A KNOT FLOER STABLE HOMOTOPY TYPE

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Abstract. Given a grid diagram for a knot or link K in S^3, we construct a filtered spectrum whose homology is the knot Floer homology of K. We conjecture that the filtered homotopy type of the spectrum is an invariant of K. Our construction does not use holomorphic geometry, but rather builds on the combinatorial definition of grid homology. We inductively define models for the moduli spaces of pseudo-holomorphic strips and disk bubbles, and patch them together into a framed flow category. The inductive step relies on the vanishing of an obstruction class that takes values in a complex of positive domains with partitions.

Contents

1. Introduction

In [11], Cohen, Jones, and Segal proposed the problem of lifting Floer homology to a Floer spectrum or pro-spectrum, in the sense of stable homotopy theory. Since then, stable homotopy refinements of Floer homology have been constructed in Seiberg-Witten theory [28, 18, 48] and symplectic geometry [12, 21, 1, 44]. In a similar vein, there is a lift of Khovanov homology to a stable homotopy type [26, 24].

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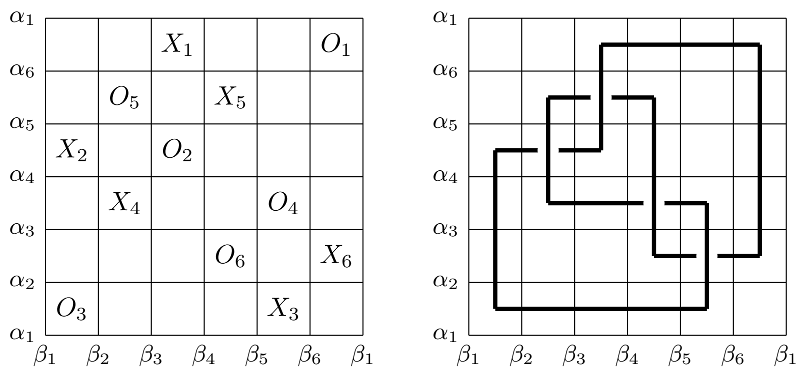

FIGURE 1. Left: A grid diagram. Right: The figure-eight knot that it represents, drawn on the same grid.

The purpose of this paper is to construct a stable homotopy refinement of knot Floer homology. Knot Floer homology was developed by Ozsváth-Szabó [39] and Rasmussen [46], and has many applications; see [29, 16, 42] for some surveys. There is also a generalization to links, called link Floer homology [41]. Knot and link Floer homology were given a combinatorial description in [31, 32]. This description is based on representing the link in terms of a grid diagram, as in Figure 1, and counting empty rectangles on the grid. When defined combinatorially from a grid, link Floer homology is sometimes called *grid homology*. The corresponding chain complex is called a *grid complex*. There are many versions of grid complexes, depending on how we keep track of the O and X markings inside the rectangles. We refer to the book [43] for an extensive treatment.

In this paper we let $\mathbb{G}$ denote a grid diagram representing a link L, and let O_1 be a fixed O-marking on $\mathbb{G}$. We will work with a variant of the grid complex which we denote by GC, where we do not count rectangles going over O_1. The complex GC is a module over $\mathbb{Z}[U_2, \dots, U_n]$, with one variable for each O-marking different from O_1. Further, GC has several Alexander filtrations, one from each component of the link. The multi-filtered chain homotopy type of GC is an invariant of the link with a distinguished component (the one containing O_1). It is closely related to the plus version of the link Floer complex; see Section 2.2 for the definitions. From GC one can recover several other versions of grid complexes, such as $\widehat{GC}$ or $\widetilde{GC}$.

From the grid diagram $\mathbb{G}$ we will construct a multi-filtered CW-spectrum $\mathcal{X}(\mathbb{G})$, whose associated chain complex is GC. The spectrum comes equipped with maps

$$U_i : \mathcal{X}(\mathbb{G}) \to \Sigma^2 \mathcal{X}(\mathbb{G}), \quad i = 2, \dots, n,$$

where Σ^2 denotes the double suspension.

Theorem 1.1. *Up to multi-filtered stable homotopy equivalence (commuting with the maps U_i), the spectrum $\mathcal{X}(\mathbb{G})$ is an invariant of the grid $\mathbb{G}$ and the distinguished marking O_1.*

A variation of this construction produces a multi-filtered spectrum $\widehat{\mathcal{X}}(\mathbb{G})$ with associated graded $g\widehat{\mathcal{X}}(\mathbb{G})$. The reduced homology of $g\widehat{\mathcal{X}}(\mathbb{G})$ is the hat flavor of grid homology (i.e., link Floer homology), $\widehat{GH}(\mathbb{G}) = \widehat{HFL}(L)$. Just as link Floer homology splits according to Alexander gradings

$h \in (\frac{1}{2}\mathbb{Z})^\ell$, the spectrum $g\widehat{\mathcal{X}}(\mathbb{G})$ decomposes into a wedge sum

$$g\widehat{\mathcal{X}}(\mathbb{G}) = \bigvee_{h \in (\frac{1}{2}\mathbb{Z})^\ell} g\widehat{\mathcal{X}}(\mathbb{G}, h)$$

such that

$$\widetilde{H}_i(g\widehat{\mathcal{X}}(\mathbb{G}, h); \mathbb{Z}) = \widehat{HFL}_i(L, h).$$

There are also tilde versions of the spectrum, $\widetilde{\mathcal{X}}(\mathbb{G})$ and $g\widetilde{\mathcal{X}}(\mathbb{G})$. The homology of the latter, $\widetilde{GH}(\mathbb{G}) = \widetilde{HFL}(\mathbb{G})$, is isomorphic to the direct sum of several copies of $\widehat{HFL}(L)$.

The construction of $\mathcal{X}(\mathbb{G})$ (and its variants) is based on first building a framed flow category; once this is done, the machinery of Cohen, Jones and Segal [11] automatically produces a spectrum. In [11], the framed flow category is obtained from moduli spaces of pseudo-holomorphic curves. In our work, we do not use any holomorphic geometry, but rather build models $\mathcal{M}([D])$ for these moduli spaces, inductively on their dimension, in a manner similar to the construction of the Khovanov stable homotopy type in [26].

The rectangles counted in the definition of grid homology are the positive domains associated to 0-dimensional moduli spaces of holomorphic strips in the symmetric product of the grid. Whereas rectangles are domains of index 1, in order to construct the spectrum $\mathcal{X}(\mathbb{G})$ we have to consider positive domains of arbitrary index. Indeed, each moduli space $\mathcal{M}([D])$ in the framed flow category is associated to an equivalence class of positive domains D on the grid, where two domains are equivalent if they differ by a periodic domain (a linear combination of vertical and horizontal annuli) which has coefficient zero on all O markings. We only consider domains D that do not cross the specified marking O_1.

1.1. **Bubbling.** The spaces $\mathcal{M}([D])$ admit compactifications $\overline{\mathcal{M}}([D])$ which correspond to moduli spaces of broken holomorphic strips. Furthermore, each $\overline{\mathcal{M}}([D])$ will be the union of spaces $\overline{\mathcal{M}}_0(D)$ associated to positive domains D in the equivalence class $[D]$, where the different $\overline{\mathcal{M}}_0(D)$ are glued along their common boundaries. These common boundaries correspond to moduli spaces of disk bubbles in symplectic geometry.

We are thus forced to also build models for the moduli spaces of bubbles. This is one of the novel aspects of our construction. Previously, stable homotopy refinements of Floer homologies have mostly been done in the absence of bubbles. (One notable exception is the work of Abouzaid and Blumberg [1], which produces a lift of Hamiltonian Floer homology to Morava K-theory allowing for bubbles.) In general situations where bubbles appear, even Floer homology is not always well-defined since the differential on the Floer complex may not square to zero.

In the link Floer complex (and, more generally, in Heegaard Floer complexes), bubbles appear but they cancel in pairs, so that the differential does square to zero. In the setting of grid diagrams, bubbles correspond to vertical and horizontal annuli, and the two annuli going through the same O-marking cancel each other out. In our construction of $\mathcal{X}(\mathbb{G})$, we implement a higher dimensional analogue of this cancellation: the spaces $\overline{\mathcal{M}}_0(D)$ by themselves are stratified spaces with a complicated structure, but after we glue them together the resulting $\overline{\mathcal{M}}([D])$ is a manifold-with-corners of the kind that is used to define a framed flow category.

To understand the strata in the compactifications $\overline{\mathcal{M}}_0(D)$, we will construct more general spaces $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$, which are models for the moduli spaces of pseudo-holomorphic strips with disk bubbles attached. The bubble configuration is described by vectors

$$\vec{N} = (N_2, \dots, N_n), \quad \vec{\lambda} = (\lambda_2, \dots, \lambda_n)$$

where N_j are non-negative integers, and λ_j is an ordered partition of N_j. The number N_j counts the bubbles going through the j^{th} O-marking. These bubbles are grouped according to the partition λ_j, with those in the same part appearing at the same height on the boundary of the pseudo-holomorphic strip.

Each $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ is a stratified space. The local models for the strata are quite interesting, being based on a stratification of the symmetric product $\mathrm{Sym}^N(\mathbb{C})$ modulo translation by $\mathbb{R}$. Specifically, we consider the stratification of $\mathrm{Sym}^N(\mathbb{C})/\mathbb{R}$ given by the signs of the imaginary parts of the N complex numbers. For example, when $N = 2$, we will encounter the Whitney umbrella

$$W = \{(a, b, c) \in \mathbb{R}^3 \mid b \leq 0, a^2 b + c^2 = 0\}.$$

We hope that these models for the moduli spaces of trajectories with bubbles are of independent interest, as they may appear in other settings. However, we warn the reader that the bubble configurations we use in this paper are different from the ones usually considered in the Gromov compactification in symplectic geometry. See Remark 8.8 for more details.

1.2. **The inductive construction.** We now sketch the construction of the spaces $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$. These will come equipped with suitable embeddings (called *neat*) in Euclidean spaces, and also with normal framings. Since the spaces $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ are not manifolds, it is not immediate what we mean by framings. We will in fact distinguish two different collections of vector fields, the *internal* and *external* framings. More details on these can be found in Section 10.

The construction of the spaces $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ goes as follows:

- We first construct them when D is trivial, and all the entries of $\vec{N}$ are 0's and 1's. In this case we define $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ to be a permutohedron, and explain how to give it a normal framing;
- We define the rest of the spaces $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ inductively on their dimension k. For the base case $k = 0$, we define them to be points, and give them suitable framings;
- For the inductive step, we suppose all spaces up to dimension $(k-1)$ have been constructed. To construct a k-dimensional space $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$, we start with its (already constructed) boundary $\partial\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ and smooth it to get a $(k-1)$-dimensional framed manifold $\partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$;
- From here we get an element $[\partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)] \in \tilde{\Omega}^{k-1}_{\mathrm{fr}}$, where $\tilde{\Omega}^{k-1}_{\mathrm{fr}}$ is a slight variant of the usual framed cobordism group $\Omega^{k-1}_{\mathrm{fr}}$ (and, in fact, is isomorphic to $\Omega^{k-1}_{\mathrm{fr}}$);
- We define a chain complex CDP_* whose generators are "positive domains with partitions," i.e., triples $(D, \vec{N}, \vec{\lambda})$. We let CDP'_* be the quotient of CDP_* by the subcomplex generated by $(D, \vec{N}, \lambda)$ where D is a chosen trivial domain, and $\vec{N}$ is made of 0's and 1's. Altogether, the classes $[\partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)]$ produce an obstruction class

$$\mathfrak{o}_k \in \mathrm{Hom}(CDP'_{k+1}, \tilde{\Omega}^{k-1}_{\mathrm{fr}});$$

- We show that $\mathfrak{o}_k$ is a cocycle, and that CDP'_* is acyclic. It follows that $\mathfrak{o}_k$ is the coboundary of some element $\mathfrak{b} \in \mathrm{Hom}(CDP'_k, \tilde{\Omega}^{k-1}_{\mathrm{fr}})$;
- We use $\mathfrak{b}$ to adjust the definition of the $(k-1)$-dimensional moduli spaces that we had previously constructed, so that all cocycles $\mathfrak{o}_k$ vanish. (We do not change the definition of any moduli spaces of dimension $(k-2)$ or lower.)
- Then $\partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ is framed null-cobordant. We fill it in arbitrarily to obtain the desired framed moduli space $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$, and continue with the induction.

A key role in this construction is played by the complex CDP'_*. To define CDP'_*, we first introduce a chain complex CD_* generated by positive domains on the grid; this is a close cousin of the complex of positive pairs CP^* used in [33, Section 4]. We then enhance CD_* by adding vectors of partitions to its generators; the result is the complex CDP_*. We show that the homology of CDP_* is supported by triples $(D, \vec{N}, \vec{\lambda})$ where D is a fixed trivial domain and $\vec{N}$ is made of 0's and 1's; hence, the quotient CDP'_* of CDP_* by these triples is acyclic. Thus, it is important that we first defined some moduli spaces by hand (to be permutohedra); otherwise we would have had to work with CDP_*, which is not acyclic.

Once the framed moduli spaces $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ are defined, the spectrum $\mathcal{X}(\mathbb{G})$ is obtained by a standard procedure from [11, 26].

We remark that for the simpler versions $g\widehat{\mathcal{X}}(\mathbb{G})$, $\widetilde{\mathcal{X}}(\mathbb{G})$ and $g\widetilde{\mathcal{X}}(\mathbb{G})$, we only use moduli spaces $\overline{\mathcal{M}}_0(D)$ for domains D that do not cross certain markings (X's, O's, or both). These spaces do not involve configurations of bubbles, because D cannot contain a full row or column. Nevertheless, if we had tried to construct only these spaces $\overline{\mathcal{M}}_0(D)$, we would have run into the problem that the analogue of CDP'_* (using domains that do not cross the X-markings) is not acyclic. Thus, even if we were only interested in the simpler versions, we still had to build all the spaces $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ and discuss bubbling.

1.3. **Further directions.** Theorem 1.1 proves a weak form of invariance for $\mathcal{X}(\mathbb{G})$: that it depends only on the grid (and its special marking), not on the other choices made in its construction. We further conjecture that the stable homotopy type of $\mathcal{X}(\mathbb{G})$ is a link invariant, that is, it is independent of the choice of grid diagram $\mathbb{G}$ representing a given link. The proof of this is beyond the scope of the present paper. Invariance of grid homology is proved in [32] by checking the Cromwell-Dynnikov moves: cyclic permutation, commutation, and stabilization. We expect that a combination of those arguments with the techniques from this paper will yield invariance for $\mathcal{X}(\mathbb{G})$. The main challenge is to prove that suitable complexes of positive domains and partitions associated to the commutation and stabilization moves are acyclic.

Another limitation of our paper is that we only consider domains that do not cross a given marking O_1. The reason for this is to ensure the acyclicity of CDP'_*. One can check that the analogue of CDP'_* using all domains on the grid is not acyclic [51]. Nevertheless, one can compute its homology and attempt to get a handle on the analogues of the obstruction classes $[\mathfrak{o}_k]$. We expect that all versions of grid homology (including those involving domains that go over O_1) admit stable homotopy refinements, in the form of spectra or pro-spectra.

1.4. **Organization of the paper.** In Section 2 we fix notation and review some facts about grid diagrams and grid homology.

In Section 3 we define the complex CD_* whose generators are positive domains on the grid.

In Section 4 we define the complex CDP_* of positive domains with partitions, we compute its homology, and introduce the acyclic quotient CDP'_*.

In Section 5 we review $\langle n \rangle$-manifolds, the type of manifolds with corners that are used in framed flow categories.

In Section 6 we discuss different notions of stratified spaces, such as Whitney and Thom-Mather stratifications.

In Section 7 we describe the local models for the stratified spaces that appear in this paper; these are generalizations of the Whitney umbrella.

In Section 8 we give examples of stratified spaces that can be associated to some simple domains on the grid.

In Section 9 we list the strata that should be included in the compactification of each space $\mathcal{M}_{\vec{N},\vec{\lambda}}(D)$.

In Section 10 we introduce the notion of neat embedding for a space $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$, and explain what we mean by internal and external framings.

In Section 11 we define the embedded framed cobordism group $\tilde{\Omega}^k_{\mathrm{fr}}$, and show that it is isomorphic to the usual Ω^k_{fr}.

Section 12 is the heart of the paper, in which we construct the spaces $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ inductively.

The case where D is trivial and $\vec{N}$ is made of 0's and 1's is relegated to Section 13, where we describe a neat embedding of the permutohedron, and give it a normal framing.

In Section 14 we review the Cohen-Jones-Segal construction of a spectrum from a framed Floer category. We then define $\mathcal{X}(\mathbb{G})$ and its variants.

In Section 15 we prove our weak invariance result, Theorem 1.1. This section also introduces the notion of maps of normally framed flow categories.

Finally, in Section 16 we give some examples. We show that the filtered equivalence class of $\mathcal{X}(\mathbb{G})$ is determined by homological data (i.e., by the link Floer complex) for several families of knots, such as thin or L-space knots.

1.5. **Conventions.** Throughout the paper $\mathbb{N}$ denotes the natural numbers including 0. We also let $\mathbb{R}_+ = [0, \infty)$.

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2. Background

2.1. **Grid diagrams.** Definitions and notions related to grid diagrams have been listed in the following enumerated list. For details, see [31, 32, 43].

(G-1) An index-n *grid diagram* $\mathbb{G}$ consists of the torus, obtained from $[0, n] \times [0, n]$ by identifying opposite edges, n 'horizontal' α-*circles*, $\alpha_1, \ldots, \alpha_n$, with α_i being the image of $[0, n] \times \{i-1\}$, and n 'vertical' β-*circles*, $\beta_1, \ldots, \beta_n$, with β_i being the image of $\{i-1\} \times [0, n]$.

(G-2) The n components of the complement of α circles are called *horizontal annuli* or *rows*, the n components of the complement of β circles are called *vertical annuli* or *columns*, and the n^2 components of the complement of α and β circles are called *square regions*.

(G-3) Grid diagrams are decorated with n O-*markings*, $O_1, \ldots, O_n$, placed in n distinct square regions so that each horizontal annulus has one O marking and each vertical annulus has one O marking. Let H_i, respectively V_i, be the horizontal, respectively vertical, annulus that contains O_i; since we are working on a torus, without loss of generality, we will assume that O_1 lies in the 'top-right' square region $(n-1, n) \times (n-1, n)$.

(G-4) We can also order and label the annuli more naturally, without regard for the position of the O's. We define the horizontal annulus $H_{(i)}$ to be the image of $[0, n] \times (i-1, i)$, and the vertical annulus $V_{(i)}$ to be the image of $(i-1, i) \times [0, n]$.

(G-5) Grid diagrams will also be decorated with n X-*markings*, $X_1, \ldots, X_n$, placed in n distinct square regions so that each horizontal annulus has one X marking and each vertical annulus has one X marking.

(G-6) By joining the O and X markings by segments in each row and column, and letting the vertical segments be overpasses, we obtain a planar diagram for a link $L \subset S^3$. (For square regions containing both an O and an X marking, we put a small unknot in that region.) We say that $\mathbb{G}$ is a grid diagram presentation for the link L.

(G-7) A *generator* or a *state* x is a unordered n-tuple $(x_1, \dots, x_n)$ of points on the torus, so that each α-circle contains some x_i and each β-circle contains some x_j. The x_i's are called the *coordinates* of x. We sometimes view x as a formal sum of its coordinates, $x_1 + x_2 + \cdots + x_n$. Generators are in one-to-one correspondence with permutations of $\{1, 2, \dots, n\}$, with permutation σ corresponding to the generator

$$x^\sigma = (\alpha_{\sigma(1)} \cap \beta_1, \alpha_{\sigma(2)} \cap \beta_2, \dots, \alpha_{\sigma(n)} \cap \beta_n).$$

The set of all generators on a grid diagram $\mathbb{G}$ is denoted $\mathbb{S} = \mathbb{S}(\mathbb{G})$.

(G-8) A *domain* D from a generator x to a generator y is a 2-chain given by a $\mathbb{Z}$-linear combination of (the closures of) the square regions, with the property that $\partial(\partial D \cap \alpha) = y - x$. In this paper, we are only interested in domains that have coefficient zero at O_1, and we will let $\mathscr{D}(x, y)$ denote the set of domains from x to y that avoid O_1.

