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A regularity result for the singular values of a transfer matrix and a quadratically convergent algorithm for computing its L_∞-norm

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Abstract: The i-th singular value of a transfer matrix need not be a differentiable function of frequency where its multiplicity is greater than one. We show that near a local maximum, however, the largest singular value has a Lipschitz second derivative, but need not have a third derivative. Using this regularity result, we give a quadratically convergent algorithm for computing the $\mathbf{L}_\infty$-norm of a transfer matrix.

Keywords: Multi-input multi-output linear system; transfer matrix; singular values; regularity of singular values, $\mathbf{L}_\infty$-norm; computation of $\mathbf{L}_\infty$-norm; quadratic convergence; $\mathbf{H}_\infty$ control.

1. Introduction

Consider the linear dynamical system

$$\dot{x} = Ax + Bu, \tag{1a}$$

$$y = Cx + Du, \tag{1b}$$

where $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, $C \in \mathbf{R}^{p\times n}$, and $D \in \mathbf{R}^{p\times m}$. The *transfer matrix* of this system is

$$H(s) = C(sI - A)^{-1}B + D. \tag{2}$$

Throughout this paper we will assume that A has no imaginary eigenvalues, so that $H(j\omega)$ is defined for all $\omega \in \mathbf{R}$. We will be concerned with the singular values [1] of the transfer matrix evaluated on the imaginary axis $\sigma_i(H(j\omega))$, where $\omega \in \mathbf{R}$. These singular values, and their associated left and right singular vectors, are useful in understanding at which frequencies, and in which output and input directions, the transfer matrix (2) is 'large' or 'small' (see e.g. [7]). One very important quantity defined in terms of the singular values is the $\mathbf{L}_\infty$-norm of the transfer matrix H,

$$\| H \|_\infty = \sup_{\omega\in\mathbf{R}} \sigma_1(H(j(\omega))). \tag{3}$$

In [2], the authors presented a bisection for computing $\|H\|_\infty$ from the matrices A, B, C, and D; an equivalent algorithm was described in [14]. This algorithm is based on a simple result (Theorem 3.1) that relates the singular values of the transfer matrix $H(j\omega)$ and the imaginary eigenvalues of an associated Hamiltonian matrix. The bisection algorithm has several advantages over brute force methods that directly use (3). At each iteration, an upper and lower bound on $\| H \|_\infty$ are maintained, so that the algorithm can compute $\| H \|_\infty$ to a *guaranteed* relative or absolute accuracy. The convergence of the algorithm is linear (with constant one-half), and independent of the input data A, B, C, D.

In this paper, we present a *quadratically* convergent algorithm for computing the $\mathbf{L}_\infty$-norm of a transfer matrix. The references [5,10,13,1] describe other approaches to modifying the bisection algorithm to make it faster. While some of these algorithms may indeed be quadratically convergent in some cases, no proof is given.

2. Regularity of the singular values as functions of frequency

Our algorithm depends critically on a regularity result for the singular value functions which extends those of MacFarlane and Hung [12]. Mac-

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[1] The i-th singular value of a complex matrix M is the nonnegative squareroot of the i-th largest eigenvalue of M^*M.

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Farlane and Hung observe that if $\sigma_i(H(j\omega_0)) > 0$ and has multiplicity one (i.e., $\sigma_k(H(j\omega_0)) \neq \sigma_i(H(j\omega_0))$ for $k \neq i$), then $\sigma_i(H(j\omega))$ is *real analytic* near ω_0, meaning it is representable by a power series in $\omega - \omega_0$ for $\omega - \omega_0$ small. This observation follows immediately from the fact that an isolated root of a polynomial whose coefficients depend analytically on a parameter is analytic in some neighborhood of the nominal parameter.

Equivalently, real analyticity of $\sigma_i(H(j\omega))$ near ω_0 means that it can be extended to an analytic function $\psi(s)$ in some neighborhood of $j\omega_0$ in the complex plane. In fact, this extension is

$$\psi(s) = \sqrt{\lambda\left(H(-s)^{T}H(s)\right)},$$

where λ is the continuation of the eigenvalue corresponding to σ_i^2 for $s = j\omega_0$, and the squareroot is the principle branch (continuation from the positive real axis). We make the obvious but important remark that $\psi(s) \neq \sigma_i(H(s))$ except when s is imaginary. [2]

When two singular values coalesce (or when one singular value becomes zero), they need not even be differentiable, let alone analytic. Moreover it is well known that the *eigenvalues* of a transfer matrix (the so-called characteristic gains) need not even be Lipschitz in ω, as in the example

$$\frac{1}{s+1}\begin{bmatrix} 0 & 1 \\ s & 0 \end{bmatrix}$$

near $\omega = 0$.

