay rate has yielded a much faster controller as compared to the controller whose results are shown in figure 3. This has come at the expense of increasing the control signal, although the gain vectors still meet the limiting bounds. Also notice that the constraints for continuity of the input signals have imposed that the first component of all gain vectors be equal.

VI. Conclusions

The main contribution the paper is to show that the problem of piecewise-affine state feedback controller synthesis can be cast as an optimization program with an infinite number of LMI constraints parameterized analitically by a vector. After a relaxation (such as, for example, gridding the domain of the vector parameterizing the LMIs), the problem can now be solved more efficiently to a point near the global optimum using available convex optimization packages. Before casting the synthesis in the format presented here, Lyapunov-based piecewise-affine state feedback controller synthesis could only be formulated as a biconvex optimization program, which is very expensive to solve globally.

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