(G-9) For any domain D, let $\mathbb{O}(D) = (O_2(D), \dots, O_n(D)) \in \mathbb{Z}^{n-1}$ be the vector that records the coefficients of D at the O-markings; that is, component $O_i(D)$ is the coefficient of D at O_i, for $2 \leq i \leq n$. Similarly, we let $\mathbb{X}(D) = (X_1(D), \dots, X_n(D)) \in \mathbb{Z}^n$ be the vector that records the coefficients of D at the X-markings.

(G-10) Given $D \in \mathscr{D}(x, y), E \in \mathscr{D}(y, z)$, by adding the underlying 2-chains, we get a domain $D * E \in \mathscr{D}(x, z)$.

(G-11) A domain is said to be *positive* if it has no negative coefficients. Let $\mathscr{D}^+(x, y) \subset \mathscr{D}(x, y)$ be the subset of positive domains. (Note that this includes the zero domain.)

(G-12) For any generators x, y, the set $\mathscr{D}(x, x)$ can be identified with $\mathscr{D}(y, y)$ by identifying the underlying 2-chains. We call either of these sets $\mathscr{P}$, the set of *periodic domains.* Further, we denote by $\mathscr{P}^+$ the subset consisting of positive periodic domains (including zero). We have

$$\mathscr{P} = \mathbb{Z}\langle H_{(1)}, \dots, H_{(n-1)}, V_{(1)}, \dots, V_{(n-1)} \rangle \qquad \mathscr{P}^+ = \mathbb{N}\langle H_{(1)}, \dots, H_{(n-1)}, V_{(1)}, \dots, V_{(n-1)} \rangle.$$

Indeed, for any periodic domain, its multiplicity at the region $H_{(i)} \cap V_{(n)}$, respectively $H_{(n)} \cap V_{(i)}$, gives the coefficient of $H_{(i)}$, respectively $V_{(i)}$, in the above formula.

(G-13) For every domain D, there is an associated integer $\mu(D)$ called its Maslov index, defined as follows. For any point p in the intersection of the α and β-circles, define the *coefficient* of D at p to be average of the coefficients of D at the four square regions adjacent to p. Then the Maslov index $\mu(D)$ is the sum of the coefficients of D at the n coordinates of x and the n coordinates of y. The Maslov index satisfies the following properties:

(a) For any $D \in \mathscr{D}(x, y), E \in \mathscr{D}(y, z)$, $\mu(D * E) = \mu(D) + \mu(E)$.

(b) For any $D \in \mathscr{D}^+(x, y)$, $\mu(D) \geq 0$.

(c) If $D \in \mathscr{D}^+(x, y)$, then $\mu(D) = 0$ if and only if $x = y$ and D is the trivial domain; let $c_x \in \mathscr{D}^+(x, x)$ denote the trivial domain.

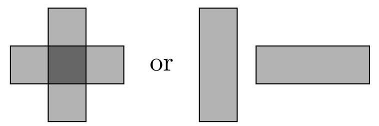
(d) If $D \in \mathscr{D}^+(x, y)$, then $\mu(D) = 1$ if and only if D is a *rectangle* in the torus: its 'bottom-left' and 'top-right' corners are coordinates of x and its 'bottom-right' and 'top-left' corners are coordinates of y; the other $(n-2)$-coordinates of x and y agree and none of them lie in D. Let

$$\mathscr{R}(x, y) = \{D \in \mathscr{D}^+(x, y) \mid \mu(D) = 1\}.$$

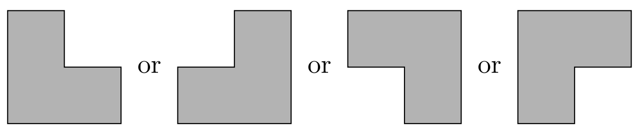
(e) If $D \in \mathscr{D}^+(x,y)$, then $\mu(D) = k$ if and only if D has a (possibly non-unique) *decomposition* into rectangles

$$D = R_1 * R_2 * \cdots * R_k \qquad R_1 \in \mathscr{R}(x = w_0, w_1), R_2 \in \mathscr{R}(w_1, w_2), \dots, R_k \in \mathscr{R}(w_{k-1}, w_k = y).$$

(This is [47, Lemma 3.5], but the proof given in that paper is wrong, so we give a correct proof as Lemma 2.1 below.) In particular, D is a positive index-2 domain if and only if it can be decomposed into two rectangles; that is, it can be two rectangles, either overlapping like a cross or disjoint,



or a hexagon in one of four possible shapes,



or a horizontal annulus, or a vertical annulus. Note, in the first six cases, D has exactly two decompositions into rectangles, while in the last two cases, D has exactly one.

(G-14) Generators carry a well-defined integer-valued grading—called the Maslov grading and denoted $\mathrm{gr}(x)$—so that for any domain $D \in \mathscr{D}(x,y)$,

$$\mathrm{gr}(x) - \mathrm{gr}(y) = \mu(D) - 2\,|\mathbb{O}(D)|\,,$$

where $|\mathbb{O}(D)| = \sum_i O_i(D)$.

(G-15) Generators also admit an Alexander grading $A(x) \in \mathbb{Z}$ with the property that for any $D \in \mathscr{D}(x,y)$,

$$A(x) - A(y) = |\mathbb{X}(D)| - |\mathbb{O}(D)|.$$

In fact, if L is a link of ℓ components, we have an Alexander multi-grading $(A_1(x), \dots, A_\ell(x)) \in (\frac{1}{2}\mathbb{Z})^\ell$ such that $A(x) = A_1(x) + \cdots + A_\ell(x)$.

(G-16) *A sign assignment* s is a function $s\colon \bigcup_{x,y} \mathscr{R}(x,y) \to \{\pm 1\}$ satisfying the following. For any $D \in \mathscr{D}^+(x,y)$ with $\mu(D) = 2$ that is not a horizontal or a vertical annulus (that is, one of the types pictured above), if $R_1 * S_1$ and $R_2 * S_2$ are the two decompositions of D into rectangles, then

$$s(R_1)s(S_1) = -s(R_2)s(S_2). \tag{2.1}$$

Furthermore, if $R * S$ is a decomposition of a horizontal annulus into rectangles, then

$$s(R)s(S) = 1, \tag{2.2}$$

and if $R * S$ is a decomposition of a vertical annulus into rectangles, then

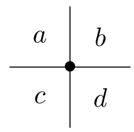
$$s(R)s(S) = -1. \tag{2.3}$$

As promised, we give a correct proof of the following lemma.

Lemma 2.1. *Let $D \in \mathscr{D}^+(x,y)$ with $\mu(D) = k$. Then D has a (possibly non-unique)* decomposition *into rectangles*

$$D = R_1 * R_2 * \cdots * R_k \qquad R_1 \in \mathscr{R}(x = w_0, w_1), R_2 \in \mathscr{R}(w_1, w_2), \dots, R_k \in \mathscr{R}(w_{k-1}, w_k = y).$$

Proof. Let p be a point of intersection of an α and a β-circle, and assume the coefficients of D at the four square regions around p are



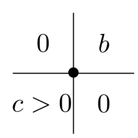
From the condition $\partial D \cap \alpha = y - x$, the following implications are immediate:

$$\begin{aligned} b+c-a-d &= 1 && \text{if } p \text{ is a coordinate of } x \text{ and not of } y;\\ b+c-a-d &= -1 && \text{if } p \text{ is a coordinate of } y \text{ and not of } x;\\ b+c-a-d &= 0 && \text{otherwise.} \end{aligned}$$

We can assume $\mu(D) = k > 0$, so D is not the constant domain. Thus, there is some coordinate of x, say x_1, such that the coefficient of D at x_1 is positive. Therefore, the coefficient of D at either the top-right or the bottom-left square region of x_1 must be non-zero. Assume, after rotating everything by π if necessary, that the coefficient of the top-right square region is non-zero. By translating in the torus if necessary, assume $x_1 = \{0\} \times \{0\}$. Consider all rectangles R that are contained in D as 2-chains (that is, the 2-chain $D - R$ has non-negative coefficients) and that have x_1 as their bottom-left corner. Partially order such rectangles by inclusion, and let $[0, k-1] \times [0, l-1]$ be a maximal element under this partial order.

Let D_0 be the restriction of D to $[0, k] \times [0, l]$. It might not be a domain, but it is a non-negative 2-chain supported on $[0, k] \times [0, l]$ such that its $(k-1)(l-1)$ coefficients in $[0, k-1] \times [0, l-1]$ are strictly positive, and at least one of the $(l-1)$ coefficients in $[k-1, k] \times [0, l-1]$ is zero, and at least one of the $(k-1)$ coefficients in $[0, k-1] \times [l-1, l]$ is zero. From non-negative 2-chains satisfying such hypothesis, we will produce some coordinate of x in $(0, k) \times (0, l)$.

This is accomplished by induction on the sum of coefficients of D_0. Consider the rightmost 0-coefficient in $[0, k-1] \times [l-1, l]$, say it is at $[i-1, i] \times [l-1, l]$ with $i \leq k-1$. Consider at the topmost 0-coefficient in $[k-1, k] \times [0, l-1]$, say it is at $[k-1, k] \times [j-1, j]$ with $j \leq l-1$. If $i = k-1$ and $j = l-1$, then the coefficients near the vertex $\{k-1\} \times \{l-1\}$ looks like



and so this point is a coordinate of x.

If not, we have either $i < k-1$ or $j < l-1$. The two cases are similar, so assume $j < l-1$. Look at the coefficients of D on two sides of the horizontal line $[0, k] \times \{j\}$, numbered as follows:

Assume none of the points on this line are coordinates of x. Then we have $a_n - b_n \geq a_{n+1} - b_{n+1}$ for all $n = 1, \ldots, k-1$. By assumption $a_k > 0$ and $b_k = 0$, so using the previous inequality, we get $a_n > b_n$ for all n. Moreover, $b_n \geq 1$ for all $n < k$, so $a_n \geq 2$ for all $n < k$.

Now consider the 2-chain D'_0 whose coefficients are same as that of D_0 except in the row $[0, k] \times [j, j+1]$; and on that row, the coefficients are $a_1 - 1, a_2 - 1, \ldots, a_k - 1$. (That is, D'_0 is obtained from D_0 by decreasing the coefficients in the row $[0,k]\times[j,j+1]$ by one.) By the inequalities obtained above, D_0' also satisfies the hypothesis, and by induction, has a coordinate of x.

Therefore, we conclude that the rectangle $[0,k-1]\times[0,l-1]$ contains another coordinate of x, say x_2. Now consider all rectangles that are contained in D, have x_1 as their bottom-left corner, and has some coordinate of x as their top-right corner. (The above argument shows that the set of such rectangles is non-empty.) Again partially order them by inclusion, and let R be the minimal element under this partial order. Then $R\in\mathscr{R}(x,z)$ for some z and $D-R\in\mathscr{D}^+(z,y)$, so we have a decomposition $D=R*(D-E)$. By induction, this produces the required decomposition of D into rectangles. □

2.2. **Grid complexes.** Let $\mathbb{G}$ be a grid diagram decorated with O- and X-markings and equipped with a sign assignment s, representing a link L; let $L_1,\dots,L_\ell$ denote the components of L. Assume the marking O_1 (which we are avoiding) lies on the link component L_1.

To $\mathbb{G}$ one can associate chain complexes in various flavors, which are typically called *grid complexes*. The chain homotopy types of these chain complexes (in appropriate senses) are invariants of (L,L_1), namely, the underlying link with a preferred component L_1. We will concentrate on the following flavor. As an Abelian group, the chain group $GC=GC(\mathbb{G})$ is freely generated by elements of the form

$$[x,j_2,\dots,j_n],\quad x\in\mathbb{S},\ j_2,\dots,j_n\in\mathbb{N}.$$

The homological grading of a generator is

$$\mathrm{gr}([x,j_2,\dots,j_n])=\mathrm{gr}(x)+2j_2+\dots+2j_n.$$

The ℓ Alexander gradings are defined as follows:

$$A_k([x,j_2,\dots,j_n])=A_k(x)+\sum_{\{i|O_i\in L_k\}}2j_i.$$

We equip GC with the structure of a module over $\mathbb{Z}[U_2,\dots,U_n]$, by letting U_i act on $[x,j_2,\dots,j_n]$ by decreasing j_i by 1, if $j_i\geq 1$; if $j_i=0$, then U_i acts by zero. Notice that U_i decreases homological grading by two, decreases the Alexander grading A_k by one if $O_i\in L_k$, and preserves the other Alexander gradings. We can alternatively describe the generators of GC as

$$U_2^{-j_2}\cdots U_n^{-j_n}x=[x,j_2,\dots,j_n].$$

The differential on GC is given by

$$\partial([x,j_2,\dots,j_n])=\sum_y\sum_{R\in\mathscr{R}(x,y)}s(R)U^{\mathbb{O}(R)}[y,j_2,\dots,j_n],$$

where we used the notation

$$U^{\mathbb{O}(R)}:=U_2^{O_2(R)}\cdots U_n^{O_n(R)}.$$

Each of the ℓ Alexander gradings are either preserved or decreased by the differential, so the complex GC admits an Alexander multi-filtration induced from these ℓ Alexander gradings on the generators.

We can endow GC with the structure of a $\mathbb{Z}[U_2',\dots,U_\ell']$ module by picking a preferred O-marking $O_{i_k}\in L_k$ and letting U_k' act as U_{i_k}, for $2\leq k\leq\ell$. (The module structure of the chain complex does depend on this additional choice of a preferred O-marking on each component.) This gives GC the structure of a graded ℓ-filtered chain complex over $\mathbb{Z}[U_2',\dots,U_\ell']$. The multi-filtered chain homotopy type of GC over $\mathbb{Z}[U_2',\dots,U_\ell']$ is an invariant of (L,L_1). We will outline a proof of this assertion below, and will assume the reader is familiar with Heegaard Floer homology basics for this part.

Grid diagrams are particular examples of Heegaard diagrams for link complements, and grid complexes correspond to link Floer complexes. From any Heegaard diagram $\mathcal{H} = (\Sigma_g, \alpha_1, \dots, \alpha_{g+n-1}, \beta_1, \dots, \beta_{g+n-1}, O_1, \dots, O_n, X_1, \dots, X_n)$ representing an ℓ-component link $L \subset S^3$, one can define a link Floer complex $\mathit{CFL}^+(\mathcal{H})$, in a similar way as we did for GC, but using pseudo-holomorphic disks instead of rectangles. We will study CFL^+ in full generality, allowing differentials to go over all basepoints, including O_1. We will assume that we have picked the preferred O-marking on L_1 to be O_1, so this becomes a module over $\mathbb{Z}[U_1', \dots, U_\ell']$ with U_1' acting as U_1. Our grid complex GC defined above is the kernel of the U_1' action, which is a subcomplex of CFL^+ of the grid diagram.

These complexes were studied in [39] when L is a knot and the Heegaard diagram has only two basepoints, in [41] for general links, but when each link component has only two basepoints, and in [31] in general; [32] and [43] study the specializations of these complexes to grid diagrams in more detail. Of these, [39] defines a version of knot Floer complex CFK^+ which is most closely related to our versions, but the later papers study a version—denoted CFL^- in [41, 31], C^- in [32], and $\boldsymbol{\mathcal{GC}}^-$ in [43]—with generators

$$U_1^{j_1} \cdots U_n^{j_n} x, \quad x \in \mathbb{S},\ j_1, \dots, j_n \in \mathbb{N},$$

and differentials going over both types of basepoints. It is proved in [31, Section 2] that the filtered chain homotopy type of CFL^- over $\mathbb{Z}[U_1', \dots, U_\ell']$ is an invariant of L.

For completeness, we include here the invariance result for the plus version.

Proposition 2.2. *The multifiltered chain homotopy type of $\mathit{CFL}^+(\mathcal{H})$ over $\mathbb{Z}[U_1', \dots, U_\ell']$ is an invariant of the link L.*

Proof. If we restrict to Heegaard diagrams with only two basepoints on each link component, the argument is entirely similar to that in [41, Theorem 4.7]; it involves checking invariance under isotopies, handleslides, and index one/two stabilizations. Note that in this case there is a single U_k variable for each component L_k, and we can call it U_k'.

Once we allow more basepoints, we also need to check invariance under index zero/three stabilizations. This was done for the minus version in [31, Section 2]. For the plus version, assuming without loss of generality that the stabilization occurs on the link component L_1 near the marking O_1, the arguments there show that the stabilized complex C' is isomorphic to a mapping cone

$$C[U_{n+1}^{-1}] \xrightarrow{U_{n+1} - U_1} C[U_{n+1}^{-1}], \tag{2.4}$$

where C is the complex for the diagram before stabilization, U_{n+1} is the new variable corresponding to the new O-marking, and U_1 is the old variable for the marking O_1. We will prove that C' is filtered chain homotopy to C, as a module over the old variables $\mathbb{Z}[U_1, \dots, U_n]$. Once this is done, the desired conclusion follows inductively: By [31, Lemma 2.4], we can choose any of the markings on a given component L_k to be the new one. Hence, when we do induction on the number of markings on L_k, we can ensure that the preferred O-marking $O_{i_k} \in L_k$ is the oldest one, and so the above filtered chain homotopy equivalences will respect the $U_{i_k} = U_k'$-action on HFL^+.

We first perform a change of basis on the stabilized complex C', viewed as the mapping cone from Equation (2.4). Keep the right $C[U_{n+1}^{-1}]$ unchanged, but for the left $C[U_{n+1}^{-1}]$, replace the generator $U_1^{-j_1} \cdots U_{n+1}^{-j_{n+1}} x$ by

$$(U_1^{-j_1} \cdots U_{n+1}^{-j_{n+1}} x)' := (U_1^{-j_1} + U_1^{-j_1+1} U_{n+1}^{-1} + \dots + U_1^{-1} U_{n+1}^{-j_1+1} + U_{n+1}^{-j_1}) U_2^{-j_2} \cdots U_{n+1}^{-j_{n+1}} x.$$

Note that the change of basis isomorphism $z \mapsto z'$ respects the homological grading and the Alexander multigrading, and also the $U_1, \dots, U_n$-actions, but not the U_{n+1}-action.

In terms of this new basis, the map in the mapping cone $C[U_{n+1}^{-1}] \to C[U_{n+1}^{-1}]$ sends a generator z' on the left to the generator $U_{n+1}z$ on the right. Therefore, this complex decomposes into two summands. The first summand is generated by elements of the form $(U_1^{-j_1} \cdots U_{n+1}^{-j_{n+1}} x)'$ with $j_{n+1} = 0$ in the left $C[U_{n+1}^{-1}]$, and the second summand is generated by the entire right $C[U_{n+1}^{-1}]$ and elements of the form $(U_1^{-j_1} \cdots U_{n+1}^{-j_{n+1}} x)'$ with $j_{n+1} > 0$ in the left $C[U_{n+1}^{-1}]$. The first summand is isomorphic to C (over $\mathbb{Z}[U_1, \dots, U_n]$, respecting all gradings), while the second summand is multifiltered chain homotopic to the zero chain complex. □

Specializing to grid diagrams, and taking the kernel of the U_1' action, we get our ℓ-filtered chain complex GC over $\mathbb{Z}[U_2', \dots, U_\ell']$, whose filtered chain homotopy type is an invariant of (L, L_1).

Here are a few other flavors of grid complexes. We may consider the intersection of the kernels of each of the U_k'-actions. This is a subcomplex $\widehat{GC} \subset GC$ generated by $[x, j_2, \dots, j_n]$ where $j_{i_k} = 0$ for all the preferred O_{i_k} markings that we picked. (If L is a knot, this complex $\widehat{GC}$ is simply our original complex GC.) By adapting the proof of Proposition 2.2 to this setting, we see that $\widehat{GC}$ is ℓ-filtered chain homotopy equivalent to $\widehat{CFL}$, the hat flavor of link Floer chain complex. In particular, if we take the homology of its associated graded complex $g\widehat{GC}$, we get $\widehat{HFL}(L)$, the hat flavor of link Floer homology.

If instead we ask for all j_i to be zero (that is, the complex is generated over $\mathbb{Z}$ by $x \in \mathbb{S}$), we have a complex denoted $\widetilde{GC}$, which is multifiltered chain homotopy equivalent to $\widehat{GC} \otimes V_1 \otimes \cdots \otimes V_{n-\ell}$, where each V_i is a direct sum of two copies of $\mathbb{Z}$ supported in certain homological and Alexander gradings. Therefore, the homology of its associated graded complex $g\widetilde{GC}$ is $\widehat{HFL}(L) \otimes V_1 \otimes \cdots \otimes V_{n-\ell}$.

3. The complex of positive domains

In this section we will study a different chain complex, CD_*, associated to grid diagrams. Unlike the grid complex, the complex CD_* does not carry any interesting topological information. Rather, it is the first step towards constructing a slightly more complicated complex, CDP_*, which will be defined in Section 4. The obstruction classes that we will encounter while constructing our CW complex will live in CDP_*.