No such behavior is possible for the singular values: by reordering them, and allowing them to become negative, we can guarantee that they are analytic for all $\omega \in \mathbf{R}$, including frequencies where the singular values are not distinct or zero. More precisely we have:

Theorem 2.1. *There are real analytic functions $f_i : \mathbf{R} \to \mathbf{R}$, $i = 1, \ldots, m$, such that for all $\omega \in \mathbf{R}$,*

$$\{\sigma_1(H(j\omega)), \ldots, \sigma_m(H(j\omega))\} = \{|f_1(\omega)|, \ldots, |f_m(\omega)|\}.$$

The functions f_i can be considered *unordered, unsigned singular values* of the transfer matrix; unlike the singular values, however, these functions have analytic extensions into a strip containing the imaginary axis.

The theorem follows immediately from perturbation theory for normal operators; see for example Theorems 1.10 (on page 82) or 6.1 (on page 138) in [11]. We remark that the left and right singular vectors associated with the f_i are also analytic in a strip containing the imaginary axis, but we will not need this fact. We are not aware of the appearance of Theorem 2.1 in the literature. [3]

Theorem 2.1 allows us to give a universal local representation of the singular value functions:

Corollary 2.2. *Given any $\omega_0 \in \mathbf{R}$ and $1 \le i \le m$, there are two functions f_- and f_+ such that in a neighborhood of ω_0, f_- and f_+ are real analytic and*

$$\sigma_i(H(j\omega)) = \begin{cases} f_-(\omega), & \omega \le \omega_0, \\ f_+(\omega), & \omega > \omega_0. \end{cases}$$

For $i = 1$, we have in addition

$$\sigma_1(H(j\omega)) = \max\{f_-(\omega), f_+(\omega)\},$$

in other words, $f_-(\omega) \ge f_+(\omega)$ for $\omega \le \omega_0$, and $f_-(\omega) \le f_+(\omega)$ for $\omega > \omega_0$.

This follows from Theorem 2.1; the f_- and f_+ are each of the form f_i or $-f_i$, where the f_i are the functions mentioned in Theorem 2.1. The second assertion in Corollary 2.2 follows from the fact that f_- and f_+ are each *some* singular value of the transfer matrix near ω_0; $\sigma_1(H(j\omega))$ must be their maximum.

We now consider $\sigma_1(H(j\omega))$ in a neighborhood of a local maximum, at, say, ω_M. Let f_- and f_+ denote the left and right analytic functions from Corollary 2.2 corresponding to $\sigma_1(H(j\omega))$ near ω_M. If $f_- = f_+$ then $\sigma_1(H(j\omega))$ is analytic at ω_M. Let us consider the case $f_+ \neq f_+$, so that $\sigma_1(H(j\omega))$ is *not* analytic at ω_M. Of course, the values of f_- and f_+ agree at ω_M, and their first derivatives at ω_M must be zero. It is less obvious but true that their *second* derivatives must also agree at ω_M. For if, say $f_-''(\omega_M) > f_+''(\omega_M)$, then in a neighborhood of ω_M, $f_-(\omega) > f_+(\omega)$ except at ω_M. This contradicts the second assertion in Corollary 2.2.

Thus the Taylor expansions of f_- and f_+ about ω_M can disagree first at order three. More

[2] It is known that the function $\sigma_1(H(s))$ is *subharmonic* in a neighborhood of the imaginary axis [3].

[3] Some hints appear in [15], however.

