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A Regularity Result for the Singular Values of a Transfer Matrix and a Quadratically Convergent Algorithm for Computing its $\mathbf{L}_\infty$-norm

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Abstract

The ith singular value of a transfer matrix, $\sigma_i(H(j\omega))$, need not be a differentiable function of ω at frequencies where its multiplicity is greater than one. However, near a local maximum, the largest singular value $\sigma_1(H(j\omega))$ has a Lipschitz second derivative, but need not have a third derivative. Based on this regularity result, we present a quadratically convergent algorithm for computing the $\mathbf{L}_\infty$-norm of a transfer matrix.

1 Introduction

We consider the *singular values*[1] of the transfer matrix of a linear (multi-input-multi-output) dynamical system, evaluated on the imaginary axis; that is, we will be concerned with $\sigma_i(H(j\omega))$, where $\omega \in \mathbf{R}$, and $H(s)$ is the transfer matrix of the system. We will assume that $H(s)$ has no poles on the imaginary axis, so that $H(j\omega)$ is defined for all $\omega \in \mathbf{R}$.

One very important quantity defined in terms of the singular values is the $\mathbf{L}_\infty$-norm of the transfer matrix H,

$$\|H\|_\infty = \sup_{\omega \in \mathbf{R}} \sigma_1(H(j\omega)). \qquad (1)$$

If $H(s)$ is stable (*i.e.*, all of its poles have negative real part), this norm coincides with the $\mathbf{H}_\infty$-norm, which is the supremum of $\sigma_1(H(s))$ over s with positive real part.

We present a ***quadratically*** convergent algorithm for computing the $\mathbf{L}_\infty$-norm of a transfer matrix. The algorithm depends critically on a regularity result for the singular value functions which extends those of MacFarlane and Hung [1]. Most of the results presented are stated without proof. The interested reader is referred to [2] or [3] for details.

2 Regularity of the Singular Values as Functions of Frequency

In [1] MacFarlane and Hung observe that if $\sigma_i(H(j\omega_0)) > 0$ and has multiplicity one, that is, is an isolated singular value, then $\sigma_i(H(j\omega))$ is *real analytic* near ω_0, meaning it is representable by a power series in $\omega - \omega_0$ for $\omega - \omega_0$ small.

However, there remains the important question of how the singular values behave when they are ***not*** isolated. Clearly when two singular values cross (or when one singular value becomes zero), they need not even be differentiable, let alone analytic. However, by reordering them, and possibly allowing them to become negative, we can guarantee that they are analytic for all $\omega \in \mathbf{R}$, including frequencies where the singular values are not distinct. More precisely we have

Theorem 1 *There are real analytic functions $f_i : \mathbf{R} \to \mathbf{R}$, $i = 1, \ldots, m$, such that for all $\omega \in \mathbf{R}$,*

$$\{\sigma_1(H(j\omega)), \ldots, \sigma_m(H(j\omega))\} = \{|f_1(\omega)|, \ldots, |f_m(\omega)|\}$$

The functions f_i can be considered unordered, unsigned 'singular values' of the transfer matrix.

Theorem 1 allows us to give a universal local representation of the singular value functions:

Corollary 1 *Given any $\omega_0 \in \mathbf{R}$ and $1 \le i \le m$, there are two real analytic functions f_- and f_+ such that in a neighborhood of ω_0,*

$$\sigma_i(H(j\omega)) = \begin{cases} f_-(\omega), & \omega \le \omega_0, \\ f_+(\omega), & \omega > \omega_0. \end{cases}$$

This local representation enables us to derive interesting regularity properties of the maximum singular value function around a local maximum. Specifically, the following theorem holds:

Theorem 2 *Suppose $\sigma_1(H(j\omega))$ has a local maximum at ω_M. Then near ω_M, we have*

$$\sigma_1(H(j\omega)) = \sigma_1(H(j\omega_M)) - a(\omega - \omega_M)^{2N} + \left\{ \begin{array}{ll} b_+(\omega - \omega_M)^{2N+1} + o((\omega - \omega_M)^{2N+1}), & \omega \ge \omega_M, \\ b_-(\omega - \omega_M)^{2N+1} + o((\omega - \omega_M)^{2N+1}), & \omega < \omega_M, \end{array} \right. \qquad (2)$$

for some $N \ge 1$, $a > 0$, and $b_- \le b_+$.

Thus, at a local maximum, we are guaranteed that the maximum singular value function is twice continuously differentiable. In fact, the maximum singular value function need not have a third derivative at a local maximum, in other words, the case $N = 1$, $b_- < b_+$ can obtain.