Definition 3.1. Given a grid diagram $\mathbb{G}$ and a sign assignment s, the *complex of positive domains*, $CD_* = CD_*(\mathbb{G})$, is freely generated over $\mathbb{Z}$ by the positive domains (avoiding O_1), with the homological grading being the Maslov index:

$$CD_k = \mathbb{Z}\langle \{(x, y, D) \mid D \in \mathscr{D}^+(x, y), \mu(D) = k\} \rangle.$$

We will usually drop x and y from the notation for a generator of CD_*, and just write it as D.

The differential $\delta\colon CD_k \to CD_{k-1}$, on a basis element $D \in \mathscr{D}^+(x, y)$, is given as follows:

$$\delta(D) = \sum_{\substack{(R,E) \in \mathscr{R}(x,w) \times \mathscr{D}^+(w,y) \\ R*E=D}} s(R)E + (-1)^k \sum_{\substack{(E,R) \in \mathscr{D}^+(x,w) \times \mathscr{R}(w,y) \\ E*R=D}} s(R)E.$$

Note that CD_* is independent of the locations of the markings $O_2, \dots, O_n$ and $X_1, \dots, X_n$.

Lemma 3.2. *The complex from Definition 3.1 is indeed a chain complex, that is, $\delta^2 = 0$.*

Proof. The proof is essentially the same as the proof that the grid complex is a chain complex.

$$\delta^2(D) = \sum_{\substack{(R,E) \in \mathscr{R}(x,w) \times \mathscr{D}^+(w,y) \\ R*E=D}} s(R)\delta(E) + (-1)^k \sum_{\substack{(E,R) \in \mathscr{D}^+(x,w) \times \mathscr{R}(w,y) \\ E*R=D}} s(R)\delta(E)$$

$$= \sum_{\substack{(R,S,F)\in\mathscr{R}(x,w)\times\mathscr{R}(w,z)\times\mathscr{D}^+(z,y)\\ R*S*F=D}} s(R)s(S)F + (-1)^{k-1} \sum_{\substack{(R,F,S)\in\mathscr{R}(x,w)\times\mathscr{D}^+(w,z)\times\mathscr{R}(z,y)\\ R*F*S=D}} s(R)s(S)F$$
$$+ (-1)^k \sum_{\substack{(S,F,R)\in\mathscr{R}(x,z)\times\mathscr{D}^+(z,w)\times\mathscr{R}(w,y)\\ S*F*R=D}} s(R)s(S)F - \sum_{\substack{(F,S,R)\in\mathscr{D}^+(x,z)\times\mathscr{R}(z,w)\times\mathscr{R}(w,y)\\ F*S*R=D}} s(R)s(S)F.$$

The second and the third terms cancel. For the first term, if the index-2 domain $R * S \in \mathscr{D}^+(x, z)$ is not a horizontal annulus or a vertical annulus, then it has a unique other decomposition which contributes with the opposite sign. Therefore, it only contributes when $x = z$ and $R * S$ is a horizontal or a vertical annulus. Similarly, the fourth term only contributes when $z = y$ and $S * R$ is a horizontal annulus or a vertical annulus. These two terms contribute with opposite signs, and hence cancel. □

Remark 3.3. A similar complex of positive pairs, denoted CP^*, is defined in [33, Section 4.1], and a certain obstruction class lives in its cohomology. Roughly, the complex CP^* is generated by pairs of generators such that there exists a positive domain between them; in other words, it is generated by positive domains modulo an equivalence relation given by adding or subtracting periodic domains. By contrast, the complex CD_* is generated by positive domains, without dividing by an equivalence relation.

We will spend the rest of this section in showing that the complex CD_* has no interesting homology.

Proposition 3.4. *The complex of positive domains, CD_*, has homology $\mathbb{Z}$ supported in grading 0, generated by the trivial domain c_x for some generator x.*

In order to prove this, we need to define a few objects and establish some of their properties, which we do in the following subsections.

3.1. **Decompositions into rectangles.**

For any domain D, let $A(D) \in \mathbb{N}^{n-1}$ be the vector recording the coefficients of D in the rightmost vertical annulus; that is, the i^{th} component of $A(D)$ is the coefficient of D at the region $H_{(i)} \cap V_{(n)}$. Similarly, let $B(D) \in \mathbb{N}^{n-1}$ be the vector recording the coefficients of D in the topmost horizontal annulus.

Lemma 3.5. *If $D \in \mathscr{D}^+(x, y)$ contains no horizontal (respectively, vertical) annulus—that is, if $D*(-H_i)$ (respectively, $D*(-V_i)$) is not a positive domain for any i—and has $A(D) \neq 0$ (respectively, $B(D) \neq 0$), then there is a decomposition $D = E * R$, with $E \in \mathscr{D}^+(x, z)$ and $R \in \mathscr{R}(z, y)$ and $A(R) \neq 0$ (respectively, $B(R) \neq 0$).*

Proof. We prove the case when D contains no horizontal annulus and $A(D) \neq 0$. The other case is similar.

By Lemma 2.1, there exists at least one decomposition of D into rectangles

$$D = R_1 * R_2 * \cdots * R_n \qquad R_1 \in \mathscr{R}(x = w_0, w_1), R_2 \in \mathscr{R}(w_1, w_2), \ldots, R_n \in \mathscr{R}(w_{n-1}, w_n = y).$$

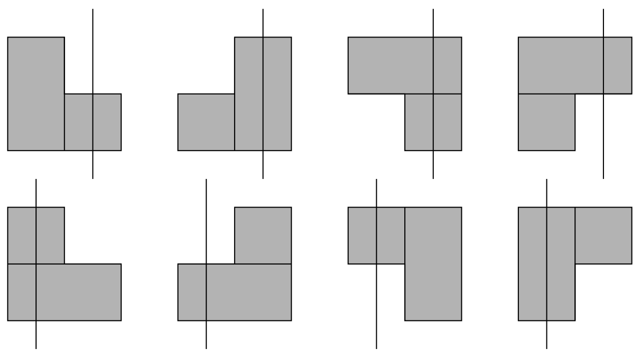
Since we assumed that $A(D) \neq 0$, there is some i such that $A(R_i) \neq 0$. Given such a decomposition $\mathfrak{m}$ of D, let $\iota(\mathfrak{m})$ be the largest such i.

We claim that if $\iota(\mathfrak{m}) \neq n$, then there is some other decomposition $\mathfrak{m}'$ with $\iota(\mathfrak{m}') = \iota(\mathfrak{m}) + 1$. If $R_1, R_2, \ldots, R_n$ are the rectangles appearing in $\mathfrak{m}$, look at the domain

$$H = R_{\iota(\mathfrak{m})} * R_{\iota(\mathfrak{m})+1} \in \mathscr{D}^+(w_{\iota(\mathfrak{m})-1}, w_{\iota(\mathfrak{m})+1}).$$

By assumption $A(R_{\iota(\mathfrak{m})}) \neq 0$ and $A(R_{\iota(\mathfrak{m})+1}) = 0$. Therefore, H is not a vertical annulus. We have already assumed that D does not contain any horizontal annulus, so H is not a horizontal annulus either. Therefore, H is either a (possibly non-disjoint) union of two rectangles or a hexagon, as pictured in Item (G-13e).

In each case, we claim that if $H = S * T$ is the (unique) other decomposition of H into rectangles with $S \in \mathscr{R}(w_{\iota(\mathfrak{m})-1}, w')$ and $T \in \mathscr{R}(w', w_{\iota(\mathfrak{m})+1})$, then $A(T) \neq 0$. This is clear in the first case when H is a union of two rectangles. In the second case, depending on the shape of H and how it intersects $V_{(n)}$, the rightmost vertical annulus (shown as a vertical line in the following figure), there are the following eight possibilities; in each case, we have shown a decomposition of $H = S * T$ with $A(T) \neq 0$. (Two of the following configurations—the second and the eighth—cannot actually appear since they do not admit any decomposition $R_{\iota(\mathfrak{m})} * R_{\iota(\mathfrak{m})+1}$ with $A(R_{\iota(\mathfrak{m})+1}) = 0$.)



Therefore, if we look at the decomposition $\mathfrak{m}'$

$$D = R_1 * \cdots * R_{\iota(\mathfrak{m})-1} * S * T * R_{\iota(\mathfrak{m})+2} * \cdots * R_n,$$

then $\iota(\mathfrak{m}') = \iota(\mathfrak{m}) + 1$. Consequently, there is some decompsition $\mathfrak{m}$ with $\iota(\mathfrak{m}) = n$. That is, D has a decomposition $E * R$, with $E \in \mathscr{D}^+(x, z)$ and $R \in \mathscr{R}(z, y)$ with $A(R) \neq 0$. □

3.2. **The partial order on generators.** Let us first introduce some notation. In the symmetric group, we will denote by τ_p the adjacent transposition $(p, p+1)$. Further, in any partially ordered set, when $y \leq x$, we denote by $[y, x]$ the interval consisting of all z with $y \leq z \leq x$.

Next, recall that the standard (strong) *Bruhat order* on the symmetric group is defined as follows. For any permutation σ, a *reduced word* for σ is a minimal decomposition of σ as a product of adjacent transpositions. All reduced words for σ have the same length, which we denote $|\sigma|$. Define $\sigma \leq \tau$ if some (not necessarily consecutive) substring of some (equivalently, every) reduced word for τ is a reduced word for σ.

Now define the following partial order on the set $\mathbb{S}$ of generators:

$$y \leq x \text{ if } \{D \in \mathscr{D}^+(x, y) \mid A(D) = B(D) = 0\} \neq \varnothing.$$

The relation between this partial order and the Bruhat order is explained below.

(P-1) If x^σ denotes the generator corresponding to the permutation σ of $\{1, 2, \ldots, n\}$ from Item (G-7), then $x^\sigma \leq x^\tau$ if and only if $\sigma \geq \tau$, that is, the above order is the *opposite* of the usual Bruhat order on the symmetric group.

(P-2) The poset has a unique maximum x^{Id}, the generator corresponding to the identity permutation. For any permutation σ, there is a unique (positive) domain $D_\sigma \in \mathscr{D}^+(x^{\mathrm{Id}}, x^\sigma)$ with

$A(D_\sigma) = B(D_\sigma) = 0$—that is, D_σ avoids the rightmost vertical annulus and the topmost horizontal annulus.

(P-3) For any reduced word $\sigma_1\sigma_2\cdots\sigma_k$ for σ, there is a decomposition of $D_\sigma = R_1 * R_2 * \cdots * R_k$ into rectangles so that, for all $1 \le i \le k$, R_i is a width-one rectangle supported in the vertical annulus $V_{(j)}$, where σ_i is the adjacent permutation $\tau_j = (j, j{+}1)$. Therefore, minimal words for σ correspond to decompositions of D_σ into width-one rectangles. In particular, $\mu(D_\sigma) = |\sigma|$.

(P-4) If $\sigma_1\sigma_2\cdots\sigma_k$ is a reduced word for σ, then $\sigma_1\sigma_2\cdots\sigma_{k-1}$ is a reduced word for $\sigma\sigma_k$. To wit, if $R_1 * R_2 * \cdots * R_k$ is the decomposition of D_σ into width-one rectangles corresponding to $\sigma_1\sigma_2\cdots\sigma_k$, then $R_1 * R_2 * \cdots * R_{k-1}$ is a decomposition of $D_{\sigma\sigma_k}$ into width-one rectangles.

(P-5) If the coordinate of x^σ on β_p lies to the bottom-left of the coordinate of x^σ on β_{p+1} for some $1 \le p < n$, then for any reduced word $\sigma_1\sigma_2\cdots\sigma_k$ of σ, $\sigma_1\sigma_2\cdots\sigma_k\tau_p$ is a reduced word for $\sigma\tau_p$. The proof is similar to that in Item (P-4). There is a rectangle $R \in \mathscr{R}(x^\sigma, x^{\sigma\tau_p})$ with width one, supported in $V_{(p)}$ and avoiding $H_{(n)}$. If $D_\sigma = R_1 * R_2 * \cdots * R_k$ is the decomposition into width-one rectangles corresponding to $\sigma_1\sigma_2\cdots\sigma_k$, then $D_{\sigma\tau_p} = R_1 * R_2 * \cdots * R_k * R$ is a decomposition into width-one rectangles corresponding to $\sigma_1\sigma_2\cdots\sigma_k\tau_p$.

(P-6) For any $1 \le p < n$, the permutation σ has a reduced word ending in the transposition τ_p if and only if the coordinate of x^σ on β_p lies to the top-left of the coordinate of x^σ on β_{p+1}.

One direction is clear. If σ has a reduced word ending in τ_p, then D_σ has a decomposition $E * R$, with $E \in \mathscr{D}^+(x^{\mathrm{Id}}, y)$ and $R \in \mathscr{R}(y, x^\sigma)$ being a width-one rectangle supported in the vertical annulus $V_{(p)}$ (and avoiding the top horizontal annulus $H_{(n)}$), and hence the coordinate of x^σ on β_p lies to the top-left of the coordinate of x^σ on β_{p+1}. The proof for the other direction is similar to Lemma 3.5. Let $\mathfrak{w} = \sigma_1\sigma_2\cdots\sigma_k$ be a reduced word for σ. By Item (P-4), $\eta_i = \sigma_1\sigma_2\cdots\sigma_i$ is a reduced word. Call a permutation to be *inverted* if its coordinate on β_p lies to the top-left of its coordinate on β_{p+1}. By assumption $\eta_k = \sigma$ is inverted, while $\eta_0 = x^{\mathrm{Id}}$ is not. Let $\iota(\mathfrak{w})$ be the smallest i, so that $\eta_i, \eta_{i+1}, \dots, \eta_k$ are all inverted. Since $\eta_{\iota(\mathfrak{w})-1}$ is not inverted, but $\eta_{\iota(\mathfrak{w})} = \eta_{\iota(\mathfrak{w})-1}\sigma_{\iota(\mathfrak{w})}$ is, we must have $\sigma_{\iota(\mathfrak{w})} = \tau_p$.

If $\iota(\mathfrak{w}) \ne k$, we will find a new reduced word $\mathfrak{w}'$ for σ with $\iota(\mathfrak{w}') = \iota(\mathfrak{w}) + 1$. Continuing, we will eventually find a word with $\iota = k$, and we will be done.

If $\sigma_{\iota(\mathfrak{w})+1}$ is a transposition that is far from τ_p, then switching $\sigma_{\iota(\mathfrak{w})} = \tau_p$ and $\sigma_{\iota(\mathfrak{w})+1}$ works; that is, $\mathfrak{w}' = \sigma_1\sigma_2\cdots\sigma_{\iota(\mathfrak{w})-1}\sigma_{\iota(\mathfrak{w})+1}\tau_p\sigma_{\iota(\mathfrak{w})+2}\cdots\sigma_k$ has $\iota(\mathfrak{w}') = \iota(\mathfrak{w}) + 1$. Now let us do the case $\sigma_{\iota(\mathfrak{w})+1} = \tau_{p-1}$ (the case τ_{p+1} is similar). Note that $\tau_p\tau_{p-1} = (p-1, p+1)\cdot\tau_p$, where $(p-1, p+1)$ denotes the non-adjacent transposition. Therefore, $\eta_{\iota(\mathfrak{w})+1} = \sigma'\tau_p$, where $\sigma' = \sigma_1\sigma_2\cdots\sigma_{\iota(\mathfrak{w})-1}\cdot(p-1, p+1)$. Since we have assumed that $\eta_{\iota(\mathfrak{w})+1}$ is also inverted, the index-two domain corresponding to $\tau_p\tau_{p-1} = (p{-}1, p{+}1)\cdot\tau_p$ looks like the third hexagon from Item (G-13e). Therefore, σ' is not inverted, and therefore, for any reduced word $\sigma'_1\sigma'_2\cdots\sigma'_{\iota(\mathfrak{w})}$ for σ', we have that $\sigma'_1\sigma'_2\cdots\sigma'_{\iota(\mathfrak{w})}\tau_p$ is a reduced word for $\eta_{\iota(\mathfrak{w})+1}$ (by Item P-5), and hence $\mathfrak{w}' = \sigma'_1\sigma'_2\cdots\sigma'_{\iota(\mathfrak{w})}\tau_p\sigma_{\iota(\mathfrak{w})+2}\cdots\sigma_k$ has $\iota(\mathfrak{w}') = \iota(\mathfrak{w}) + 1$.

(P-7) If σ does not have a reduced word ending in the transposition τ_p, then for any reduced word $\sigma_1\sigma_2\cdots\sigma_k$ of σ, we have that $\sigma_1\sigma_2\cdots\sigma_k\tau_p$ is a reduced word for $\sigma\tau_p$. This follows immediately from Items (P-5) and (P-6).

3.3. **Plausible triples.** Given two partially ordered sets $S_1, \dots, S_m$, the *product partial order* on $S_1 \times \cdots \times S_m$ is given by

$$(s_1, \dots, s_m) \le (s'_1, \dots, s'_m) \iff (s_i \le s'_i \text{ for all } i).$$

We will give $\mathbb{N}^{n-1}$ the product partial order coming from its factors.

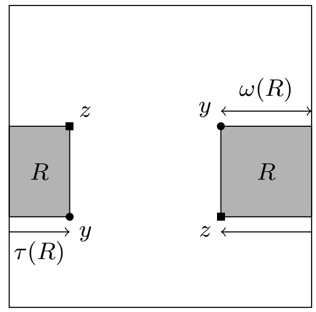


FIGURE 2. The shaded rectangle R is an A-witness.

In the proof of Proposition 3.4 that will be given in Section 3.4, we will filter positive domains according to the vectors $A(D), B(D)$ that capture their multiplicties on the rightnots column and topmost row. In the process, given a triple (a, b, y), with $y \in \mathbb{S}$ and $(a, b) \in \mathbb{N}^{n-1} \times \mathbb{N}^{n-1}$, we will be interested in the set of generators

$$G^{a,b,y} = \{x \in \mathbb{S} \mid \exists D \in \mathscr{D}^+(x, y), A(D) = a, B(D) = b\}.$$

This is an upward closed subset: that is, if $x \in G^{a,b,y}$ and $x \leq x'$, then $x' \in G^{a,b,y}$. Therefore, $G^{a,b,y}$ always contains x^{Id}. Moreover, if $D \in \mathscr{D}^+(x, y)$ with $A(D) \leq a$ and $B(D) \leq b$, then $x \in G^{a,b,y}$, since there exists a (unique) periodic domain $E \in \mathscr{P}^+$ with $(A(E), B(E)) = (a - A(D), b - B(D))$, and therefore, $D * E \in \mathscr{D}^+(x, y)$ satisfies the required condition. That is, $G^{a,b,y}$ has an alternate description

$$G^{a,b,y} = \{x \mid \exists D \in \mathscr{D}^+(x, y), A(D) \leq a, B(D) \leq b\}.$$

In particular, since $c_y \in \mathscr{D}^+(y, y)$, the set $G^{a,b,y}$ contains y, and hence all z with $z \geq y$.

We would like to understand in what cases $G^{a,b,y}$ contains more elements than just those in the interval $[y, x^{\mathrm{Id}}]$. An example is shown in Figure 2, where $z \leq y$ but the rectangle R with $A(R) = a > 0$ and $B(R) = 0$ makes it so that $z \in G^{a,0,y}$.

It turns out that the specific condition we need is *plausibility*, as defined below. Consider triples (a, b, y) with y a generator and $(a, b) \in \mathbb{N}^{n-1} \times \mathbb{N}^{n-1}$. Call such a triple *$A$-plausible* (respectively, *B-plausible*) if there exist z and $R \in \mathscr{R}(z, y)$ with $0 < A(R) \leq a$ (respectively, $0 < B(R) \leq b$); call such rectangles *A-witnesses* (respectively, *B-witnesses*). Assign to any such witness R a pair $(\omega(R), \tau(R)) \in \mathbb{N}^2$, where $\omega(R)$ is number of vertical annuli to the left of $V_{(n)}$, including $V_{(n)}$, (respectively, horizontal annuli below $H_{(n)}$, including $H_{(n)}$) that R intersects, and $\tau(R)$ is the horizontal width (respectively, vertical height) of R. See again Figure 2.

Lemma 3.6. *If (a, b, y) is neither A-plausible nor B-plausible, and $D \in \mathscr{D}^+(x, y)$ with $(A(D), B(D)) = (a, b)$, then $x \geq y$.*

Proof. Look at decompositions $D = E * F$ where $E \in \mathscr{P}^+$ and $F \in \mathscr{D}^+(x, y)$, and consider the one that maximizes the Maslov index of E. Then F does not contain any vertical annulus or any horizontal annulus.