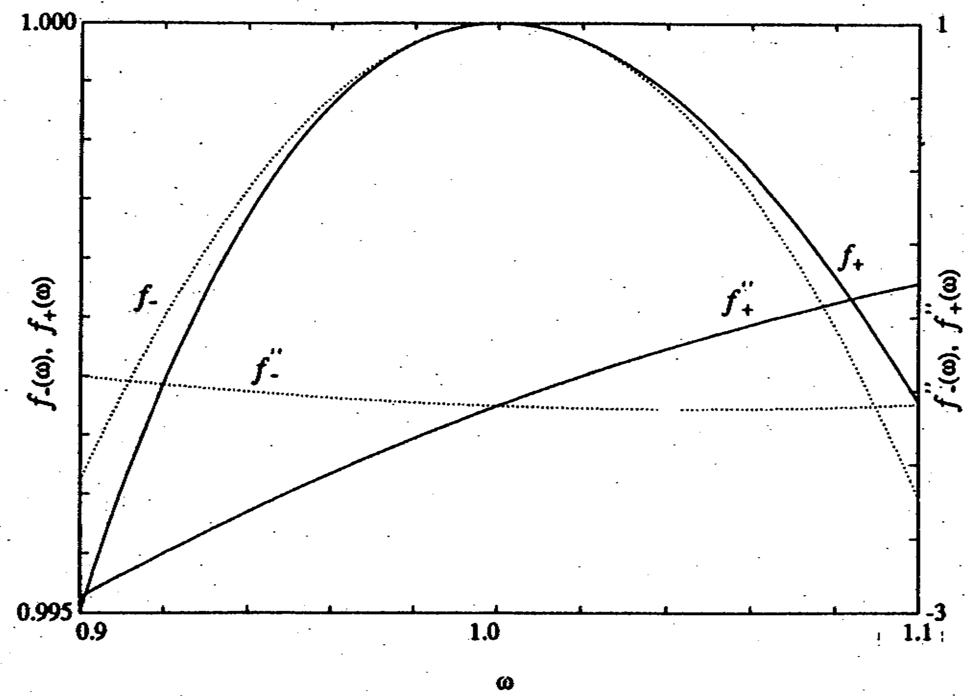

Fig. 1. Singular values and their second derivatives near $\omega_M = 1$ for the example system.

generally, the same argument shows that the Taylor expansions of f_- and f_+ about ω_M disagree first at some *odd* order: for some $P \geq 1$,

$$f_-^{(j)}(\omega_M) = f_+^{(j)}(\omega_M), \quad j = 0, \ldots, 2P;$$
$$f_-^{(2P+1)}(\omega_M) < f_+^{(2P+1)}(\omega_M).$$

Thus the maximum singular value is $2P$ times continuously differentiable, but the derivative of order $2P+1$ does not exist. Moreover, since ω_M is a local maximum of both $f_-(\omega)$ and $f_+(\omega)$, the first nonconstant term in their Taylor series expansion around ω_M *must* be an even, negative function of $\omega - \omega_M$. that is, it must of the form $-a(\omega - \omega_M)^{2N}$ for some N with $1 \leq N \leq P$ and $a > 0$. We summarize these remarks in the following theorem:

Theorem 2.3. *Suppose $\sigma_1(H(\mathrm{j}\omega))$ has a local maximum at ω_M. Then near ω_M, we have*

$$\sigma_1(H(\mathrm{j}\omega)) = \sigma_1(H(\mathrm{j}\omega_M)) - a(\omega - \omega_M)^{2N} + \begin{cases} b_+(\omega - \omega_M)^{2N+1}, & \omega \geq \omega_M, \\ b_-(\omega - \omega_M)^{2N+1}, & \omega < \omega_M \end{cases} + \mathrm{o}\left((\omega - \omega_M)^{2N+1}\right), \tag{4}$$

for some $N \geq 1$, $a > 0$, and $b_- \leq b_+$.

Thus, at a local maximum, we are guaranteed that the maximum singular value is twice continuously differentiable. In fact, the maximum singular value function need not have a third derivative at a local maximum, in other words, the case $N = P = 1$, $b_- < b_+$ can obtain. An example is the two-input two-output system with transfer matrix

$$H(s) = \begin{bmatrix} H_0(s) & 0 \\ 0 & H_0(s^{-1}) \end{bmatrix}, \tag{5}$$

where

$$H_0(s) = \frac{\sqrt{3}\, s^2 + \sqrt{2}\, s}{2s^2 + 2s + 1}. \tag{6}$$

The singular values of (5) are the magnitudes of the diagonal entries; both have a global maximum of one at $\omega_M = 1$. For (5) we have

$$f_-(\omega) = |H_0(\mathrm{j}\omega)|, \qquad f_+(\omega) = |H_0((\mathrm{j}\omega)^{-1})| \tag{7}$$

as the maximum singular value, in a neighborhood left and right of $\omega_M = 1$, respectively. Figure 1 shows (7) and their second derivatives.