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[1] The ith singular value of a complex matrix M is the nonnegative squareroot of the ith largest eigenvalue of M^*M.

3 A quadratically convergent algorithm for computing $\|H\|_\infty$

In [4] (see also [5], [6] etc), a relation between the singular values of a transfer matrix and a certain Hamiltonian matrix is established. This relation enables us to compute the the frequencies for which, given a real $\gamma > 0$, *any* singular value of the transfer matrix equals γ. With a little more work, we can separate the frequencies corresponding to different singular values of the transfer matrix, and therefore for any $\gamma > \sigma_1(D)$ we can compute every solution of each equation,

$$\sigma_i(H(j\omega)) = \gamma, \qquad i = 1, \ldots, m. \tag{3}$$

Based just on this property, we can devise a simple bisection algorithm, once we find a lower bound γ_L such that $\sigma_1(H(j\omega)) = \gamma_L$ has some solution, and an upper bound γ_U such that $\sigma_1(H(j\omega)) = \gamma_U$ has no solutions. The convergence in this case would be *linear*, since the error in estimating the global maximum at the end of each step equals half the error at the end of the previous step.

But since we have the additional regularity property of the maximum singular value function, we can devise a *quadratically* convergent algorithm for computing the global maximum.

The Quadratically Convergent Algorithm:

repeat {

find the frequency intervals $\mathbf{I}_1, \ldots, \mathbf{I}_l$ *where* $\sigma_1(H(j\omega)) > \gamma$;

for each $\mathbf{I}_k$ *set* $\omega_k = midpoint(\mathbf{I}_k)$;

$\gamma = (1+\epsilon)\max_k \sigma_1(H(j\omega_k))$

} *until* { $l = 0$ }

γ is initialized to any number between $\sigma_1(D)$ and $\|H\|_\infty$. On exit, we have $\gamma \le \|H\|_\infty < \gamma(1+\epsilon)$; therefore the algorithm guarantees a relative error less than ϵ.

4 Convergence Properties

First, the algorithm always converges. Referring to the algorithm, we define

$$\gamma(i) = \gamma, \quad \text{at iteration } i \tag{4}$$

and

$$V(i) = \max_k \text{ length}(\mathbf{I}_k), \quad \text{at iteration } i. \tag{5}$$

Then we have

Theorem 3 $V(i+1) \le V(i)/2$.

Since there are at most $n/2$ intervals at each iteration, the total length of the intervals converges to zero; convergence of γ to $\|H\|_\infty$ follows from uniform continuity of $\sigma_1(H(j\omega))$. To summarize,

Theorem 4 $\gamma(i) \longrightarrow \|H\|_\infty$ *as* $i \longrightarrow \infty$.

Next, the convergence is *always* at least *quadratic*, a direct consequence of the regularity result in theorem 2. Let

$$\Omega_{max} = \{\omega_1, \ldots, \omega_r\} \tag{6}$$

be the set of frequencies that are global maximizers of the maximum singular value, that is,

$$\Omega_{max} = \{\omega \mid \sigma_1(H(j\omega)) = \|H\|_\infty\}. \tag{7}$$

Let N_j, a_j, b_{+j}, and b_{-j} be the constants in the local representation of $\sigma_1(H(j\omega))$ near σ_j given by equation (2), for $j = 1, \ldots, r$. Then we have:

Theorem 5

$$\lim_{i\to\infty} \frac{\|H\|_\infty - \gamma(i+1)}{(\|H\|_\infty - \gamma(i))^2} = \min_j \frac{1}{a_j}\left(\frac{b_{+j}+b_{-j}}{4a_jN_j}\right)^{2N_j} \tag{8}$$

Remark:

It can also be shown that $V(i)$ converges quadratically to zero.

For the case where the number of poles n of $H(s)$ is much larger than either the number of inputs or outputs, the algorithm typically takes approximately $100n^3$ flops to compute $\|H\|_\infty$ to double precision, which appears to be a remarkable improvement over existing methods.

5 Conclusion

The maximum singular value of a transfer matrix, while generally not even differentiable, is always at least twice continuously differentiable near a local maximum. We have presented a quadratically convergent algorithm for computing $\|H\|_\infty$ based on this regularity result.

The algorithm is quite general, and works whenever we are able to compute the intervals in which a continuous function exceeds any given number. If the function has a continuous second derivative, the algorithm has quadratic convergence.

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