By Lemma 3.5, if $(A(F), B(F)) \neq (0, 0)$, then F has a decomposition $G * R$, with $G \in \mathscr{D}^+(x, z)$ and $R \in \mathscr{R}(z, y)$ with $(A(R), B(R)) \neq (0, 0)$. Then R is a witness, which contradicts the hypothesis. Therefore, we must have $(A(F), B(F)) = (0, 0)$. Then $x \geq y$ due to the domain F, and we are done. ∎

Lemma 3.7. *If (a, b, y) is A-plausible, and $D \in \mathscr{D}^+(x, y)$ with $(A(D), B(D)) = (a, b)$, then there exists an A-witness $R \in \mathscr{R}(z, y)$ and $E \in \mathscr{D}^+(x, z)$ with $E * R = D$. We may choose R to be one that minimizes ω among all A-witnesses. In fact, we may choose R to be the (unique) A-witness R_0 that minimizes the pair (ω, τ), ordered lexicographically, among all A-witnesses.*

Analogous statements hold if (a, b, y) is B-plausible.

Proof. Let us only consider the case for A-plausible. The other case is similar. We prove this by induction on the Maslov index of D. There are three statements in the problem, and for clarity, we write them out. Each statement is weaker than the next.

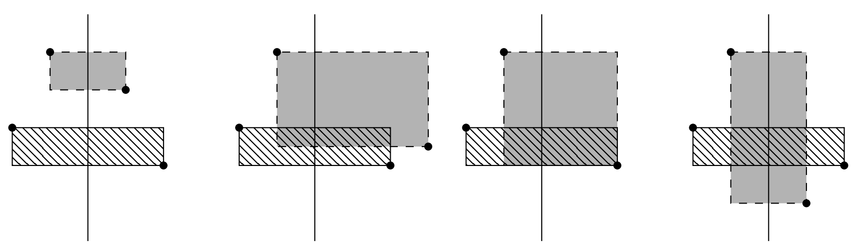
(P_1^n) If (a, b, y) is A-plausible, and $D \in \mathscr{D}^+(x, y)$ with $\mu(D) = n$ and $(A(D), B(D)) = (a, b)$, then there exists an A-witness $R \in \mathscr{R}(z, y)$ and $E \in \mathscr{D}^+(x, z)$ with $E * R = D$.

(P_2^n) If (a, b, y) is A-plausible, and $D \in \mathscr{D}^+(x, y)$ with $\mu(D) = n$ and $(A(D), B(D)) = (a, b)$, then there exists an A-witness $R \in \mathscr{R}(z, y)$ and $E \in \mathscr{D}^+(x, z)$ with $E * R = D$, and R minimizes ω among all A-witnesses.

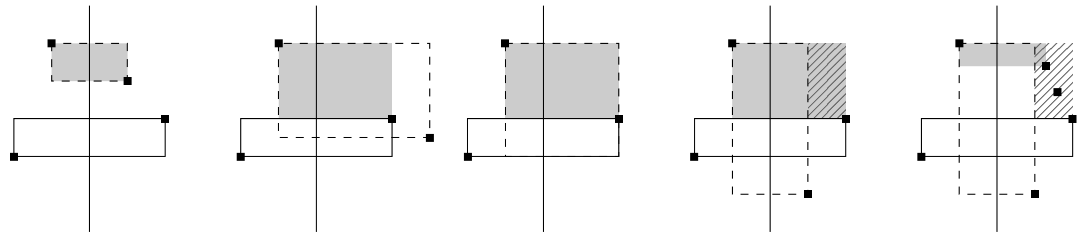
(P_3^n) If (a, b, y) is A-plausible, and $D \in \mathscr{D}^+(x, y)$ with $\mu(D) = n$ and $(A(D), B(D)) = (a, b)$, then there exists an A-witness $R \in \mathscr{R}(z, y)$ and $E \in \mathscr{D}^+(x, z)$ with $E * R = D$, and R minimizes (ω, τ) among all A-witnesses.

The base case for the induction is either vacuous or trivial, depending on whether one starts at $n = 0$ or $n = 1$. We will do induction on n, and at each step, we will first get (P_1^n), then (P_2^n), and then (P_3^n). For this, we will make use of the following implications.

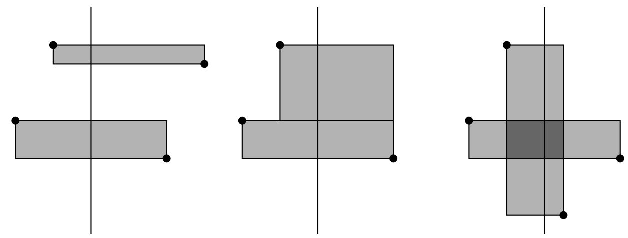
(Ind-1) $(P_1^n) \wedge (P_2^{n-1}) \Rightarrow (P_2^n)$. Consider the decomposition $D = E * R$ as provided by (P_1^n), with $R \in \mathscr{R}(z, y)$. Consider an A-witness S that minimizes ω. If $\omega(R) = \omega(S)$, we are done. Otherwise $\omega(S) < \omega(R)$; therefore, the top-left corner of S lies outside R, and the configuration of S (shaded), R (striped), and y-coordinates (dots) looks like one of the follows (the vertical column $V_{(n)}$ is once again shown as a vertical line).



In each case, the domain $E \in \mathscr{D}^+(x, z)$ produces an A-plausible triple $(z, A(E), B(E))$ and the following A-witness (shaded) has minimum ω, which equals $\omega(S)$. The coordinates of z are shown as black squares. The fourth case has been subdivided into two cases: in the first subcase, there are no z coordinates in the interior of the striped rectangle, while in the second subcase, there are a few; in that subcase, the A-witness uses the leftmost of those extra z-coordinates.

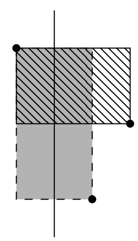


Therefore, by (P_2^{n-1}), we have a decomposition $E = F * T$, with $F \in \mathscr{D}^+(x, w)$ and $T \in \mathscr{R}(w, z)$ with $\omega(T) = \omega(S)$. Therefore, the Maslov index 2 domain $H = T * R \in \mathscr{D}^+(w, y)$ looks like one of the following (the decomposition $T * R$ and the y-coordinates are also shown).

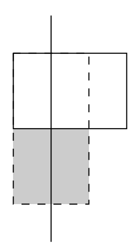


In each case, the other decomposition $H = T' * R'$ satisfies $\omega(R') = \omega(S)$. Therefore, we have a decomposition $D = (F * T') * R'$, with R' an A-witness minimizing ω.

(Ind-2) $(P_2^n) \wedge (P_3^{n-1}) \Rightarrow (P_3^n)$. The proof is similar to (but easier than) the previous proof. Consider the decomposition $D = E * R$ as provided by (P_2^n), with $R \in \mathscr{R}(z, y)$ minimizing ω. Consider an A-witness S that minimizes (ω, τ), ordered lexicographically. We must have $\omega(R) = \omega(S)$. If in addition, $\tau(R) = \tau(S)$, we are done. Otherwise $\tau(S) < \tau(R)$; therefore, the configuration of S (shaded), R (striped), and y-coordinates (dots) looks as follows.



Therefore, the domain $E \in \mathscr{D}^+(x, z)$ is A-plausible and the following A-witness T (shaded) has minimum (ω, τ), which equals $(\omega(S), \tau(S))$.



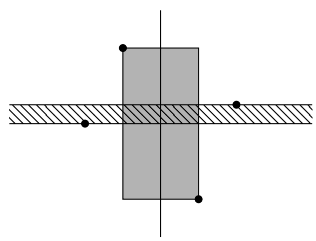
By (P_3^{n-1}), we have a decomposition $E = F * T$ with $F \in \mathscr{D}^+(x,w)$ and $T \in \mathscr{R}(w,z)$. The Maslov index 2-domain $H = T * R \in \mathscr{D}^+(w,y)$ is a hexagon, and the other decomposition $H = T' * R'$ satisfies $(\omega(R'), \tau(R')) = (\omega(S), \tau(S))$. Therefore, we have a decomposition $D = (F * T') * R'$, with R' the unique A-witness minimizing (ω, τ).

(Ind-3) $(P_1^{n-1}) \wedge (P_3^n) \Rightarrow (P_1^{n+1})$. Let D be the given domain with $\mu(D) = n+1$, and consider the A-witness R for D which minimizes (ω, τ). Since R minimizes (ω, τ), none of the y-coordinates can lie in the interior of R.

If D does not contain any horizontal annuli, we are done by Lemma 3.5. Therefore, assume D contains some horizontal annulus H. If H is disjoint from the interior of R, then $D * (-H)$ is a domain with $\mu = n-1$, which is still A-plausible since it still contains the A-witness R. Therefore, by (P_1^{n-1}), it admits a decomposition $E * T$ with $E \in \mathscr{D}^+(x,w), T \in \mathscr{R}(w,y)$ with $A(T) \neq 0$; consequently, D has a decomposition $(E * H) * T$ and we are done.

Therefore, we may assume that D contains some horizontal annulus H that intersects R. Let $H = S * T$ with $S \in \mathscr{R}(y,w), T \in \mathscr{R}(w,y)$ be the unique decomposition of H into rectangles. Exactly one of $A(S)$ and $A(T)$ is non-zero. If $A(T) \neq 0$, we are done, since D then has a decomposition $(D * (-T)) * T$. So we may assume $A(S) \neq 0$ and $A(T) = 0$.

Therefore, the configuration of the A-witness R (shaded), the horizontal annulus H (striped), and the y-coordinates (dots) looks as follows.



Therefore, the domain $D * (-T) \in \mathscr{D}^+(x,w)$ is A-plausible, with R still being the A-witness that minimizes (ω, τ). By (P_3^n), $D * (-T)$ contains the rectangle R, and therefore, D contains R as well. $\square$

3.4. **Proof of acyclicity.** Given a triple (a,b,y), with y a generator and $(a,b) \in \mathbb{N}^{n-1} \times \mathbb{N}^{n-1}$, in Section 3.3 we defined the following set of generators

$$\begin{aligned} G^{a,b,y} &= \{x \in \mathbb{S} \mid \exists D \in \mathscr{D}^+(x,y), A(D) = a, B(D) = b\} \\ &= \{x \in \mathbb{S} \mid \exists D \in \mathscr{D}^+(x,y), A(D) \leq a, B(D) \leq b\}. \end{aligned}$$

This is an upward closed subset that contains y.

Lemma 3.8. *The set $G^{a,b,y}$ has a unique minimum $m^{a,b,y}$, so that $G^{a,b,y}$ is the interval $[m^{a,b,y}, x^{\mathrm{Id}}]$. Furthermore, $m^{a,b,y}$ equals x^{Id} if and only if $a = b = 0$ and $y = x^{\mathrm{Id}}$.*

Proof. We prove this by induction on (a,b), viewed as an element of the poset $\mathbb{N}^{2n-2}$ under the product partial order. For the base case, we have $G^{0,0,y} = [y, x^{\mathrm{Id}}]$, and so it has a unique minimum $m^{0,0,y} := y$.

Now consider the case $(a,b) \neq (0,0)$. For the first part, if (a,b,y) is neither A-plausible nor B-plausible, then it follows from Lemma 3.6 that $m^{a,b,y}$ exists and equals y. On the other hand, if (a,b,y) is A-plausible (respectively, B-plausible), and $R_0 \in \mathscr{R}(z,y)$ is the unique A-witness (respectively B-witness) that minimizes (ω, τ), then it follows from Lemma 3.7 (using induction on (a,b)) that $m^{a,b,y}$ exists and equals $m^{a-A(R_0),b,z}$ (respectively, $m^{a,b-B(R_0),z}$).

For $(a,b) \neq (0,0)$, the second part follows from the first part. Let $D \in \mathscr{D}^+(y,y)$ with $(A(D),B(D)) = (a,b)$, and consider some decomposition of D into rectangles:

$$D = R_1 * R_2 * \cdots * R_n \qquad R_1 \in \mathscr{R}(y = w_0, w_1), R_2 \in \mathscr{R}(w_1, w_2), \ldots, R_n \in \mathscr{R}(w_{n-1}, w_n = y).$$

Clearly, $w_i \in G^{a,b,y}$ for all i because of the domain $R_{i+1} * \cdots * R_n \in \mathscr{D}^+(w_i, y)$. Since $(a,b) \neq (0,0)$, D is non-trivial, and therefore, there is at least one rectangle, and consequently, the set $\{w_0, \ldots, w_n\}$ contains at least two elements. Therefore, $G^{a,b,y}$ is not the one-element set $\{x^{\mathrm{Id}}\}$. $\square$

We are now ready to prove Proposition 3.4.

Proof of Proposition 3.4. The idea of the proof is to construct a sequence of filtrations on the chain complex, and to prove that various associated graded complexes are acyclic.

For domain $D \in CD_*$, let $(A(D),B(D))$ be its filtration grading in the product partial order on $\mathbb{N}^{2n-2}$. It is clear that the differential either preserves (A,B) or lowers it. For $(a,b) \in \mathbb{N}^{n-1} \times \mathbb{N}^{n-1}$, let $CD_*^{a,b}$ be the associated graded complex in filtration grading (a,b). We will prove that $CD_*^{a,b}$ is acyclic if $(a,b) \neq (0,0)$, and $CD_*^{0,0}$ has homology $\mathbb{Z}$ generated by $c_{x^{\mathrm{Id}}}$.

Now put a new filtration grading on $CD_*^{a,b}$ as follows. For any domain $D \in \mathscr{D}^+(x,y)$ with $(A(D),B(D)) = (a,b)$, define its filtration grading to be y, viewed as an element of the poset from Section 3.2. The differential δ on *the associated graded complex* $CD_*^{a,b}$ either preserves y or increases it. Now let $CD_*^{a,b,y}$ be associated graded complex consisting of only those domains that end at y. Now, it is enough to show that $CD_*^{a,b,y}$ is acyclic, unless $a = b = 0$ and $y = x^{\mathrm{Id}}$. When $a = b = 0$ and $y = x^{\mathrm{Id}}$, the homology is clearly $\mathbb{Z}$, generated by the trivial domain $c_{x^{\mathrm{Id}}}$.

The complex $CD_*^{a,b,y}$ is generated by domains $D \in \mathscr{D}^+(x,y)$ with $(A(D),B(D)) = (a,b)$. Note, if there is such a domain, then $x \in G^{a,b,y}$ by definition, and conversely, for any $x \in G^{a,b,y}$, there is a unique such positive domain D. Note that $G^{a,b,y} = [m^{a,b,y}, x^{\mathrm{Id}}]$ by Lemma 3.8. Therefore, the complex $CD_*^{a,b,y}$ is isomorphic to the following complex (which resembles the grid complex from Section 2.2). It is generated by the elements of $[m^{a,b,y}, x^{\mathrm{Id}}]$, and the differential on a generator is given by

$$\delta(x) = \sum_{\substack{m^{a,b,y} \leq z < x \\ R \in \mathscr{R}(x,z) \\ A(R)=B(R)=0}} s(R) z.$$

If in the above formula we have $m^{a,b,y} = x^{\sigma}$, $x = x^{\theta}$ and $z = x^{\eta}$, for some permutations σ, θ, η, then the condition

$$m^{a,b,y} \leq z < x \text{ and } \exists R \in \mathscr{R}(x,z) \text{ with } A(R) = B(R) = 0$$

is equivalent to

$$\sigma \geq \eta > \theta \text{ and } |\eta| = |\theta| + 1 \text{ and } R = (-D_{\theta}) * D_{\eta}.$$

Therefore, the complex $CD_*^{a,b,y}$ is isomorphic to the following. It is generated by permutations in $[\mathrm{Id}, \sigma]$, and the differential on a generator is given by

$$\delta(\theta) = \sum_{\substack{\sigma \geq \eta > \theta \\ |\eta| = |\theta|+1}} s((-D_{\theta}) * D_{\eta}) \eta.$$

Now fix some reduced word $\sigma_1 \sigma_2 \cdots \sigma_k$ for σ. If $(a,b) \neq (0,0)$, then $m^{a,b,y} \neq x^{\mathrm{Id}}$ (once again, using Lemma 3.8), and hence $k > 0$. Let σ_k be the transposition $\tau_p = (p, p+1)$. Define a grading on permutations by declaring its value on θ to be $|\theta| - 1$ if θ has a reduced word ending in τ_p, and

$|\theta|$ otherwise. Since the differential increases the length $|\cdot|$ by one, this defines a filtration grading on above complex.

We claim that the associated graded complex is a direct sum of two-generator acyclic complexes, and hence is acyclic. If η and θ are in the same filtration grading and η appears in $\delta(\theta)$, then the filtration grading must be $|\eta|-1=|\theta|$. Therefore, η has a reduced word, say $\mathfrak{w}$ of length ℓ, ending in τ_p; since $\theta<\eta$ with $|\theta|=|\eta|-1$, θ has a reduced word $\mathfrak{w}'$ of length $\ell-1$ which is a sub-word of $\mathfrak{w}$. But since θ does not have any reduced word ending in τ_p, $\mathfrak{w}'$ must be obtained from $\mathfrak{w}$ by deleting τ_p from the end. That is, $\eta=\theta\tau_p$, and there is a (width-one) rectangle from x^θ to x^η.

On the other hand, if θ does not have a reduced word ending in τ_p, and $\theta\leq\sigma$, consider some reduced word $\mathfrak{w}$ for θ that is a sub-word of $\sigma_1\sigma_2\cdots\sigma_k$, and hence a sub-word of $\sigma_1\sigma_2\cdots\sigma_{k-1}$. By Item (P-7), $\mathfrak{w}\tau_p$ is a reduced word for $\theta\tau_p$; since it is a sub-word of $\sigma_1\sigma_2\cdots\sigma_k$, $\theta\tau_p\leq\sigma$. Similarly, if θ has a reduced word ending in τ_p, by Item (P-4), removing τ_p from the end produces a reduced word for $\theta\tau_p$, and hence $\theta\tau_p<\theta$; so if $\theta\leq\sigma$, $\theta\tau_p\leq\sigma$ as well. In either case, if $\theta\leq\sigma$, $\theta\tau_p\leq\sigma$. Therefore, θ and $\theta\tau_p$ span an acyclic summand of the associated graded complex. Therefore, the associated graded complex is acyclic, and this concludes the proof. □

4. The complex of positive domains with partitions

4.1. **Ordered partitions.** For $N\geq 0$, denote by $\mathrm{Part}(N)$ the set of ordered partitions of N as sums of positive integers. Thus, an element $\lambda\in\mathrm{Part}(N)$ is of the form

$$\lambda=(\lambda_1,\dots,\lambda_m),\ m\geq 0,\ \sum\lambda_j=N.$$

We denote by $\ell(\lambda)=m$ the *length* of the partition. The quantity $N-\ell(\lambda)$ is called the *co-length.*

The number of ordered partitions of N is 2^{N-1} for $N\geq 1$, and 1 for $N=0$. Indeed, to each $\lambda\in\mathrm{Part}(N)$ we can uniquely associate an $(N-1)$-tuple

$$\epsilon(\lambda)=(\epsilon_1(\lambda),\dots,\epsilon_{N-1}(\lambda))\in\{0,1\}^{N-1} \tag{4.1}$$

as follows: Consider N objects (represented by bullets) in a row, with the first λ_1 in the first partition class, the next λ_2 in the second class, etc. We place a 0 between objects in the same class, and a 1 between objects in a different class. For example, the partition $2+3+1$ corresponds to the string 01001:

$$(\bullet\ 0\ \bullet)\ 1\ (\bullet\ 0\ \ \bullet\ \ 0\ \bullet)\ 1\ (\bullet)$$

For $\lambda=(\lambda_1,\dots,\lambda_m)\in\mathrm{Part}(N)$ and $\lambda'=(\lambda'_1,\dots,\lambda'_{m'})\in\mathrm{Part}(N')$, we define their *concatenation*

$$\lambda*\lambda'=(\lambda_1,\dots,\lambda_m,\lambda'_1,\dots,\lambda'_{m'})\in\mathrm{Part}(N+N'). \tag{4.2}$$

For $\lambda,\lambda'\in\mathrm{Part}(N)$, we write $\lambda\geq\lambda'$ if λ is a *refinement* of $\lambda'=(\lambda'_1,\dots,\lambda'_m)$, that is, if there are partitions of each λ'_j such that their concatenation gives λ. We have

$$\lambda\geq\lambda'\iff\epsilon(\lambda)\geq\epsilon(\lambda'),$$

where on the right hand side we used the product partial order on $\{0,1\}^{N-1}$.