3. A quadratically convergent algorithm for computing $\|H\|_\infty$

We first show how to compute the frequency intervals in which the maximum singular value exceeds any given number. We recall a result from [2].

Theorem 3.1. *Suppose* $\gamma > \sigma_1(D)$ *and* $\omega \in \mathbf{R}$. *Define the Hamiltonian matrix*

$$M_\gamma = \begin{bmatrix} A & 0 \\ 0 & -A^{\mathrm{T}} \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & -C^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} -D & \gamma I \\ \gamma I & -D^{\mathrm{T}} \end{bmatrix}^{-1} \begin{bmatrix} C & 0 \\ 0 & B^{\mathrm{T}} \end{bmatrix}.$$

Then $\det(M_\gamma - \mathrm{j}\omega I) = 0$ *if and only if for some* i, $\sigma_i(H(\mathrm{j}\omega)) = \gamma$.

Thus, the imaginary eigenvalues of M_γ are exactly the frequencies for which *some* singular value of the transfer matrix *equals* γ; the endpoints of the frequency intervals where the maximum singular value exceeds γ must be among these. For a multi-input multi-output (MIMO) system ($m > 1$ or $p > 1$), it is not possible to identify the intervals on which $\sigma_1(H(\mathrm{j}\omega)) > \gamma$, given only the imaginary eigenvalues of M_γ: Figure 2 gives an example which illustrates this difficulty.

With a little work, we can separate the frequencies corresponding to different singular values of the transfer matrix, so that for any γ we can compute every solution of each equation,

$$\sigma_i(H(\mathrm{j}\omega)) = \gamma, \quad i = 1, \ldots, m.$$

Suppose the imaginary eigenvalues of M_γ are $\mathrm{j}\omega_1, \ldots, \mathrm{j}\omega_r$. One method is to compute the signature of

$$\gamma^2 I - H(\mathrm{j}\omega_k)^* H(\mathrm{j}\omega_k)$$

for each k, e.g. compute a lower triangular L and a diagonal Σ such that

$$\gamma^2 I - H(\mathrm{j}\omega_k)^* H(\mathrm{j}\omega_k) = L \Sigma L^*.$$

Then, $\sigma_i(H(\mathrm{j}\omega_k)) = \gamma$ if and only if Σ has at least i nonnegative entries and fewer than i positive entries.

In particular, we can compute every solution of the equation $\sigma_1(H(\mathrm{j}\omega)) = \gamma$ by computing the imaginary eigenvalues of M_γ, and discarding those for which

