RECENT ADVANCES IN ROBUST CONTROL
DORATO AND YEDAVALLI, EDITORS
IEEE PRESS, 1990

Paper 2.16

Numerical Solution of a Two-disk Problem

Stephen A. Norman* and Stephen P. Boyd†
Information Systems Laboratory, Department of Electrical Engineering
Stanford University, Stanford, California 94305

1 Introduction

A *multidisk problem* in frequency domain control system synthesis is a problem in which the designer wishes to find a feedback compensator that minimizes the H_∞-norm of a certain transfer matrix related to the system being designed, subject to constraints on the H_∞-norms of one or more other transfer matrices related to the system. These problems typically arise when a designer is asked to minimize a performance index such as the gain from a disturbance to a plant output while providing robust stabilization of the plant and/or satisfying constraints on other performance indices.

There are no known analytical solutions to general multidisk problems, although there are partial and approximate solutions [BH79, Kwa85] as well as an explicit solution to a special case [OF86]. In [TP88], Ting and Poolla propose and demonstrate a technique for finding suboptimal solutions to linear two-disk problems.

The purpose of this short paper is to show that while multidisk problems may not have analytical solutions, they can often be posed as infinite-dimensional convex programming problems, and so can be readily solved numerically. In fact, a program called **qdes**, described in [BBB*88], can be used to generate and solve finite-dimensional approximations to such infinite-dimensional programming problems. **Qdes** was used to solve the example two-disk problem considered in [TP88]; the optimal performance index was found to be over four times better than the "approximate solution" found in [TP88]. This short paper outlines how **qdes** was used to solve this example two-disk problem.

*Research supported in part by a NSERC (Canada) 1967 Science and Engineering Scholarship.

†Research supported in part by NSF under ECS-85-52465, AFOSR under 89-0228, Boeing Electronics Company under LF0937, and Bell Communications Research.

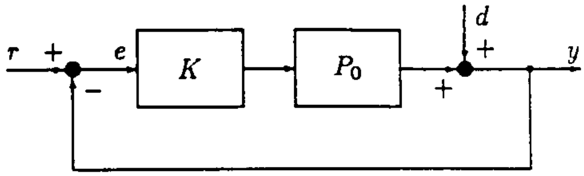

Figure 1: Block diagram for the example problem.

2 Example two-disk problem

The problem, as stated in [TP88], is

$$\inf_{K \text{ stabilizing } P_0} \|W_1(I+P_0K)^{-1}\|_\infty$$

subject to

$$\|W_2P_0K(I+P_0K)^{-1}\|_\infty \le 1,$$

where

$$\begin{aligned} P_0(s) &= \frac{s-2}{s-12} \\ W_1(s) &= \frac{1}{30}\left(\frac{s+6}{s+1}\right) \\ W_2(s) &= \frac{1}{3}\left(\frac{s+1}{s+2}\right)\left(\frac{(s+6)^2}{s^2+2s+37}\right). \end{aligned}$$

The objective is to minimize the W_1-weighted H_∞-norm of the gain from the disturbance d to the plant output y, as shown in Figure 1, and the constraint is obtained from the specification that K stabilize all plants in the family

$$\mathbf{C} = \{P_0(I+\Delta W_2) : \|\Delta\|_\infty < 1\}.$$

The derivation of the H_∞ constraint from this robustness specification is described in [DS81, Zam81].

It is easy to pose the problem as a convex optimization problem. In the terminology of

Reprinted from *1989 American Control Conference*, vol. 2, pp. 1745–1747, June 1989.

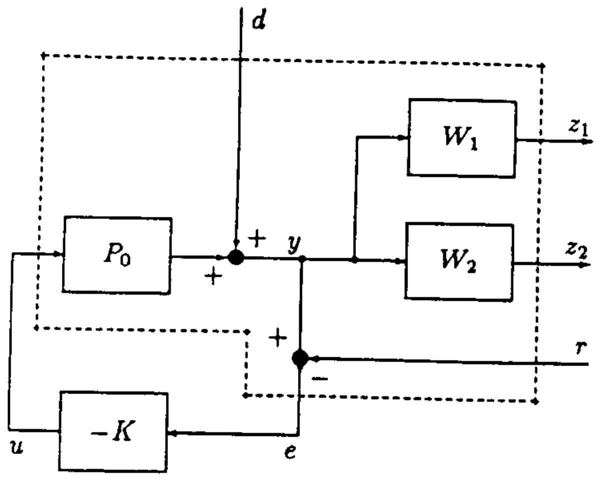

Figure 2: Problem posed in terms of H_∞-norms of closed-loop transfer functions.

[BBB*88], u is the single *control input* and e is the single *measured variable*. As shown in Figure 2, the problem can be stated as that of finding the compensator K that stabilizes P_0 and minimizes the H_∞-norm of the closed-loop transfer function from the *exogenous input* d to the *regulated variable* z_1, subject to the constraint that the H_∞-norm of the closed-loop transfer function from the exogenous input r to the regulated variable z_2 be less than or equal to one. In general any multidisk problem of the form

> Minimize the H_∞-norm of one closed-loop transfer matrix subject to upper bounds on the H_∞-norms of other closed-loop transfer matrices

is a convex programming problem.