Definition 4.1. If $\lambda,\lambda'\in\mathrm{Part}(N)$ are such that $\lambda\geq\lambda'$, we say that λ is *finer* than λ', and λ' is *coarser* than λ. If $\lambda\geq\lambda'$ and $\ell(\lambda')=\ell(\lambda)-1$, we say that λ' is an *elementary coarsening* of λ. We denote by $\mathrm{EC}(\lambda)$ the set of elementary coarsenings of λ.

If $\lambda' = (\lambda'_1, \dots, \lambda'_m) \in \mathrm{EC}(\lambda)$, then there is an index k and $\lambda_k^1, \lambda_k^2 \geq 1$ such that

$$\lambda = (\lambda'_1, \dots, \lambda'_{k-1}, \lambda_k^1, \lambda_k^2, \lambda'_{k+1}, \dots, \lambda'_m), \quad \lambda_k^1 + \lambda_k^2 = \lambda'_k.$$

We define the *sign* of the elementary coarsening to be

$$s(\lambda, \lambda') = (-1)^{m+1-k} = (-1)^{\ell(\lambda)-k}.$$

Alternatively, note that there is a unique $i \in \{1, \dots, N-1\}$ such that $\epsilon_i(\lambda) = 1$ and $\epsilon_i(\lambda') = 0$; and for all $j \neq i$, we have $\epsilon_j(\lambda) = \epsilon_j(\lambda')$. We have

$$s(\lambda, \lambda') = (-1)^{1+\epsilon_{j+1}(\lambda)+\cdots+\epsilon_{N-1}(\lambda)}.$$

Definition 4.2. If $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathrm{Part}(N)$, a *unit enlargement* of λ is a partition $\lambda' \in \mathrm{Part}(N+1)$ of the form

$$\lambda' = (\lambda_1, \dots, \lambda_{k-1}, 1, \lambda_k, \dots, \lambda_m)$$

for some $k \in \{1, \dots, m\}$. The sign of the unit enlargement is defined to be

$$s(\lambda, \lambda') = (-1)^{m+1-k} = (-1)^{\ell(\lambda')-k}.$$

The set of unit enlargements of λ is denoted $\mathrm{UE}(\lambda)$.

Definition 4.3. If $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathrm{Part}(N)$, the *initial reduction* of λ is the partition

$$\lambda^- := (\lambda_2, \dots, \lambda_m) \in \mathrm{Part}(N - \lambda_1).$$

The *final reduction* of λ is the partition

$$\lambda^+ := (\lambda_1, \dots, \lambda_{m-1}) \in \mathrm{Part}(N - \lambda_m).$$

The reductions are not well-defined when $N = 0$ (and λ is the empty partition). We define the sets

$$\mathrm{IR}(\lambda) := \begin{cases} \{\lambda^-\} & \text{if } N > 0, \\ \varnothing & \text{if } N = 0; \end{cases} \quad \text{and} \quad \mathrm{FR}(\lambda) := \begin{cases} \{\lambda^+\} & \text{if } N > 0, \\ \varnothing & \text{if } N = 0. \end{cases}$$

Let us now extend the order on partitions to the case where their sum may be different.

Definition 4.4. Suppose $\lambda \in \mathrm{Part}(N)$ and $\lambda' \in \mathrm{Part}(N')$ for $N' \geq N$. We write $\lambda \geq \lambda'$ if there is a partition $\eta \in \mathrm{Part}(N')$ such that $\eta \geq \lambda'$ and η is obtained from λ by $N' - N$ unit enlargements.

For example, we have $\lambda = (2, 1) > \eta = (1, 2, 1) > \lambda' = (1, 3)$.

Remark 4.5. If $\lambda \in \mathrm{Part}(N)$ and $\lambda' \in \mathrm{Part}(N')$ satisfy $\lambda \geq \lambda'$, we must have the inequality

$$N - \ell(\lambda) \leq N' - \ell(\lambda'). \tag{4.3}$$

In other words, the co-length of partitions decreases with respect increasing partitons.

4.2. **The new complex.** We now define a slightly more complicated complex, $CDP_* = CDP_*(\mathbb{G})$, associated to a grid diagram $\mathbb{G}$ and a sign assignment s. We will call it the *complex of positive domains with partitions.* As an Abelian group, CDP_* is feely generated by triples $(D, \vec{N}, \vec{\lambda})$ with

$$D \in \mathscr{D}^+(x, y), \vec{N} = (N_2, \dots, N_n) \in \mathbb{N}^{n-1}, \vec{\lambda} = (\lambda_2, \dots, \lambda_n), \lambda_j = (\lambda_{j,1}, \dots, \lambda_{j,m_j}) \in \mathrm{Part}(N_j).$$

The intuition is that a triple of this type will be associated to a configuration consisting of the following:

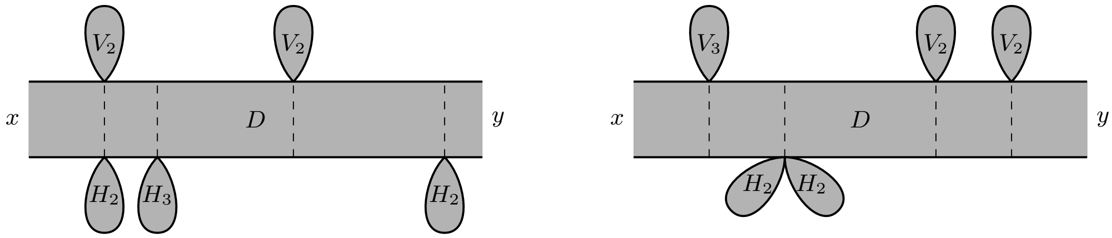


FIGURE 3. Two pseudo-holomorphic strips with bubbles. We do not distinguish between bubbles with domain H_j or V_j, nor do we record the relative heights of bubbles that pass through different O_j markings; so both pictures correspond to the triple $(D, (N_2 = 4, N_3 = 1), (\lambda_2 = (2, 1, 1), \lambda_3 = (1)))$.

- A pseudo-holomorphic strip in $\mathrm{Sym}^n(T^2)$ with domain D. This consists of a map
$$u\colon (\mathbb{R} \times [0, 1], \mathbb{R} \times 0, \mathbb{R} \times 1) \to (\mathrm{Sym}^n(T^2), \alpha_1 \times \cdots \times \alpha_n, \beta_1 \times \cdots \times \beta_n),$$
up to translation in the first $\mathbb{R}$ factor, which is J-holomorphic with respect to certain (1-parameter family of) almost complex structures on $\mathrm{Sym}^n(T^2)$ with $\lim_{s\to-\infty} u(s, t) = x$, $\lim_{s\to+\infty} u(s, t) = y$ and whose underlying 2-chain on T^2 is D.
- Several disk bubbles attached to the strip. There are N_j bubbles going through the marking O_j, and each can have as domain either the row H_j or the column V_j. If the domain of a bubble is H_j (respectively, V_j), it is attached to the strip at some point $(s, 0)$ (respectively, $(s, 1)$), and s is called the *height* of the bubble (which is only well-defined up to an overall translation). We do not distinguish between bubbles with domain H_j and bubbles with domain V_j, and just record the total number N_j of such bubbles.
- Partitioning of the bubbles according to their heights. These bubbles are allowed to occur at the same height, and $\lambda_j = (\lambda_{j,1}, \ldots, \lambda_{j,m_j})$ is the partitioning of these N_j bubbles so that the bubbles in the same partition class are attached to the strip at the same height. Further, the ordering of the partition classes corresponds to the ordering of the heights—the first $\lambda_{j,1}$ bubbles have the smallest height (closest to x), and the last λ_{j,m_j} bubbles have the largest height (closest to y). For $j \neq j'$, we do not record the relative heights of the N_j bubbles (with domain H_j or V_j) and the $N_{j'}$ bubbles (with domain $H_{j'}$ or $V_{j'}$).

See Figure 3.

For future reference, set
$$|\vec{N}| = N_2 + \cdots + N_n$$
$$|\vec{\lambda}| = \ell(\lambda_2) + \cdots + \ell(\lambda_n).$$
and let $e_2, \ldots, e_n$ denote the standard unit vectors in $\mathbb{N}^{n-1}$.

The grading on CDP_* is given by
$$\mathrm{gr}(D, \vec{N}, \vec{\lambda}) = \mu(D) + |\vec{\lambda}| \tag{4.4}$$
The differential $\delta\colon CDP_k \to CDP_{k-1}$ has four kinds of terms:

- **Type I** terms, given by taking out a rectangle from the domain, just as in the complex CD_*;
- **Type II** terms, given by boundary degenerations, i.e., taking out a row H_j or a column V_j from the domain D, and at the same time increasing N_j by one, and changing λ_j by a unit enlargement;

- **Type III** terms, given by an elementary coarsening of one of the partitions λ_j. This corresponds to two groups of bubbles at two different heights reaching the same height.
- **Type IV** terms, given by taking the initial or final reduction of one of the partitions λ_j. This corresponds to removing a boundary degeneration, in the limit as its height goes to $-\infty$ (for initial reductions) or $+\infty$ (for final reductions).

Precisely, we can write

$$\delta = \delta^{\mathrm{I}} + \delta^{\mathrm{II}} + \delta^{\mathrm{III}} + \delta^{\mathrm{IV}} \tag{4.5}$$

such that, for $D \in \mathscr{D}^+(x,y)$, we have

$$\delta^{\mathrm{I}}(D,\vec{N},\vec{\lambda}) = \sum_{\substack{(R,E)\in\mathscr{R}(x,w)\times\mathscr{D}^+(w,y)\\ R*E=D}} s(R)(E,\vec{N},\vec{\lambda}) + (-1)^{\mu(D)} \sum_{\substack{(E,R)\in\mathscr{D}^+(x,w)\times\mathscr{R}(w,y)\\ E*R=D}} s(R)(E,\vec{N},\vec{\lambda}). \tag{4.6}$$

$$\begin{aligned}\delta^{\mathrm{II}}(D,\vec{N},\vec{\lambda}) = {}& (-1)^{\mu(D)+1}\sum_{j=1}^{n}(-1)^{|(\lambda_{j+1},\dots,\lambda_n)|}\sum_{\substack{E\in\mathscr{D}^+(x,y)\\ E*H_j=D}}\sum_{\lambda'_j\in\mathrm{UE}(\lambda_j)} s(\lambda_j,\lambda'_j)(E,\vec{N}+\vec{e}_j,\vec{\lambda}')\\ &+(-1)^{\mu(D)}\sum_{j=1}^{n}(-1)^{|(\lambda_{j+1},\dots,\lambda_n)|}\sum_{\substack{E\in\mathscr{D}^+(x,y)\\ E*V_j=D}}\sum_{\lambda'_j\in\mathrm{UE}(\lambda_j)} s(\lambda_j,\lambda'_j)(E,\vec{N}+\vec{e}_j,\vec{\lambda}').\end{aligned} \tag{4.7}$$

$$\delta^{\mathrm{III}}(D,\vec{N},\vec{\lambda}) = (-1)^{\mu(D)+1}\sum_{j=1}^{n}(-1)^{|(\lambda_{j+1},\dots,\lambda_n)|}\sum_{\lambda'_j\in\mathrm{EC}(\lambda_j)} s(\lambda_j,\lambda'_j)(D,\vec{N},\vec{\lambda}'). \tag{4.8}$$

$$\begin{aligned}\delta^{\mathrm{IV}}(D,\vec{N},\vec{\lambda}) = {}& (-1)^{\mu(D)+1}\sum_{j=1}^{n}(-1)^{|(\lambda_j,\dots,\lambda_n)|}\sum_{\lambda'_j\in\mathrm{IR}(\lambda_j)} (D,\vec{N}-\lambda_{j,1}\vec{e}_j,\vec{\lambda}')\\ &+(-1)^{\mu(D)+1}\sum_{j=1}^{n}(-1)^{|(\lambda_{j+1},\dots,\lambda_n)|}\sum_{\lambda'_j\in\mathrm{FR}(\lambda_j)} (D,\vec{N}-\lambda_{j,m_j}\vec{e}_j,\vec{\lambda}').\end{aligned} \tag{4.9}$$

In the expressions (4.7), (4.8) and (4.9) we used the notation

$$\vec{\lambda}' = (\lambda_1,\dots,\lambda_{j-1},\lambda'_j,\lambda_{j+1},\dots,\lambda_n).$$

Lemma 4.6. *The complex CDP_* defined above is indeed a chain complex, i.e., $\delta^2=0$.*

Proof. We claim that each of δ^{I}, δ^{II} and δ^{III} squares to zero, and that any two of these differentials anti-commute with each other. We also claim that δ^{IV} anti-commutes with δ^{I} and δ^{II}, and that we have

$$(\delta^{\mathrm{IV}})^2 + \delta^{\mathrm{III}}\delta^{\mathrm{IV}} + \delta^{\mathrm{IV}}\delta^{\mathrm{III}} = 0. \tag{4.10}$$

Together, these claims will show that $\delta^2 = 0$.

Let us start with the differential δ^{I}. This gave the complex CD_*, and fact that $(\delta^{\mathrm{I}})^2=0$ was established in Lemma 3.2.

To see that $(\delta^{\mathrm{II}})^2=0$, note that in the expression $(\delta^{\mathrm{II}})^2(D,\vec{N},\vec{\lambda})$ we encounter terms of two kinds. Some are of the form

$$\pm(E,\vec{N}+\vec{e}_i+\vec{e}_j,\vec{\lambda}''),\quad i>j \tag{4.11}$$

such that E is obtained from D by deleting a (vertical or horizontal) annulus going through O_i and another annulus through O_j. Also, $\vec{\lambda}'' = (\lambda''_1, \dots, \lambda''_n)$ is obtained from $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$ by doing unit enlargements to λ_i and λ_j. The terms of the form (4.11) come in pairs, corresponding to the order in which we delete the two annuli (and do the respective unit enlargements). The presence of the sign $(-1)^{|(\lambda_{j+1},\dots,\lambda_n)|}$ guarantees that these terms cancel in pairs.

Second, we also have terms of the form

$$\pm(E, \vec{N} + 2\vec{e}_j, \vec{\lambda}'')$$

where E is obtained from D by deleting two annuli through the same O_j, and $\vec{\lambda}''$ is obtained from $\vec{\lambda}$ by doing two unit enlargements to the same partition λ_j. Again, these terms cancel in pairs, due to presence of the signs $s(\lambda_j, \lambda'_j)$ and $s(\lambda'_j, \lambda''_j)$, where λ'_j is the intermediate partition. This completes the proof that $(\delta^{\mathrm{II}})^2 = 0$.

The proof that $(\delta^{\mathrm{III}})^2 = 0$ is similar, with elementary coarsenings instead of unit enlargements. Once again, the signs $(-1)^{|(\lambda_{j+1},\dots,\lambda_n)|}$ and $s(\lambda_j, \lambda'_j)$ ensure that the resulting terms cancel out in pairs.

The same kind of argument can be used to show that

$$(\delta^{\mathrm{II}}\delta^{\mathrm{III}} + \delta^{\mathrm{III}}\delta^{\mathrm{II}})(D, \vec{N}, \vec{\lambda}) = 0.$$

Next, let us check that

$$(\delta^{\mathrm{I}}\delta^{\mathrm{II}} + \delta^{\mathrm{II}}\delta^{\mathrm{I}})(D, \vec{N}, \vec{\lambda}) = 0.$$

Here we obtain terms of the form $\pm(E, \vec{N} + \vec{e}_j, \vec{\lambda}')$, where E is obtained from D by deleting a rectangle R and an annulus H_j or V_j, and $\vec{\lambda}'$ is obtained from $\vec{\lambda}$ by doing a unit enlargement to $\vec{\lambda}_j$. The terms come in pairs, corresponding to which of the operations δ^{I} and δ^{II} we do first. To see that they cancel out, observe that they get the same sign contributions from the factor $(-1)^{\mu(D)}$ in (4.6); the same goes for the factors $(-1)^{|(\lambda_{j+1},\dots,\lambda_n)|}$ and $s(\lambda_j, \lambda'_j)$ in (4.7). However, the contributions due to the factor $(-1)^{\mu(D)}$ in (4.7) differ: for one term we get $(-1)^{\mu(D)}$, and for the other $(-1)^{\mu(E)}$, where $\mu(D) = \mu(E) + 1$.

The proofs that

$$(\delta^{\mathrm{I}}\delta^{\mathrm{III}} + \delta^{\mathrm{III}}\delta^{\mathrm{I}})(D, \vec{N}, \vec{\lambda}) = 0,$$
$$(\delta^{\mathrm{I}}\delta^{\mathrm{IV}} + \delta^{\mathrm{IV}}\delta^{\mathrm{I}})(D, \vec{N}, \vec{\lambda}) = 0$$

are similar. The cancellations are due to the signs $(-1)^{\mu(D)}$ in (4.8) and (4.9).

Next, we check that

$$(\delta^{\mathrm{II}}\delta^{\mathrm{IV}} + \delta^{\mathrm{IV}}\delta^{\mathrm{II}})(D, \vec{N}, \vec{\lambda}) = 0.$$

On the left hand side we obtain terms of the form $\pm(E, \vec{N} + \vec{e}_j - \lambda_{i,1}\vec{e}_i, \vec{\lambda}'')$ (from a unit enlargement combined with an initial reduction, in either order) and $\pm(E, \vec{N} + \vec{e}_j - \lambda_{i,m_i}\vec{e}_i, \vec{\lambda}'')$ (from a unit enlargement combined with a final reduction, in either order). These terms cancel in pairs as follows:

- When $i \neq j$, the term from a unit enlargement followed by a reduction cancels with the one where the operations are done in the opposite order. The signs of the terms differ due to the presence of the $(-1)^{|(\lambda_{j+1},\dots,\lambda_n)|}$ in (4.7) and (4.9);
- When $i = j$, the term from a unit enlargement in position k, which is not the first ($k \geq 2$), followed by an initial reduction, cancels with the one from the initial reduction followed by a unit

enlargement in position $k-1$. This is because of the extra sign $(-1)^{\ell(\lambda_j)}$ in the initial reduction term in (4.9);

- When $i = j$, the term from a unit enlargement in position k, which is not the last $(k < m_j)$, followed by a final reduction, cancels with the one from the final reduction followed by the same unit enlargement in position k. This is because of the sign $s(\lambda_j, \lambda'_j)$ in (4.7);
- When $i = j$, the term from a unit enlargement in the first position, followed by an initial reduction, cancels with the one from a unit enlargement in the last position, followed by a final reduction. Indeed, the latter term is $(D, \vec{N}, \vec{\lambda})$ and the former term is $(-1)^{m_j}(-1)^{m_j+1}(D, \vec{N}, \vec{\lambda}) = -(D, \vec{N}, \vec{\lambda})$.

Finally, we prove Equation (4.10). In the expression

$$((\delta^{\mathrm{IV}})^2 + \delta^{\mathrm{III}}\delta^{\mathrm{IV}} + \delta^{\mathrm{IV}}\delta^{\mathrm{III}})(D, \vec{N}, \vec{\lambda})$$

we encounter terms of the form $(D, \vec{N}, \vec{\lambda}'')$, where $\vec{\lambda}''$ is obtained from $\vec{\lambda}$ either by a combination of an elementary coarsening and a reduction, or by two reductions. Most of the time, these terms cancel each other in pairs corresponding to reversing the order of the two operations. There are, however, two special cases:

- The term obtained by doing an elementary coarsening by combining the first two pieces of the partition λ_j, followed by an initial reduction of that partition, cancels with the term obtained by doing two initial reductions of λ_j;
- Similarly, the term obtained by doing an elementary coarsening by combining the last two pieces of the partition λ_j, followed by a final reduction of that partition, cancels with the term obtained by doing two final reductions of λ_j.

Checking that the signs of the paired terms differ is a straightforward exercise. □

Recall from Proposition 3.4 that the simpler complex CD_* has homology generated by the constant domain $c_{x^{\mathrm{Id}}}$, for the generator x^{Id}. In fact, the span $\langle c_{x^{\mathrm{Id}}} \rangle \cong \mathbb{Z}$ is a subcomplex of CD_*, and its quotient complex is acyclic. We will now establish a similar result for CDP_*.

Definition 4.7. We denote by $CDP^\dagger_* \subset CDP_*$ the subcomplex generated by triples $(c_{x^{\mathrm{Id}}}, \vec{N}, \vec{\lambda})$ with $\vec{N}$ made only of 0's and 1's. We let CDP'_* be the quotient complex $CDP_*/CDP^\dagger_*$.