$$\gamma^2 I - H(\mathrm{j}\omega)^* H(\mathrm{j}\omega)$$

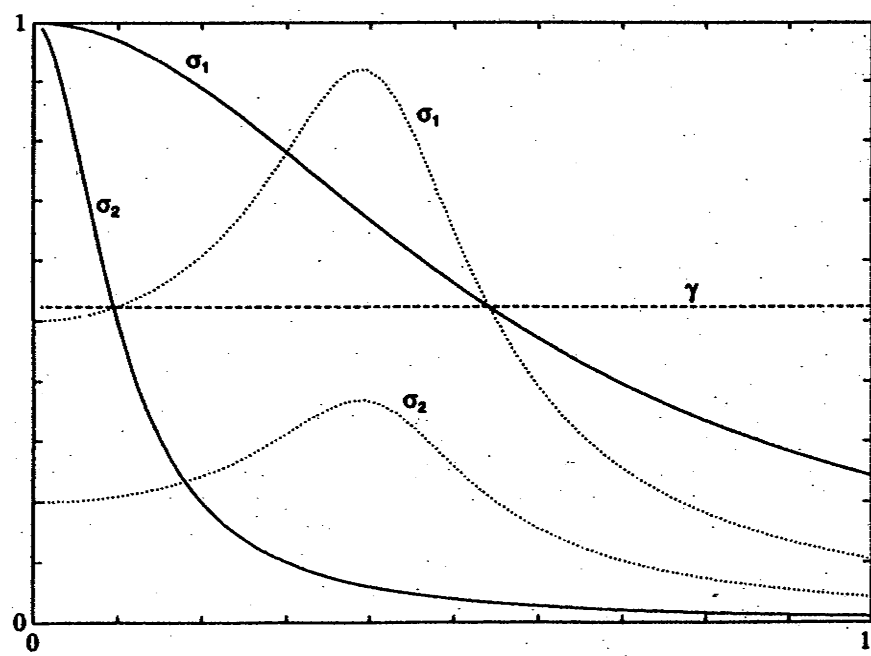

Fig. 2. The dark lines show the singular values of one transfer matrix; the dotted lines show the singular values of another transfer matrix. These systems have identical imaginary eigenvalues of M_γ, but different intervals in which $\sigma_1(H(\mathrm{j}\omega)) > \gamma$. Thus, the intervals in which $\sigma_1(H(\mathrm{j}\omega)) > \gamma$ cannot be determined from the imaginary eigenvalues of M_γ alone.

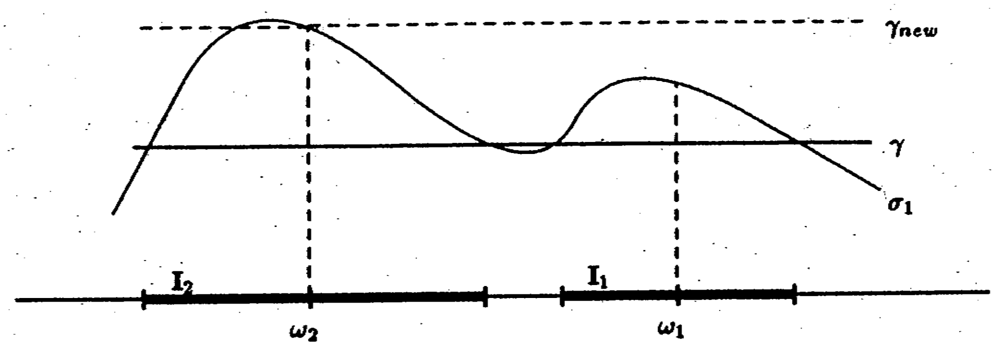

Fig. 3. One iteration of the algorithm: γ_{new} is the updated value of γ.

is not positive semidefinite. These solutions are exactly the endpoints of the intervals in which $\sigma_1(H(j\omega)) > \gamma$. If ω_q and ω_r are consecutive solutions of $\sigma_1(H(j\omega)) = \gamma$, then on the interval (ω_q, ω_r), either $\sigma_1(H(j\omega)) > \gamma$ or $\sigma_1(H(j\omega)) < \gamma$, which is readily determined by checking the signature of

$$\gamma^2 I - H\left(\tfrac{1}{2}j(\omega_q + \omega_r)\right)^* H\left(\tfrac{1}{2}j(\omega_q + \omega_r)\right).$$

We can now describe the algorithm for computing $\|H\|_\infty$:

$\gamma \leftarrow$ any number between $\sigma_{max}(D)$ and $\|H\|_\infty$;
repeat{
  find the frequency intervals $\mathbf{I}_1, \ldots, \mathbf{I}_l$
    where $\sigma_1(H(j\omega)) > \gamma$;
  for each $\mathbf{I}_k$ set $\omega_k = \text{midpoint}(\mathbf{I}_k)$;
  $\gamma = \max_k \sigma_1(H(j\omega_k))$
}

Figure 3 shows one iteration of this algorithm. We will prove that γ always converges monotonically and quadratically to $\|H\|_\infty$. The set $\{\omega_1, \ldots, \omega_l\}$ (l may change at each iteration) approximates the set of frequencies that are global maximizers of the maximum singular value,

$$\Omega_{max} = \{\omega \mid \sigma_1(H(j\omega)) = \|H\|_\infty\}. \tag{8}$$

This convergence is also monotone and quadratic in the following sense: the set $\bigcup_k \mathbf{I}_k$ monotonically and quadratically converges to Ω_{max}.