The solution of the problem using **qdes** proceeds as follows. A stable coprime factorization of P_0 as $P_0 = ND^{-1}$ with $XN + YD = I$, taken from [TP88], is

$$N(s) = \left(\frac{s-2}{s+6}\right); \quad D(s) = \left(\frac{s-12}{s+6}\right);$$

$$X(s) = 1.8; \text{ and } Y(s) = -0.8.$$

The set of achievable stable closed-loop transfer matrices from the exogenous input vector $[d \;\; r]^T$ to the regulated variable vector $[z_1 \;\; z_2]^T$ can be parametrized as

$$\{T_1 + T_2 Q T_3 : Q \text{ stable}\}$$

where the parameter Q is a SISO transfer function and T_1, T_2, and T_3 are stable transfer matrices of appropriate sizes computed from the coprime factorization as shown in [BBB*88]. The compensator corresponding to a given parameter Q is

$$K = (X + DQ)(Y - NQ)^{-1}.$$

The version of **qdes** available to solve this problem is for use with discrete-time systems, so the problem is converted to an equivalent discrete-time problem by mapping the $j\omega$-axis in the S-plane on to the unit circle in the z-plane with the substitution (bilinear transformation) $s = 20\left(\frac{z+1}{z-1}\right)$. The solution space is made finite-dimensional by assuming that the optimal parameter $Q(z)$ is closely approximated by a finite impulse response (FIR) filter with 20 taps.

The fragment of **qdes** source code found below shows how the problem is specified to **qdes**.

```
minimize {
        max_Mag_H[Z1][D];
}

subject_to {
        max_Mag_H[Z2][R] <= 1.0;
}
```

This fragment specifies the objective and constraint; the dimensions of the problem and the values of T_1, T_2, and T_3 must also be specified. The output of **qdes** is a list of the FIR filter coefficients corresponding to the optimal $Q(z)$.

For the two-disk problem, **qdes** finds a 20-tap Q which satisfies the constraint. The objective function value for the approximation to the two-disk problem is 0.1864. Increasing the number of taps in the FIR filter $Q(z)$ to 40 and 80 does not significantly improve the solution; this gives some confidence that the solution based on the 20-tap $Q(z)$ is very nearly optimal. The compensator $K(z)$ obtained from $Q(z)$ is a 23rd-order stable compensator, which has a 20th-order minimal realization. The Hankel singular values of $K(z)$ are

$$1.427,\ 1.371,\ 0.821,\ 0.040,\ 0.033,\ \ldots\ .$$

It seems likely that the compensator model can be reduced to third-order, and this turns out to be the case. Using simple balance-and-truncate model reduction a third-order compensator was obtained that performed as well as the original

20th-order design. The application of the substitution (inverse bilinear transformation) $z = \left(\frac{s+20}{s-20}\right)$ gives the compensator

$$K(s) = -1.5345 \frac{(s-0.084796)(s^2+1.5166s+35.658)}{(s+0.86685)(s^2+1.3714s+24.576)}.$$

With this compensator

$$\|W_1(I+P_0K)^{-1}\|_\infty = 0.1995$$

and

$$\|W_2P_0K(I+P_0K)^{-1}\|_\infty = 0.9929.$$

For comparison, the "approximate solution" in [TP88] has

$$\|W_1(I+P_0K)^{-1}\|_\infty = 0.7998$$

and

$$\|W_2P_0K(I+P_0K)^{-1}\|_\infty = 1.$$

3 Conclusion

While there are no known analytical techniques for solving general multidisk problems, many multidisk problems are susceptible to effective numerical solutions.

References

[BBB*88] S. Boyd, V. Balakrishnan, C. Barratt, N. Khraishi, X. Li, D. Meyer, and S. Norman. A new CAD method and associated architectures for linear controllers. *IEEE Trans. Aut. Control*, AC-33(3):268–283, March 1988.

[BH79] J. Ball and W. Helton. Interpolation with outer functions and gain equalization in amplifiers. In *International Symposium on Mathematical Theory of Networks and Systems (Delft University of Technology, The Netherlands)*, page 41, 1979.

[DS81] J. C. Doyle and G. Stein. Multivariable feedback design: Concepts for a classical/modern synthesis. *IEEE Trans. Aut. Control*, AC-26(1):4–16, February 1981.

[Kwa85] H. Kwakernaak. Minimax frequency domain performance and robustness optimization of linear feedback systems. *IEEE Trans. Aut. Control*, AC-30(10):994–1004, October 1985.

[OF86] S. D. O'Young and B. A. Francis. Optimal performance and robust stabilization. *Automatica*, 22(2):171–183, 1986.

[TP88] T. Ting and K. Poolla. Upper bounds and approximate solutions for multidisk problems. *IEEE Trans. Aut. Control*, AC-33(8):783–786, August 1988.

[Zam81] G. Zames. Feedback and optimal sensitivity: Model reference transformations, multiplicative seminorms, and approximate inverses. *IEEE Trans. Aut. Control*, AC-26(2):301–320, April 1981.