Proposition 4.8. (*a*) *The complex CDP'_* is acyclic, and therefore the inclusion of $CDP^\dagger_*$ in CDP_* is a quasi-isomorphism.*

(*b*) *For a grid diagram $\mathbb{G}$ of size n, the homology of $CDP^\dagger_*$ (and hence also of CDP_*) is isomorphic to $\mathbb{Z}^{2^{n-1}}$. Its rank in degree k is $\binom{n}{k}$.*

Proof. (*a*) As in the proof of Proposition 3.4, we filter the complex CDP'_* by the quantity

$$(A(D), B(D)) = (a, b) \in \mathbb{N}^{2n-2}$$

capturing the multiplicities of the domain D in the rightmost column and the topmost row. This is a bounded below increasing filtration, and in the associated graded, the differential has no more Type II terms. Then, note that the quantity $|\vec{N}|$ is kept constant by Type I and III terms, and decreased by Type IV terms. Thus, we can filter the associated graded complex by $|\vec{N}|$ (also a bounded below increasing filtration), and in the new associated graded, Type IV terms also disappear. Next, again following the proof of Proposition 3.4, we filter with respect to the endpoint y of the domain D. The resulting associated graded complex breaks as a direct sum of finite dimensional complexes

$$CDP^{a,b,y,\vec{N}}_*$$

generated by triples $(D, \vec{N}, \vec{\lambda})$ with $D \in \mathscr{D}^+(x, y)$ such that $(A(D), B(D)) = (a, b)$. It suffices to show that all of these complexes are acyclic.

When $y \neq x^{\mathrm{Id}}$ or $(a, b) \neq (0, 0)$, we filter $CDP_*^{a,b,y,\vec{N}}$ with respect to the quantity $|\ell(\vec{\lambda})|$, and get rid of the Type III terms in the differential. We are left with only Type I terms. The resulting associated graded is a direct sum of complexes of the form $CD_*^{a,b,y}$, which were shown to be acyclic in the proof of Proposition 3.4. We deduce that $CDP_*^{a,b,y,\vec{N}}$ is acyclic.

So we are only left with case when $y = x^{\mathrm{Id}}$ and $(a, b) = (0, 0)$. Then the domain D has to be the constant domain $c_{x^{\mathrm{Id}}}$, and our complex $CDP_*^{0,0,x^{\mathrm{Id}},\vec{N}}$ has only Type III terms in the differential. Here, $\vec{N} = (N_2, \dots, N_n) \in \mathbb{N}^n$ and, because of how we defined CDP'_*, we only consider the case when at least one N_i is ≥ 2. We find that $CDP_*^{0,0,x^{\mathrm{Id}},\vec{N}}$ is the tensor product of complexes $CDP_*(c_{x^{\mathrm{Id}}}, N_j)$, for $j = 2, \dots, n$, where $CDP_*(c_{x^{\mathrm{Id}}}, N_j)$ is generated by all the partitions of N_j. By the Künneth formula, it suffices to show that $CDP_*(c_{x^{\mathrm{Id}}}, N_j)$ is acyclic when $N_j \geq 2$.

Let us represent the partitions of N_j by sequences $(\epsilon_1, \dots, \epsilon_{N_j - 1})$ as in (4.1). We see that $CDP_*(c_{x^{\mathrm{Id}}}, N_j)$ is a hypercube complex, with the differential decreasing one of the ϵ_k by 1. In fact, we can describe $CDP_*(c_{x^{\mathrm{Id}}}, N_j)$ as the tensor product of $N_j - 1$ complexes of the form $\mathbb{Z} \xrightarrow{\cong} \mathbb{Z}$, which are acyclic. Thus, $CDP_*(c_{x^{\mathrm{Id}}}, N_j)$ is acyclic for $N_j \geq 2$, and the conclusion follows.

(b) Note that the differential on $CDP_*^\dagger$ only has terms of Type IV, corresponding to initial or final reductions. We can identify the generators of $CDP_*^\dagger$ with sequences $\vec{N} = (N_2, \dots, N_n) \in \{0, 1\}^{n-1}$. The terms in the differential come in pairs, corresponding to an initial and final reduction that do the same thing: change a value of N_j from 1 to 0. The paired terms come with opposite signs, because $\ell(\lambda_j) = 1$. We get that $CDP_*^\dagger$ is the tensor product of $(n-1)$ copies of the complex $\mathbb{Z} \xrightarrow{0} \mathbb{Z}$. The calculation of the homology of $CDP_*^\dagger$ now follows from the Künneth formula. ☐

5. $\langle n \rangle$-MANIFOLDS

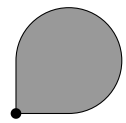
5.1. **Definitions and examples.** We recall the definition of an $\langle n \rangle$-manifold, following Jänich [17]; see also [23] and [26, Section 3.1]. We will borrow the terminology from [24, Definiton 3.2].

We say that a map from a subset $S \subset \mathbb{R}^k$ to $\mathbb{R}^n$ is *smooth* if it is the restriction of a smooth map defined on an open set containing S. In particular, this allows us to define diffeomorphisms between open subsets of $\mathbb{R}_+^k$. Then, following Cerf [10] and Douady [14], we define a k-dimensional *manifold with corners* to be a topological space X along with a maximal atlas, where an atlas is a collection of charts (U, ϕ), where $U \subseteq X$ is open and ϕ is a homeomorphism from U to an open subset of $\mathbb{R}_+^k$, such that the sets U cover X, and, for any two charts (U, ϕ) and (V, ψ), the map

$$\phi \circ \psi^{-1} \colon \psi(U \cap V) \to \phi(U \cap V)$$

is a diffeomorphism. For $x \in X$, let $c(x)$ denote the number of coordinates in $\phi(x)$ which are zero, for some (and hence any) chart (U, ϕ) with $x \in U$. The codimension-i boundary of X is the subspace $\{x \in X \mid c(x) = i\}$; the usual boundary ∂X is the closure of the codimension-1 boundary. A *facet* is the closure of a connected component of the codimension-1 boundary of X. A *multifacet* of X is a (possibly empty) union of disjoint facets of X.

A k-dimensional *multifaceted manifold* is a k-dimensional manifold with corners X such that every $x \in X$ belongs to exactly $c(x)$ facets of X. For example, a simple polytope in $\mathbb{R}^m$ is a multifaceted manifold. By contrast, the following "teardrop" manifold with corners



is not a multifaceted manifold, because the codimension-2 corner belongs to a single facet.

A k-dimensional $\langle n\rangle$*-manifold* X is a k-dimensional multifaceted manifold, together with an ordered n-tuple $(\partial_1 X,\dots,\partial_n X)$ of multifacets of X such that

- $\bigcup_i \partial_i X = \partial X$ and
- $\partial_i X \cap \partial_j X$ is a multifacet of both $\partial_i X$ and $\partial_j X$ for all $i \neq j$.

For a subset $I \subset \{1,\dots,n\}$, we write

$$\partial_I X \coloneqq \bigcap_{i\in I} \partial_i X, \qquad \mathring{\partial}_I X \coloneqq \partial_I X \setminus \bigcup_{I \subsetneq J} \mathring{\partial}_J X. \tag{5.1}$$

The subsets $\mathring{\partial}_I X$ are called the *strata* of X, and $\partial_I X$ are the *closed strata*.

Example 5.1. The n-dimensional hypercube $X = [0,1]^n$ is an $\langle n\rangle$-manifold, with $\partial_i X$ being the union of the two facets given by setting the i^{th} coordinate to either 0 or 1.

Example 5.2. Consider the $(n-1)$-dimensional permutohedron Π_n, defined as the convex hull of all points in $\mathbb{R}^n$ whose coordinates are a permutation of $(1,2,3,\dots,n)$. This is an $\langle n-1\rangle$-manifold, with the facets being the convex hulls of points for permutations that preserve a given partition of $\{1,2,3,\dots,n\}$ into two subsets A and B. The boundary $\partial_i \Pi_n$ consists of those facets for which $|A| = i$ and $|B| = n-i$. We refer to [55, Example 0.10], [5, Section 2] or [24, Section 3.3] for more details.

5.2. **Neat embeddings and smoothings.** Let

$$\mathbb{E}(n,N) = \mathbb{R}^n_+ \times \mathbb{R}^N$$

for some $N, n \geq 0$. We will describe a class of embeddings of $\langle n\rangle$-manifolds into $\mathbb{E}(n,N)$, called *neat*. Neat embeddings of $\langle n\rangle$-manifolds were defined by Laures in [23] and used by Lipshitz and Sarkar in [26] to construct a Khovanov stable homotopy type. Our definition here will be slightly different, in that we require more than the intersections of strata with the boundaries of $\mathbb{E}(n,N)$ being perpendicular; we ask that that the strata contain small product neighborhoods of a special form near the boundaries.

Let X be a $\langle n\rangle$-manifold. Let $t_1,\dots,t_n$ be the coordinates on $\mathbb{E}(n,N)$ corresponding to the $\mathbb{R}_+$ factors. We view $\mathbb{E}(n,N)$ as a $\langle n\rangle$-manifold, with $\partial_I \mathbb{E}(n,N)$ being given by $t_i = 0$ for $i \in I$. We also let $\nu_\epsilon(\partial_I \mathbb{E}(n,N))$ be an ϵ-neighborhood of $\partial_I \mathbb{E}(n,N)$, given by $t_i \in [0,\epsilon)$ for $i \in I$. Finally, we let

$$\pi_I \colon \mathbb{E}(n,N) \to \partial_I \mathbb{E}(n,N)$$

be the orthogonal projection.

Definition 5.3. A smooth embedding of the $\langle n\rangle$-manifold X into $\mathbb{E}(n,N)$ is called *neat* if

(1) It respects the strata, i.e., for every i, we have $\partial_i X = X \cap \partial_i \mathbb{E}(n,N)$.
(2) For every $I \subset \{1,\dots,n\}$, there exists $\epsilon > 0$ such that

$$\nu_\epsilon(\partial_I \mathbb{E}(n,N)) \cap X = \nu_\epsilon(\partial_I \mathbb{E}(n,N)) \cap \pi_I^{-1}(\partial_I X).$$

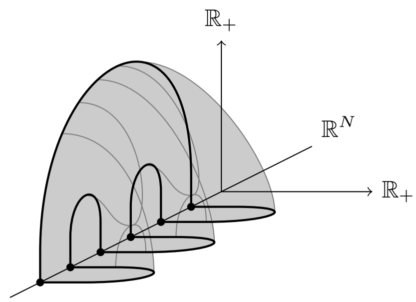


FIGURE 4. A neat embedding of the permutohedron Π_2. The hexagon is puffed out in the interior of $\mathbb{R}_+^2 \times \mathbb{R}^N$. The boundary of the hexagon is drawn thick, and some curves in the interior of the hexagon are shown by thin lines (to aid visualization).

Remark 5.4. The condition that the embedding be smooth makes sense in terms of maps of smooth manifolds with corners. However, once we assume conditions (1) and (2), we can rephrase smoothness by simply asking for a topological embedding such that its restriction to every stratum $\mathring{\partial}_I X$ is a smooth embedding into the corresponding stratum $\mathring{\partial}_I \mathbb{E}(n, N)$.

Example 5.5. The permutohedron Π_2 is a hexagon, and Figure 4 shows a neat embedding of that hexagon. The edges are perpendicular to $\mathbb{R}^N$ at vertices, and in fact contain small perpendicular intervals. We then fill in the hexagon so that, near an edge contained in one of the two hyperplanes $0 \times \mathbb{R}_+ \times \mathbb{R}^N$ or $\mathbb{R}_+ \times 0 \times \mathbb{R}^N$, it contains the product of that edge and an interval $[0, \epsilon)$ in the direction perpendicular to that hyperplane.

Proposition 5.6. *Let X be an $\langle n \rangle$-manifold such that ∂X is compact. Then X admits a neat embedding into $\mathbb{E}(n, N)$ for some N.*

Proof. This was proved by Laures in [23, Proposition 2.1.7], for X compact. He used his definition of neat embedding, which only required the intersections of strata with the boundaries of $\mathbb{E}(n, N)$ to be perpendicular. However, an inspection of his proof shows that the resulting embedding is neat in our sense. Further, the compactness condition can be weakened to ∂X being compact. Indeed, the proof proceeds by constructing collar neighborhoods of the strata (by integrating vector fields), then neatly embedding a neighborhood of ∂X, and then extending the embedding to the interior. The last step can also be done when X is not compact, in a similar way to the proof that ordinary smooth manifolds can be embedded in Euclidean space. □

Remark 5.7. With a little more work, one can also drop the compactness assumption on ∂X in Proposition 5.6. However, we will not need this more general statement.

Note that the boundary of $\mathbb{E}(n, N)$ is given by the equation $t_1 t_2 \cdots t_n = 0$, which can be smoothed into $t_1 t_2 \cdots t_n = \delta$. Neat embeddings allow us to smooth the boundaries on $\langle n \rangle$-manifolds in a similar fashion.

Definition 5.8. Let X be a compact $\langle n \rangle$-manifold. Pick a neat embedding of X into some $\mathbb{E}(n, N)$, and a value $\delta > 0$ smaller than all ϵ appearing (for different I) in Definition 5.3. Then, the subset

$$\mathrm{sm}[X] = X \cap \{(t_1, t_2, \ldots, t_n, x) \in \mathbb{E}(n, N) \mid t_1 t_2 \cdots t_n \geq \delta\}$$

is a smooth manifold with boundary, called a *smoothing* of X. The boundary $\partial\,\mathrm{sm}[X]$ is called the *smoothed boundary* of X.

6. Stratified spaces

There are many different definitions of stratified spaces in the literature. We will start with the following simple minded one.

Definition 6.1. A *stratified space* is a topological space X together with a locally finite decomposition of X into disjoint subsets, called *strata*, such that each stratum is equipped with the structure of a smooth manifold. The decomposition is called a *stratification* of X.

It will be helpful to know that the stratified spaces we will encounter in this paper satisfy certain properties; in particular, we will need to be able to smooth the boundary of each stratum, to obtain manifolds with boundary. This can be done for *Thom-Mather stratified spaces.* In turn, to show that a space is Thom-Mather stratified, it suffices to show that it is Whitney stratified, so we will start by defining *Whitney stratifications.*

6.1. Whitney stratified spaces.

Whitney stratified spaces were defined in [54]. See [15] for another exposition.

Definition 6.2. Let $X, Y \subseteq \mathbb{R}^n$ be smooth submanifolds, and let $x \in X$. We say that Y is *Whitney regular over X at x* if, whenever two sequences (x_i) of points in X and (y_i) of points in Y, with $x_i \neq y_i$, are such that:

- both sequences (x_i) and (y_i) converge to x,
- the sequences of tangent spaces $T_{y_i}Y$ converge to a subspace $T \subseteq \mathbb{R}^n$, and
- the secant lines $\overleftrightarrow{x_i y_i}$ converge to a line $L \subseteq \mathbb{R}^n$,

then L is contained in T.

Definition 6.3. Let M be a smooth m-dimensional manifold, $X, Y \subset M$ be smooth submanifolds, and $x \in X$. We say that Y is *Whitney regular over X at x* if their images in $\mathbb{R}^n$ are so, under one (and therefore under any) coordinate chart for M at x.

We define the *bad set* $B(X, Y)$ to be the set of $x \in X$ such that Y is not Whitney regular over X at x. We say that Y is *Whitney regular over X* if $B(X, Y) = \varnothing$.

Definition 6.4. Let M be a smooth manifold, and $V \subseteq M$ a subset. A *Whitney stratification* of V is a stratification such that all the strata are smooth submanifolds of M, and they are regular over each other.

The commonly given example of a non-Whitney stratification is that of the Whitney umbrella $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 z = y^2\}$, where one stratum X is the z-axis and the other stratum Y is its complement. Then Y is not Whitney regular over X at the origin. However, if we make the origin into a separate stratum of its own, we get a Whitney stratification. We will come back to the Whitney umbrella in Section 7.2; see Figure 7.

More generally, Thom [52] showed that all semialgebraic sets admits Whitney stratifications. Recall that the class of *semialgebraic sets* of $\mathbb{R}^n$ is the smallest Boolean algebra of subsets of $\mathbb{R}^n$ which contains all sets of the form

$$\{x \in \mathbb{R}^n \mid f(x) > 0\}$$

with $f\colon \mathbb{R}^n \to \mathbb{R}$ a polynomial function. For our purposes, we will need the following three results:

Proposition 6.5 (Proposition (2.1) in [15]). *Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a polynomial map, and $V \subseteq \mathbb{R}^n$ a semialgebraic set. Then the image $f(V)$ is semialgebraic.*

Proof. The Tarski-Seidenberg theorem says that the conclusion is true when f is a linear projection. For the general statement, consider the graph of f in $\mathbb{R}^n \times \mathbb{R}^m$. This is semialgebraic, and its linear projection to $\mathbb{R}^m$ is $f(V)$. □

Proposition 6.6 (Whitney's theorem [54, 53]; Proposition (2.6) in [15]). *Let X, Y be semialgebraic smooth submanifolds of $\mathbb{R}^n$. Then the bad set $B(X, Y)$ is semialgebraic, of dimension strictly smaller than the dimension of X.*

Proposition 6.7 (Proposition (1.2) in [15]). *Let $V_1, \dots, V_m$ be Whitney stratified spaces. Then the product stratification on $V_1 \times \cdots \times V_m$ (consisting of Cartesian products of the strata in each V_i) is a Whitney stratification.*

6.2. **Thom-Mather stratified spaces.** The following definition is based on [34, Section 8].

Definition 6.8. A *Thom-Mather stratified space* is a triple $(V, \mathscr{S}, \mathscr{T})$ satisfying the following axioms:

(A-1) V is a Hausdorff, locally compact, second countable topological space;

(A-2) $\mathscr{S}$ is a family of locally closed subsets of V, such that V is the disjoint union of the members of $\mathscr{S}$. The members of $\mathscr{S}$ are called the *strata* of V, and their closures are called the *closed strata*.

(A-3) Each stratum of V (with the induced topology from V) is a topological manifold, and additionally equipped with a C^∞ structure;

(A-4) The family $\mathscr{S}$ is locally finite.

(A-5) If $X, Y \in \mathscr{S}$ and $X \cap \overline{Y} \neq \varnothing$, then $X \subseteq \overline{Y}$. If this is the case, we write $X \leq Y$. If $X \leq Y$ and $X \neq Y$, we write $X < Y$.

(A-6) $\mathscr{T}$ is a collection of triples (T_X, π_X, ρ_X), one for each $X \in \mathscr{S}$, where T_X is an open neighborhood of X in V (called a *tubular neighborhood*), $\pi_X \colon T_X \to X$ is a continuous retraction, and $\rho_X \colon T_X \to [0, \infty)$ a continuous function.

(A-7) $X = \{v \in T_X \mid \rho_X(v) = 0\}$.

(A-8) For $X, Y \in \mathscr{S}$, denote

$$T_{X,Y} = T_X \cap Y, \quad \pi_{X,Y} = \pi_X|_{T_{X,Y}}, \quad \rho_{X,Y} = \rho_X|_{T_{X,Y}}.$$

Then, we require that for any distinct strata X and Y the mapping

$$(\pi_{X,Y}, \rho_{X,Y})\colon T_{X,Y} \to X \times (0, \infty)$$

is a smooth submersion.

(A-9) For any strata $X, Y,$ and Z, we have

$$\pi_X \pi_Y(v) = \pi_X(v), \quad \rho_X \pi_Y(v) = \rho_X(v)$$

whenever both sides of the respective equation are defined.

(A-10) If $X, Y \in \mathscr{S}$ satisfy $T_{X,Y} \neq \varnothing$, then $X \leq Y$.

(A-11) If $X, Y \in \mathscr{S}$ are such that $T_X \cap T_Y \neq \varnothing$, then X and Y are comparable, i.e., we have $X \leq Y$ or $Y \leq X$.

Remark 6.9. The terminology used in [34] is *abstract stratified set*. This is required to only satisfy the conditions (A-1)–(A-9), but it is noted there that every such set is equivalent to one that also satisfies (A-10) and (A-11).

Remark 6.10. The function ρ_X is called the *tubular function* of X. Roughly, it is meant to play the role of the distance to X.

Remark 6.11. As noted in [34], the assumptions (A-1)–(A-10) above also have the following implications:

- The relation $\leq$ is a partial order on $\mathscr{S}$;
- $X \leq Y$ if and only if $T_{X,Y} \neq \varnothing$;
- X and Y are comparable if and only if $T_X \cap T_Y \neq \varnothing$.