Given a prespecified relative tolerance ε, one possible stopping criterion is

$$\text{until}\{\sigma_1(H(j\omega)) = \gamma(1+\varepsilon) \text{ has no solutions}\}.$$

This stopping criterion guarantees a relative error less than ε: on exit we have

$$\gamma \le \|H\|_\infty < \gamma(1+\varepsilon).$$

Implementing this stopping criterion directly would require one additional Hamiltonian eigenvalue computation every iteration. It is more efficient to directly incorporate the stopping criterion into the algorithm as follows:

$\gamma \leftarrow$ any number between $\sigma_{max}(D)$ and $\|H\|_\infty$;
repeat{
  find the frequency intervals $\mathbf{I}_1, \ldots, \mathbf{I}_l$
    where $\sigma_1(H(j\omega)) > \gamma(1+\varepsilon)$;
  for each $\mathbf{I}_k$ set $\omega_k = \text{midpoint}(\mathbf{I}_k)$;
  $\gamma = \max_k \sigma_1(H(j\omega_k))$
} until $\{l = 0\}$

On exit, we have $\gamma \le \|H\|_\infty \le \gamma(1+\varepsilon)$.

4. Proof of global convergence

We first prove that the algorithm always converges. Referring to the algorithm, we define

$$\gamma(i) = \gamma, \quad \text{at iteration } i, \tag{9}$$

and

$$V(i) = \max_k \text{length}(\mathbf{I}_k), \quad \text{at iteration } i. \tag{10}$$

Then we have:

Theorem 4.1. $V(i+1) \le \frac{1}{2}V(i)$.

Proof. Let $\mathbf{I}_1^{(i)}, \ldots, \mathbf{I}_{l_i}^{(i)}$ denote the frequency intervals in which $\sigma_1(H(j\omega)) > \gamma$ at iteration i of the algorithm, so that

$$V(i) = \max_k \text{length}\left(\mathbf{I}_k^{(i)}\right).$$

Each interval $\mathbf{I}_k^{(i+1)}$ is contained in one of the

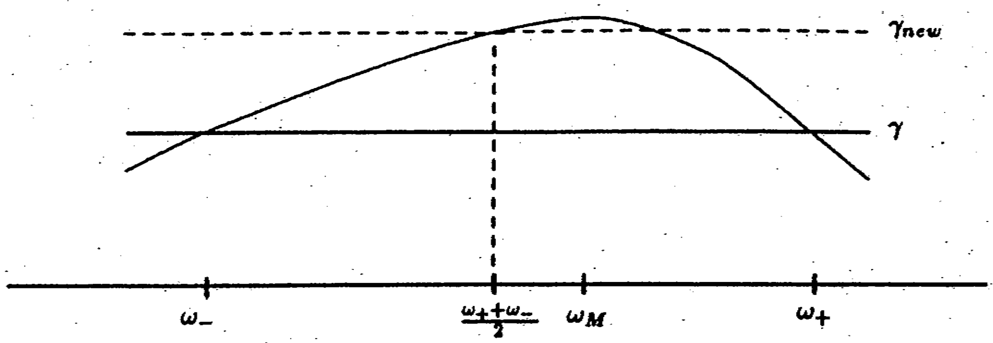

Fig. 4. One iteration of the algorithm near the global maximizer ω_M.

intervals $\mathbf{I}_1^{(i)}, \ldots, \mathbf{I}_{l_i}^{(i)}$; moreover, each interval $\mathbf{I}_k^{(i+1)}$ cannot contain any of the midpoints of the intervals $\mathbf{I}_k^{(i)}$, since at these frequencies we have

$$\sigma_1(H(\mathrm{j}\omega)) \le \gamma(i+1),$$

whereas in the intervals $\mathbf{I}_k^{(i+1)}$ we have

$$\sigma_1(H(\mathrm{j}\omega)) > \gamma(i+1).$$

Thus, each interval at iteration $i+1$ is contained in either the left or right half of an interval from iteration i. The theorem follows immediately. □

Since there at most $\frac{1}{2}n$ intervals at each iteration, the total length of the intervals converges to zero; convergence of γ to $\| H \|_\infty$ follows from uniform continuity of $\sigma_1(H(\mathrm{j}\omega))$. To summarize:

Theorem 4.2. *$\gamma(i) \to \| H \|_\infty$ as $i \to \infty$.*

5. Proof of quadratic convergence

We now show that the convergence is *always* at least *quadratic*, a direct consequence of the regularity result in Theorem 2.3. Let

$$\Omega_{\max} = \{ \omega_1, \ldots, \omega_r \} \tag{11}$$

be the set of maximizing frequencies (see (8)). Let N_j, a_j, b_{+j}, and b_{-j} be the constants in the local representation of $\sigma_1(H(\mathrm{j}\omega))$ near ω_j given by equation (4), for $j = 1, \ldots, r$. Then we have:

Theorem 5.1.

$$\lim_{i\to\infty} \frac{\| H \|_\infty - \gamma(i+1)}{(\| H \|_\infty - \gamma(i))^2} = \min_j \frac{1}{a_j} \left(\frac{b_{+j} + b_{-j}}{4 a_j N_j} \right)^{2N_j}. \tag{12}$$

Remark. It can also be shown that $V(i)$ converges quadratically to zero.