Remark 6.12. Another implication is that if x is a point in a k-dimensional stratum X, then there is a neighborhood V_x of x in V homeomorphic to $\mathbb{R}^k \times C(L)$, where L is a stratified space and $C(L) = \big(L \times [0,1)\big)/\big(L \times \{0\}\big)$ is the open cone on L. Specifically, we can take V_x to be the preimage of a chart in X under the map π_X. If we identify

$$L \cong \{0\} \times L \times \{1/2\} \subset \mathbb{R}^k \times C(L) \cong V_x,$$

then the stratification of L is given by intersections with the strata Y such that $X < Y$.

The space L is called the *link* of X at x. We will refer to $\mathbb{R}^k \times C(L)$ as the *local model* of V around x, and to $C(L)$ as the *local model in the normal directions*.

Remark 6.13. Given a stratum X in a Thom-Mather stratified space, its closure $\overline{X}$ and the boundary $\partial X = \overline{X} \setminus X$ have induced Thom-Mather stratifications. The boundary ∂X is the union of all strata Y such that $Y < X$.

We now turn to examples of Thom-Mather stratified spaces. First, an $\langle n \rangle$-manifold X can be made into a Thom-Mather stratified space as follows. Let $\partial_i X$, for $i = 1, \dots, n$, be the distinguished collection of multifacets, and $\partial_I X$ and $\mathring{\partial}_I X$ be as in Equation (5.1). We let $\mathring{\partial}_I X$ be the strata in X and $\partial_I X$ are their closures. Observe that

$$\partial_J X \subseteq \partial_I X \iff I \subseteq J.$$

The tubular neighborhood of a stratum $\mathring{\partial}_I X$ in an $\langle n \rangle$-manifold X can be constructed by integrating a smooth vector field that is transverse to $\mathring{\partial}_I X$, and vanishes at the boundary of $\mathring{\partial}_I X$.

Example 6.14. Consider the quadrant $X = \mathbb{R}^2_+$, and view it as a $\langle 2 \rangle$-manifold with $\partial_1 X = 0 \times \mathbb{R}_+$ and $\partial_2 X = \mathbb{R}_+ \times 0$. There are four strata:

$$\mathring{\partial}_\varnothing X = (0,\infty) \times (0,\infty), \quad \mathring{\partial}_{\{1\}} X = 0 \times (0,\infty), \quad \mathring{\partial}_{\{2\}} X = (0,\infty) \times 0, \quad \mathring{\partial}_{\{1,2\}} X = 0 \times 0.$$

Their tubular neighborhoods are shown in Figure 5: that of $\mathring{\partial}_{\{1,2\}} X$ is the lightly shaded quarter-disk, those of $\mathring{\partial}_{\{1\}} X$ and $\mathring{\partial}_{\{2\}} X$ are darkly shaded, and the tubular neighborhood of $\mathring{\partial}_\varnothing X$ is $\mathring{\partial}_\varnothing X$ itself.

Remark 6.15. In [23, Lemma 2.1.6] it is proved that the *closed* strata $\partial_I X$ in an $\langle n \rangle$-manifold admit a system of collar neighborhoods, of the form $\mathbb{R}^{|I|} \times \partial_I X$. These are different from the tubular neighborhoods that we consider in their paper, but serve similar purposes. The collar neighborhoods from [23, Lemma 2.1.6] were used in the construction of the Khovanov stable homotopy type in [26]. We do not use them here because they do not admit a straightforward generalization to other stratified spaces.

A larger class of Thom-Mather stratified spaces is provided by the following result.

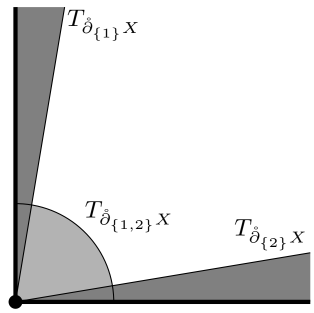


FIGURE 5. The Thom-Mather stratified space $X = [0, \infty)^2$, with tubular neighborhoods.

Theorem 6.16 (Thom [52], Mather [34]). *The strata in Whitney stratified spaces admit tubular neighborhoods as in Definition 6.8, and therefore Whitney stratified spaces can be turned into Thom-Mather stratified spaces.*

6.3. **Smoothings.** We now explain how one can smooth the boundary of a stratum in a Thom-Mather stratified space. The lemma below is key: it allows us to find neighborhoods of the boundary that are submanifolds with corners.

Lemma 6.17. *Let X be an n-dimensional stratum in a Thom-Mather stratified space $(V, \mathscr{S}, \mathscr{T})$. For any stratum $Y \subseteq \partial X$, choose $\epsilon_Y > 0$ sufficiently small, inductively on the dimension of Y, so that $\epsilon_Y \ll \epsilon_Z$ when $Z < Y$. Consider the following closed neighborhood of ∂X in $\overline{X}$:*

$$\mathcal{N} = \bigcup_{Y<X} (\rho_Y^{-1}([0, \epsilon_Y]) \cap \overline{X}).$$

Then, the complement

$$M := \overline{X} \setminus \operatorname{int}(\mathcal{N})$$

is an n-dimensional $\langle n \rangle$-manifold.

Proof. It is convenient to set $\rho_Y(x) = \infty$ where ρ_Y is undefined, i.e., for $x \notin T_Y$. Then we can write

$$M = \overline{X} \cap \bigcap_{Y<X} \rho_Y^{-1}([\epsilon_Y, \infty]).$$

For $i = 1, \dots, n$, set

$$\partial_i M = \{x \in M \mid \rho_Y(x) = \epsilon_Y \text{ for some } Y < X \text{ with } \dim Y = i - 1\}.$$

To see that this turns M into an $\langle n \rangle$-manifold, pick any point $x \in \partial M = \bigcup_i \partial_i M$, and consider the strata $Y < X$ that satisfy $\rho_Y(x) = \epsilon_Y$. Using (A-11), we see that $\leq$ is a total order on these strata, so the strata have different dimensions and we can label them by $Y_1 < Y_2 < \cdots < Y_k$. Thus, x lies at the intersection of the boundaries $\partial_i M$ where $i = \dim Y_j$ for some j. Furthermore, near x, the subset $M \subseteq X$ is given by the inequalities

$$\rho_{Y_i} \geq \epsilon_{Y_i}, \ i = 1, \dots, k.$$

We claim that x is a codimension k corner, that is, the local model for M near x is $0 \in \mathbb{R}^{n-k} \times \mathbb{R}^k_+$. For this, it suffices to show that the map

$$(\rho_{Y_1,X}, \rho_{Y_2,X}, \dots, \rho_{Y_k,X}) \colon U(x) \to \mathbb{R}^k$$

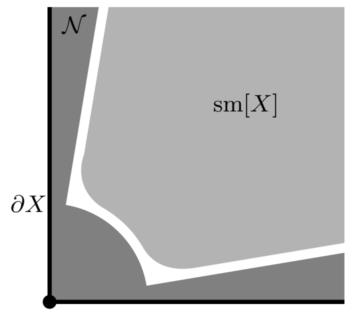


FIGURE 6. A smoothing of the stratified space from Figure 5.

is a submersion at x. (Here, $U(x)$ is a neighborhood of x in X.)

We prove by induction on $j \leq k$ that

$$(\rho_{Y_1,X}, \rho_{Y_2,X}, \dots, \rho_{Y_j,X})\colon U(x) \to \mathbb{R}^j$$

is a submersion at x. The base case $j = 1$ follows from (A-8). For the inductive step, using (A-9) , we write

$$(\rho_{Y_1,X}, \rho_{Y_2,X}, \dots, \rho_{Y_j,X}) = (\rho_{Y_1,Y_j}, \rho_{Y_2,Y_j}, \dots, \rho_{Y_{j-1},Y_j}, \mathrm{id}) \circ (\pi_{Y_j,X}, \rho_{Y_j,X}).$$

Note that $(\pi_{Y_j,X}, \rho_{Y_j,X})$ is a submersion by (A-8). Using the inductive hypothesis for $j-1$ and the fact that the composition of submersions is a submersion, the claim follows. □

Definition 6.18. Let X be an n-dimensional stratum of a Thom-Mather stratified space, such that ∂X is compact. We define the *smoothing of* X to be the n-dimensional smooth manifold with boundary

$$\mathrm{sm}[X] := \mathrm{sm}[M]$$

where $M \subseteq X$ is the $\langle n \rangle$-manifold from Lemma 6.17, and $\mathrm{sm}[M] \subseteq M$ was defined in Definition 5.8.

See Figure 6 for an example.

7. LOCAL MODELS

In this paper we will work with a certain kind of stratified spaces, where we have boundaries and corners as in $\langle n \rangle$-manifolds, but we also allow a different type of boundary, modeled on "generalized Whitney umbrellas" and called the *special boundary.* When we glue several of these spaces along their special boundaries, we will obtain an $\langle n \rangle$-manifold.

The spaces will be constructed in Section 12. For now, we will limit ourselves to describing the local models that appear in their stratifications.

7.1. The spaces I_N.

Let us first consider the space

$$I_N = \mathrm{Sym}^N(\mathbb{R})/\mathbb{R},$$

where $N \geq 0$ and Sym^N denotes the N^{th} symmetric product, and $\mathbb{R}$ acts by simultaneous translation on all factors. Recall from Section 4.1 that $\mathrm{Part}(N)$ denotes the set of ordered partitions of N. For

$$\lambda = (\lambda_1, \dots, \lambda_m) \in \mathrm{Part}(N),$$

let $I(\lambda) \subseteq I_N$ be the subset consisting of N-tuples of real numbers (modulo $\mathbb{R}$) such that the first λ_1 of these numbers coincide (take the same value x_1), the next λ_2 coincide (taking a value x_2), and so on, with $x_i < x_j$ for $i < j$. The decomposition

$$I_N = \bigsqcup_{\lambda \in \mathrm{Part}(N)} I(\lambda) \tag{7.1}$$

gives I_N the structure of a stratified space, with strata $I(\lambda)$.

If $N > 0$, the dimension of $I(\lambda)$ is $\ell(\lambda) - 1$, where $\ell(\lambda) = m$ is the length of the partition. The local model of I_N around a point in $I(\lambda)$ is

$$\mathbb{R}^{m-1} \times I_{\lambda_1} \times \cdots \times I_{\lambda_m}.$$

The refinement order on partitions introduced in Section 4.1 is relevant to the decomposition of I_N, because it tells us which strata are in the closures of other strata:

$$I(\lambda) \leq I(\mu) \iff \lambda \leq \mu. \tag{7.2}$$

Example 7.1. When $N = 0$ or 1, there is a unique partition of N, and in both cases I_N is a point. When $N = 2$, we have $I_2 \cong [0, \infty)$, with the strata in the decomposition being $I_2(2) = \{0\}$ and $I_2(1,1) = (0, \infty)$. When $N = 3$, one can check that $I_3 \cong [0, \infty)^2$, with $I_3(3)$ being the origin, $I_3(1,2)$ and $I_3(2,1)$ being the two half-lines on the boundary, and $I_3(1,1,1) \cong (0, \infty)^2$. When $N \geq 4$, the topology of I_N is more complicated; see [7].

7.2. **The spaces Z_N.** Next, consider the space

$$Z_N = \mathrm{Sym}^N(\mathbb{C})/\mathbb{R},$$

where $\mathbb{R}$ acts on the $\mathbb{C}$ factors by translating the real parts. If we let the coordinates on each copy of $\mathbb{C}$ be

$$z_j = x_j + iy_j, \ j = 1, \dots, N,$$

note that $\mathrm{Sym}^N(\mathbb{C})$ can be identified with $\mathbb{C}^N$ using the elementary symmetric polynomials in $z_1, \dots, z_N$:

$$s_1 = \sum_j z_j, \quad s_2 = \sum_{j<l} z_j z_l, \ \dots, \ s_N = \prod_j z_j. \tag{7.3}$$

Further, dividing by $\mathbb{R}$ is equivalent to setting $\mathrm{Re}(s_1) = x_1 + \cdots + x_N = 0$. Therefore, if $N > 0$, we can identify Z_N with $\mathbb{R}^{2N-1}$, with real coordinates being

$$\mathrm{Im}(s_1), \mathrm{Re}(s_2), \mathrm{Im}(s_2), \dots, \mathrm{Re}(s_N), \mathrm{Im}(s_N). \tag{7.4}$$

We put a stratification on $Z_N \cong \mathbb{R}^{2N-1}$, with the strata being given by the signs of the imaginary parts $y_j = \mathrm{Im}(z_j)$, as well as by which real coordinates coincide for the indices j with $y_j = 0$. Precisely, consider the decomposition

$$Z_N = \bigsqcup_{\substack{p^- + p^0 + p^+ = N \\ \lambda \in \mathrm{Part}(p^0)}} Z(p^-, p^0, p^+; \lambda),$$

with $Z(p^-, p^0, p^+; \lambda)$ consisting of the multisets $\{z_1, \dots, z_N\}$ where p^- of the y_j's are less than zero, p^0 are zero, p^+ are greater than zero, and the p^0 coordinates x_j (those for which $y_j = 0$) are split according to the partition λ, as in the decomposition (7.1) of the space I_{p^0}. Observe that

$$Z(p^-, p^0, p^+; \lambda) \leq Z(q^-, q^0, q^+; \mu) \iff (p^- \leq q^-, p^+ \leq q^+ \text{ and } \lambda \leq \mu), \tag{7.5}$$

where we used the order on partitions introduced in Definition 4.4. We will denote the closure of $Z(p^-, p^0, p^+; \lambda)$ by $\overline{Z}(p^-, p^0, p^+; \lambda)$.

We have

$$\dim Z(p^-, p^0, p^+; \lambda) = 2p^- + 2p^+ + \ell(\lambda) - 1. \tag{7.6}$$

For example, there is a unique zero dimensional stratum, namely $Z(0, N, 0; N)$.

Observe that the codimension of a stratum $Z(p^-, p^0, p^+; \lambda) \subset Z_N$ is $2p^0 - \ell(\lambda)$, which is at least p^0. In particular, there are $N + 1$ codimension zero strata, corresponding to $p^0 = 0$.

We let $Z(p^-, p^0, p^+)$ be the union of $Z(p^-, p^0, p^+; \lambda)$ over all $\lambda \in \operatorname{Part}(p^0)$. In particular, $Z(0, N, 0)$ is our old space I_N. Also, note that when $p^0 = 0$ or 1, there is a unique partition (p^0), so $Z(p^-, p^0, p^+) = Z(p^-, p^0, p^+; (p^0))$.

Example 7.2. When $N = 0$, Z_0 is a point. When $N = 1$, we have

$$Z_1 = \operatorname{Sym}^1(\mathbb{C})/\mathbb{R} \cong \mathbb{R},$$

with the three strata $(-\infty, 0), \{0\}$, and $(0, \infty)$.

Example 7.3. When $N = 2$, we look at $Z_2 = \operatorname{Sym}^2(\mathbb{C})/\mathbb{R}$, with the coordinates on the two copies of $\mathbb{C}$ being

$$z_1 = x_1 + iy_1, \quad z_2 = x_2 + iy_2.$$

We identify $\operatorname{Sym}^2(\mathbb{C})$ with $\mathbb{C}^2$ using the symmetric polynomials $z_1 + z_2$ and $z_1 z_2$. After dividing by $\mathbb{R}$-translation (that is, setting $x_1 + x_2 = 0$), we are left with three real coordinates on Z_2:

$$\begin{aligned} a &= \operatorname{Im}(z_1 + z_2) = y_1 + y_2, \\ b &= \operatorname{Re}(z_1 z_2) = -(x_1^2 + y_1 y_2), \\ c &= \operatorname{Im}(z_1 z_2) = x_1(y_2 - y_1). \end{aligned}$$

Let $W \subset Z_2$ be the hypersurface given by the condition that at least one of z_1 and z_2 be real, i.e., $y_1 y_2 = 0$. From here we get $b = -x_1^2 \le 0$ and $c = \pm x_1 a$, so $a^2 b + c^2 = 0$:

$$W = \{(a, b, c) \in \mathbb{R}^3 \mid b \le 0, a^2 b + c^2 = 0\}.$$

This is the Whitney umbrella shown in Figure 7. The complement of W in Z_2 splits into three connected components:

$$\begin{aligned} Z(2, 0, 0) &= \{(a, b, c) \in \mathbb{R}^3 \mid a^2 b + c^2 < 0 \text{ and } a < 0\}, \\ Z(1, 0, 1) &= \{(a, b, c) \in \mathbb{R}^3 \mid a^2 b + c^2 > 0 \text{ or } b > 0\}, \\ Z(0, 0, 2) &= \{(a, b, c) \in \mathbb{R}^3 \mid a^2 b + c^2 < 0 \text{ and } a > 0\}, \end{aligned}$$

corresponding to none, one, or two of the y_j coordinates being positive, and the rest negative.

The codimension-1 strata are the two halves of W:

$$\begin{aligned} Z(1, 1, 0) &= \{(a, b, c) \in \mathbb{R}^3 \mid a^2 b + c^2 = 0, a < 0\}, \\ Z(0, 1, 1) &= \{(a, b, c) \in \mathbb{R}^3 \mid a^2 b + c^2 = 0, a > 0\}. \end{aligned}$$

This leaves the strata corresponding to $y_1 = y_2 = 0$, which give the half-line

$$Z(0, 2, 0) = \{(0, b, 0) \in \mathbb{R}^3 \mid b \le 0\}.$$

This is further decomposed into the codimension-2 stratum $Z(0, 2, 0; (1, 1))$, which is the open half-line, and the stratum $Z(0, 2, 0; (2))$, which is just the point $\{(0, 0, 0)\}$.

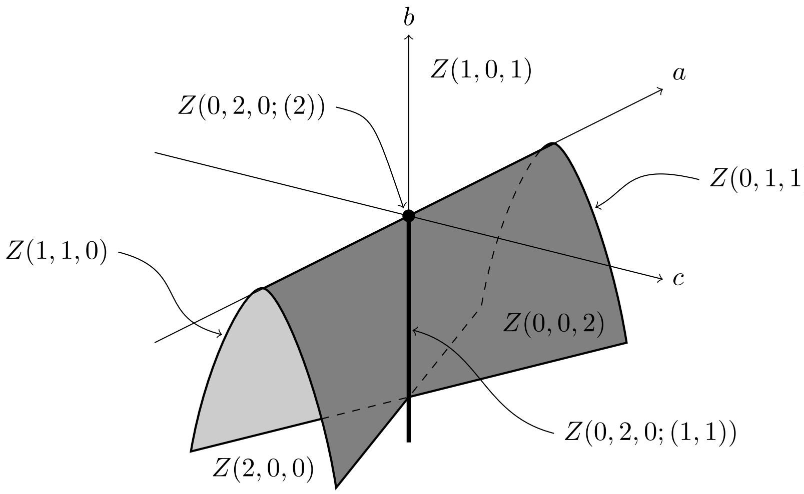


FIGURE 7. The Whitney umbrella inside $Z_2 \cong \mathbb{R}^3$.

7.3. **Models for internal framings.** We will now construct explicit framings for the normal bundles to the strata in Z_N; assume $N > 0$. These will be models for the internal framings in Section 10 later.

Convention 7.4. A framing of a vector bundle is defined to be a smoothly varying, ordered basis of the fibers. (We do not ask it to be orthonormal.) The normal bundle to a submanifold $X \subset V$ is defined to be the quotient TV/TX, which we can identify with any complement of TX in TV. Throughout this paper, when talking about a framing of the normal bundle, we will always mean that such a complement has been chosen, and we consider a framing of it; thus, the frame consists of vectors in TV. We do not ask for the complement to be the orthogonal complement. Of course, from a framing as above one can get one of the orthogonal complement, and/or an orthonormal framing, by using the Gram-Schmidt process. However, it is convenient to have the extra flexibility.

Let $\widetilde{Z}_N = \mathrm{Sym}^N(\mathbb{C})$, which we can identify with $\mathbb{R} \times Z_N$ by letting the first coordinate be $\mathrm{Re}(s_1)$ in the notation of Equation (7.3). The space $\widetilde{Z}_N$ has a stratification with strata

$$\widetilde{Z}(p^-, p^0, p^+; \lambda) \cong \mathbb{R} \times Z(p^-, p^0, p^+; \lambda). \tag{7.7}$$

The normal bundle of $Z(p^-, p^0, p^+; \lambda)$ in Z_N is identified with the normal bundle of $\widetilde{Z}(p^-, p^0, p^+; \lambda)$ in $\widetilde{Z}_N$, so it suffices to frame the latter.