Proof. We will give the proof for the case when $\Omega_{\max}$ is a singleton, say $\Omega_{\max} = \{ \omega_M \}$; our proof is readily extended to the general case. To simplify notation, we will drop the subscripts j (unnecessary since we assume there is only one maximizing frequency) and write γ for $\gamma(i)$ and γ_{new} for $\gamma(i+1)$.

The representation (4) of $\sigma_1(H(\mathrm{j}\omega))$ implies the existence of a neighborhood around ω_M, with $\sigma_1(H(\mathrm{j}\omega))$ strictly monotonic increasing for $\omega < \omega_M$ and strictly monotonic decreasing for $\omega > \omega_M$. Thus we may solve locally for the inverse functions $\omega_+(\gamma)$ and $\omega_-(\gamma)$: for $\| H \|_\infty - \gamma$ small and positive,

$$\sigma_1(H(\mathrm{j}\omega_+(\gamma))) = \gamma, \quad \omega_+(\gamma) \ge \omega_M;$$
$$\sigma_1(H(\mathrm{j}\omega_-(\gamma))) = \gamma, \quad \omega_-(\gamma) \le \omega_M.$$

The frequency interval in the algorithm is thus $(\omega_-(\gamma), \omega_+(\gamma))$ for γ close to $\| H \|_\infty$, and we have

$$\gamma_{\text{new}} = \sigma_1\big(H\big(\tfrac{1}{2}\mathrm{j}(\omega_-(\gamma) + \omega_+(\gamma))\big)\big),$$

as shown in Figure 4.

The inverse functions ω_- and ω_+ have Puiseux series representations (see e.g. [6, p. 246] or [11]):

$$\begin{aligned}\omega_-(\gamma) = \omega_M &+ \alpha_-(\| H \|_\infty - \gamma)^{1/2N} \\ &+ \beta_-(\| H \|_\infty - \gamma)^{2/2N} \\ &+ o\big((\| H \|_\infty - \gamma)^{2/2N}\big),\end{aligned} \tag{13}$$

$$\begin{aligned}\omega_+(\gamma) = \omega_M &+ \alpha_+(\| H \|_\infty - \gamma)^{1/2N} \\ &+ \beta_+(\| H \|_\infty - \gamma)^{2/2N} \\ &+ o\big((\| H \|_\infty - \gamma)^{2/2N}\big).\end{aligned} \tag{14}$$

From

$$\sigma_1(H(j\omega_-)) = \sigma_1(H(j\omega_+)) = \gamma$$

we find that

$$\alpha_- = -a^{-1/2N}, \qquad \beta_- = \frac{b_-}{2N} a^{-(1+1/N)},$$

$$a_+ = a^{-1/2N}, \qquad \beta_+ = \frac{b_+}{2N} a^{-(1+1/N)}.$$

Adding equations (13) and (14), we get

$$\frac{(\omega_+ + \omega_-)}{2} = \omega_M + a^{-(1+1/N)}\left(\frac{b_- + b_+}{4N}\right) \cdot (\|H\|_\infty - \gamma)^{1/N} + o\left((\|H\|_\infty - \gamma)^{1/N}\right).$$

Substituting in (4), we obtain

$$\|H\|_\infty - \gamma_{\text{new}} = \frac{1}{a}\left(\frac{b_- + b_+}{4aN}\right)^{2N} (\|H\|_\infty - \gamma)^2 + o\left((\|H\|_\infty - \gamma)^2\right).$$

The conclusion (12) follows immediately. □

6. Conclusion

In [2] we show how to compute other quantities of interest such as the maximum of the maximum singular value over a given frequency band, or the minimum dissipation of a transfer matrix. The algorithm described in this paper is readily modified to compute these quantities as well.

In our experience, the algorithm converges substantially faster than bisection methods. However, the task of a careful numerical analysis considering the effects of roundoff error in the computations, remains.

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