Fix a small $\epsilon > 0$. Consider any stratum $\widetilde{Z}(p^-, p^0, p^+; \lambda) \subset \widetilde{Z}_N$ and a point z in the stratum. Assume $\lambda = (\lambda_1, \dots, \lambda_m)$. The point z consists of an unordered tuple

$$\{z_1 = x_1 + iy_1, \dots, z_N = x_N + iy_N\}$$

such that $p^{\pm}$ of them have imaginary part positive/negative, and the remaining p^0 numbers—say $z_1, \dots, z_{p^0}$—have imaginary part 0 with their real parts satisfying

$$x_1 = \dots = x_{\lambda_1} < x_{\lambda_1+1} = \dots = x_{\lambda_1+\lambda_2} < \dots < x_{p^0-\lambda_m+1} = \dots = x_{p^0}.$$

For convenience, let us relabel $z_1, \dots, z_{p^0}$ as $z_{1,1}, \dots, z_{1,\lambda_1}, z_{2,1}, \dots, z_{2,\lambda_2}, \dots, z_{m,1}, \dots, z_{m,\lambda_m}$, so the above inequality becomes

$$x_{1,1} = \dots = x_{1,\lambda_1} < x_{2,1} = \dots = x_{2,\lambda_2} < \dots < x_{m,1} = \dots = x_{m,\lambda_m}.$$

We will pick a smoothly varying $z' \in \widetilde{Z}(p^-, p^0, p^+; (1, 1, \dots, 1))$ that is ϵ-close to z. For specificity, pick a smoothly varying ϵ_z which is less than $\min\{\epsilon, x_{2,1} - x_{1,1}, \dots, x_{m,1} - x_{m-1,1}\}$, and define z' to be the unordered tuple $\{z'_{1,1} = x'_{1,1}, \dots, z'_{m,\lambda_m} = x'_{m,\lambda_m}, z_{p^0+1}, \dots, z_N\}$ with

$$x'_{j,l} = x_{j,1} + \frac{(2l - \lambda_j - 1)\epsilon_z}{2\lambda_j}.$$

(In words, $z'_{1,1}, \dots, z'_{m,\lambda_m}$ still have imaginary part 0, but their real parts are distinct, and for each j, the real parts $x'_{j,1}, \dots, x'_{j,\lambda_j}$ are equally spaced in an interval of size $< \epsilon_z$ centered at $x_{j,1}$.)

Recall that $\widetilde{Z}_N \cong \mathbb{R}^{2N}$ has real coordinates $\mathrm{Re}(s_1), \mathrm{Im}(s_1), \dots, \mathrm{Re}(s_N), \mathrm{Im}(s_N)$ and so the tangent bundle has a basis given by the infinitesimal variations $\delta \mathrm{Re}(s_1), \delta \mathrm{Im}(s_1), \dots, \delta \mathrm{Re}(s_N)$, $\delta \mathrm{Im}(s_N)$, which are the unit vectors along those coordinate directions. However, near the point z' we will choose a different coordinate system, *tailored to the point* z', and consequently a different basis of the tangent bundle.

Consider disjoint neighborhoods $U_{1,1} \ni z'_{1,1}, \dots, U_{m,\lambda_m} \ni z'_{m,\lambda_m}$ in $\mathbb{C}$ and a neighborhood $U \ni \{z_{p^0+1}, \dots, z_N\}$ in $\mathrm{Sym}^{N-p^0}(\mathbb{C})$ which is disjoint from each $U_{j,l} \times \mathrm{Sym}^{N-p^0-1}(\mathbb{C})$. Then

$$W := U_{1,1} \times \dots \times U_{m,\lambda_m} \times U = \prod_{l=1}^{\lambda_1} U_{1,l} \times \dots \times \prod_{l=1}^{\lambda_m} U_{m,l} \times U$$

is an open neighborhood of z' in $\widetilde{Z}_N = \mathrm{Sym}^N(\mathbb{C})$.

Any point $w \in W$ has coordinates $w_{j,l} = u_{j,l} + iv_{j,l} \in U_{j,l}$ and an unordered tuple $\{w_{p^0+1} = u_{p^0+1} + iv_{p^0+1}, \dots, w_N = u_N + iv_N\} \in U$. Let $t_1, \dots, t_{N-p^0}$ be the elementary symmetric polynomials in $w_{p^0+1}, \dots, w_N$. Then our tailored coordinates on U are $\mathrm{Re}(t_1), \mathrm{Im}(t_1), \dots, \mathrm{Re}(t_{N-p^0}), \mathrm{Im}(t_{N-p^0})$, similar to Equation (7.4). However, for each j, we pick different tailored coordinates on $\prod_{l=1}^{\lambda_j} U_{j,l}$:

$$v_{j,1}, u_{j,2} - u_{j,1}, v_{j,2}, u_{j,3} - u_{j,2}, \dots, u_{j,\lambda_i} - u_{j,\lambda_{i-1}}, v_{j,\lambda_i}, \sum_{l=1}^{\lambda_j} u_{j,l}.$$

For notational convenience, let $\Delta_{j,l} = u_{j,l+1} - u_{j,l}$. Together, these give tailored local coordinates on W, which we reorder and write as:

$$\begin{gathered} v_{1,1}, \Delta_{1,1}, v_{1,2}, \Delta_{1,2}, \dots, \Delta_{1,\lambda_1-1}, v_{1,\lambda_1}, \\ \dots \\ v_{m,1}, \Delta_{m,1}, v_{m,2}, \Delta_{m,2}, \dots, \Delta_{m,\lambda_m-1}, v_{m,\lambda_m}, \\ \sum_{l=1}^{\lambda_1} u_{1,l}, \dots, \sum_{l=1}^{\lambda_m} u_{m,l}, \mathrm{Re}(t_1), \mathrm{Im}(t_1), \dots, \mathrm{Re}(t_{N-p^0}), \mathrm{Im}(t_{N-p^0}). \end{gathered}$$

Their infinitesimal variations give a basis for the tangent bundle of W. Of these, infinitesimals of the form $\delta v_{j,l}$ form a basis of the normal bundle of $\widetilde{Z}(p^-, p^+, p^0; (1, \dots, 1))$, while the rest form a basis for its tangent bundle.

Note that $\widetilde{Z}(p^-, p^+, p^0; \lambda) < \widetilde{Z}(p^-, p^+, p^0; (1, \dots, 1))$; as $\epsilon \to 0$, the point $z' \in \widetilde{Z}(p^-, p^+, p^0; (1, \dots, 1))$ approaches $z \in \widetilde{Z}(p^-, p^+, p^0; \lambda)$. Each of the $2N$ vectors listed above has a limit, and in the limit, the vectors from the last row

$$\lim_{\epsilon\to 0}\delta\sum_{l=1}^{\lambda_1} u_{1,l}|_{z'},\dots,\lim_{\epsilon\to 0}\delta\sum_{l=1}^{\lambda_m} u_{m,l}|_{z'},\lim_{\epsilon\to 0}\delta\operatorname{Re}(t_1)|_{z'},\dots,\lim_{\epsilon\to 0}\delta\operatorname{Im}(t_{N-p^0})|_{z'}$$

give a basis for the tangent space of $\widetilde{Z}(p^-,p^+,p^0;\lambda)$ at z. Similarly, the limits of the other vectors give normal vectors of $\widetilde{Z}(p^-,p^+,p^0;\lambda)$ in $\widetilde{Z}_N$ at z, but they do not give a basis. Specifically, we get

$$\begin{aligned} &\lim_{\epsilon\to 0}\delta v_{j,l_1}|_{z'}=\lim_{\epsilon\to 0}\delta v_{j,l_2}|_{z'} && \text{for all } j,l_1,l_2\text{, and}\\ &\lim_{\epsilon\to 0}\delta\Delta_{j,l}|_{z'}=0 && \text{for all } j,l. \end{aligned}$$

Instead we pick a path $\tau\colon[0,1]\to\widetilde{\widetilde{Z}}(p^-,p^0,p^+;(1,1,\dots,1))$ with $\tau(0)=z$, $\tau(1)=z'$, and $\tau((0,1])$ lying inside the contractible subspace of $\widetilde{Z}(p^-,p^0,(1,1,\dots,1))$ with coordinates satisfying

$$\begin{aligned} v_{j,l}&=0 && \text{for all } 1\le j\le m, 1\le l\le\lambda_j,\\ \sum_{l=1}^{\lambda_j-1}\Delta_{j,l}&<\epsilon_z && \text{for all } 1\le j\le m,\\ w_j&=z_j && \text{for all } j>p^0. \end{aligned}$$

Pick an isotopy of $\widetilde{Z}_N$ supported in a small neighborhood of this path that sends $\tau(1)=z'$ to $\tau(0)=z$.

Definition 7.5. The *standard frame* for the normal bundle to $\widetilde{Z}(p^-,p^0,p^+;\lambda)$ in $\widetilde{Z}_N$ (equivalently, $Z(p^-,p^0,p^+;\lambda)$ in Z_N) at any point z is given by the images under the above isotopy of the following vectors in $T_{z'}\widetilde{Z}_N$ (in this order):

$$\begin{gathered} \delta v_{1,1},-\delta\Delta_{1,1},\delta v_{1,2},-\delta\Delta_{1,2},\dots,-\delta\Delta_{1,\lambda_1-1},\delta v_{1,\lambda_1},\\ \cdots\\ \delta v_{m,1},-\delta\Delta_{m,1},\delta v_{m,2},-\delta\Delta_{m,2},\dots,-\delta\Delta_{m,\lambda_m-1},\delta v_{m,\lambda_m}. \end{gathered}$$

While this depends on the choice of ϵ, the smoothly varying function ϵ_z, the path τ staying within an ϵ_z neighborhood, and the isotopy supported near τ, each of these choices come from a contractible space, and so the different standard frames at z coming from different choices are canonically isotopic.

Remark 7.6. We pause here to justify our choice of these complicated tailored coordinate systems and the consequent standard frames. Consider any two strata $Z(p^-,p^0,p^+;\lambda)<Z(q^-,q^0,q^+;\mu)$ of consecutive dimensions. From Equations (7.5) and (7.6), we conclude that there are only three possibilities

$$\begin{aligned} (p^-,p^0,p^+)&=(q^-,q^0,q^+)\text{ and }\lambda\in\operatorname{EC}(\mu),\\ (p^-,p^0,p^+)&=(q^--1,q^0+1,q^+)\text{ and }\lambda\in\operatorname{UE}(\mu),\\ (p^-,p^0,p^+)&=(q^-,q^0+1,q^+-1)\text{ and }\lambda\in\operatorname{UE}(\mu). \end{aligned}$$

In each case, for any point $z\in Z(p^-,p^0,p^+;\lambda)$, one of the normal vectors in the standard frame at z is an inward or outward normal vector to $Z(q^-,q^0,q^-;\mu)$; and for any nearby $z'\in Z(q^-,q^0,q^+;\mu)$, the standard frame at z' is composed of the remaining vectors of the standard frame at z (up to canonical isotopy). For instance, in the first case, if

$$\mu=(\mu_1,\dots,\mu_m),\qquad\lambda=(\mu_1,\dots,\mu_{k-1},\mu_k+\mu_{k+1},\mu_{k+2},\dots,\mu_m),$$

then the normal vector $-\delta\Delta_{k,\mu_k}$ in the standard frame is the outward normal to $Z(q^-, q^0, q^+; \mu)$. In the second (respectively third) case, if

$$\mu = (\mu_1, \dots, \mu_m), \qquad \lambda = (\mu_1, \dots, \mu_{k-1}, 1, \mu_k, \dots, \mu_m),$$

then the normal vector $\delta v_{k,1}$ in the standard frame is the outward (respectively inward) normal to $Z(q^-, q^0, q^+; \mu)$.

Example 7.7. Consider the stratified space $Z_2 = \mathrm{Sym}^2(\mathbb{C})/\mathbb{R}$ from Example 7.3, viewed as the hyperplane $\mathrm{Re}(s_1) = 0$ of the space $\widetilde{Z}_2 = \mathrm{Sym}^2(\mathbb{C})$ which has real coordinates

$$\begin{aligned} \mathrm{Re}(s_1) &= x_1 + x_2 \\ a = \mathrm{Im}(s_1) &= y_1 + y_2 \\ b = \mathrm{Re}(s_2) &= x_1x_2 - y_1y_2 \\ c = \mathrm{Im}(s_2) &= x_1y_2 + x_2y_1; \end{aligned}$$

its various strata $Z(p^-, p^0, p^+; \lambda)$ are the intersections of the corresponding strata $\widetilde{Z}(p^-, p^0, p^+; \lambda)$ with this hyperplane.

The stratum $\widetilde{Z}(1,1,0)$ is a codimension one stratum. Near points in $Z(1,1,0)$ we use tailored coordinates (y_1, x_1, x_2, y_2). For points actually on $Z(1,1,0)$, we have $y_1 = x_1 + x_2 = 0$ and $y_2 < 0$, so the coordinates are $(y_1 = 0, x_1 = c/a, x_2 = -c/a, y_2 = a)$. The standard frame at any point $(a,b,c) \in Z(1,1,0)$ consists of the single vector δy_1; since

$$\begin{aligned} \delta a &= \delta y_1 \\ \delta b &= -y_2\delta y_1 = -a(\delta y_1) \\ \delta c &= x_2\delta y_1 = -c/a(\delta y_1), \end{aligned}$$

the standard frame is the vector $(1, -a, -c/a)$. Observe that this is a normal vector pointing towards $Z(1,0,1)$ and away from $Z(2,0,0)$.

A similar calculation shows that for any point (a,b,c) in the other codimension one stratum $Z(0,1,1)$, the standard frame is again $(1,-a,-c/a)$, which is now a normal vector pointing towards $Z(0,0,2)$ and away from $Z(1,0,1)$.

Near the one-dimensional stratum $Z(0,2,0;(1,1))$, the tailored local coordinates are (y_1, y_2, x_1, x_2). For points actually on $Z(0,2,0;(1,1))$, $y_1 = y_2 = x_1 + x_2 = 0$ and $x_1 < x_2$, so the coordinates are $(y_1 = 0, y_2 = 0, x_1 = -\sqrt{-b}, x_2 = \sqrt{-b})$. The standard frame at any point $(0,b,0) \in Z(0,2,0;(1,1))$ is $[\delta y_1, \delta y_2]$; we have

$$\begin{aligned} \delta a &= \delta y_1 + \delta y_2 \\ \delta b &= -y_2\delta y_1 - y_1\delta y_2 = 0 \\ \delta c &= x_1\delta y_2 + x_2\delta y_1 = -\sqrt{-b}(\delta y_2) + \sqrt{-b}(\delta y_1), \end{aligned}$$

and hence, the standard frame is $[(1,0,\sqrt{-b}), (1,0,-\sqrt{-b})]$. The two vectors point towards the two sheets of $Z(0,1,1)$ entering $Z(0,2,0;(1,1))$, and away from the two sheets of $Z(1,1,0)$ entering $Z(0,2,0;(1,1))$. Moroever, for any nearby point in $Z(1,1,0)$ or $Z(0,1,1)$, the standard frame is $(1,-a,-c/a)$, which approaches one of the two vectors in the standard frame as we approach $Z(0,2,0;(1,1))$ (that is, as $a, c \to 0$ with $c/a = \pm\sqrt{-b}$).

Finally for the zero-dimensional stratum $Z(0,2,0;(2))$, we pick a nearby point $z' = \{x_1 + iy_1, x_2 + iy_2\} \in Z(0,2,0;(2))$ with $y_1 = y_2 = x_1 + x_2 = 0$ and $x_1 < x_2$, but use tailored coordinates

$(y_1, \Delta_1 = x_2 - x_1, y_2, \mathrm{Re}(s_1) = x_1 + x_2)$ near z'; in terms of b, the tailored coordinates of z' are $(y_1 = 0, \Delta_1 = 2\sqrt{-b}, y_2 = 0, \mathrm{Re}(s_1) = 0)$. We consider the frame $[\delta y_1, -\delta\Delta_1, \delta y_2]$; we calculate

$$\begin{aligned} a &= y_1 + y_2 \\ \delta a &= \delta y_1 + \delta y_2 \\ b &= x_1x_2 - y_1y_2 = (\mathrm{Re}(s_1)^2 - \Delta_1^2)/4 - y_1y_2 \\ \delta b &= -\Delta_1(\delta\Delta_1)/2 - y_2(\delta y_1) - y_1(\delta y_2) = -\sqrt{-b}(\delta\Delta_1) \\ c &= x_1y_2 + x_2y_1 = y_2(\mathrm{Re}(s_1) - \Delta_1)/2 + y_1(\mathrm{Re}(s_1) + \Delta_1)/2 \\ \delta c &= (y_1 - y_2)(\delta\Delta_1)/2 + (\mathrm{Re}(s_1) + \Delta_1)(\delta y_1)/2 + (\mathrm{Re}(s_1) - \Delta_1)(\delta y_2)/2 = \sqrt{-b}(\delta y_1 - \delta y_2), \end{aligned}$$

so the frame is given by $[(1, 0, \sqrt{-b}), (0, \sqrt{-b}, 0), (1, 0, -\sqrt{-b})]$. Instead of taking the limit as $b \to 0$ (which would not produce a 3-dimensional frame), we take the path $(0, bt, 0)$, $t \in [0, 1]$, choose an isotopy supported near this path that sends $(0, b, 0)$ to $(0, 0, 0)$, and consider the image of this frame under this isotopy. Assuming the isotopy near the path is an affine translation in the a, b, c coordinates, the standard frame at $(0, 0, 0)$ is also given by $[(1, 0, \sqrt{-b}), (0, \sqrt{-b}, 0), (1, 0, -\sqrt{-b})]$. This depends on b, but b varies on the contractible space $(-\epsilon, 0)$ for some $\epsilon > 0$, and the different frames for different choices of b are canonically isotopic. Observe that the second vector points away from the stratum $Z(0, 2, 0; (1, 1))$; and by construction, the other two vectors are canonically isotopic to the frame $[(1, 0, \sqrt{-b}), (1, 0, -\sqrt{-b})]$ at z', which is the standard frame at z'.

7.4. **Local models from Z_N.** Consider a stratum

$$Y = Z(p^-, p^0, p^+; \lambda) \subseteq Z_N$$

with $\lambda = (\lambda_1, \dots, \lambda_m)$. Around any point $y \in Y$, consider the affine subspace generated by the standard frame for the normal bundle to Y, as in Definition 7.5. This subspace is the *local model for Y in the normal directions*, and we have an isomorphism

$$\mathcal{L}(Y) := Z_{\lambda_1} \times \cdots \times Z_{\lambda_m} \tag{7.8}$$

where the subspace of $\mathcal{L}(Y)$ generated by the vectors $\delta v_{j,1}, -\delta\Delta_{j,1}, \dots, \delta v_{j,\lambda_j}$ is isomorphic to Z_{λ_j} by the linear map that sends these vectors to the standard frame of the unique 0-dimensional stratum inside Z_{λ_j}.

The above decomposition allows us to understand the local models around y inside all the other strata. Using the local coordinate system around y, we can identify $\mathcal{L}(Y)$ with a small disk inside Z_N, transverse to Y at y and of complementary dimension. The stratification of each Z_{λ_i} induces a product stratification of $\mathcal{L}(Y)$, and the stratification of Z_N, intersected with $\mathcal{L}(Y)$, also produces a stratification of $\mathcal{L}(Y)$. These two stratifications are canonically isomorphic. Indeed, consider another stratum $X = Z(q^-, q^0, q^+; \mu)$ with $Y \leq X$. Then the intersection

$$\mathcal{L}(Y; X) := \mathcal{L}(Y) \cap X$$

is the union of the following strata in the product stratification of $\mathcal{L}(Y)$:

$$Z_{\lambda_1}(q_1^-, q_1^0, q_1^+; \mu^1) \times \cdots \times Z_{\lambda_m}(q_m^-, q_m^0, q_m^+; \mu^m), \qquad q_i^- + q_i^0 + q_i^+ = \lambda_i, \mu^i \in \mathrm{Part}(\lambda_i),$$

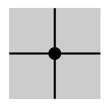
satisfying

$$q^- = p^- + \sum q_i^-, \qquad q^0 = \sum q_i^0, \qquad q^+ = p^+ + \sum q_i^+ \qquad \mu = \mu^1 * \cdots * \mu^m,$$

where $*$ is the concatenation of partitions from (4.2). We call $\mathcal{L}(Y, X)$ the *local model for Y in the normal directions inside X*.

Example 7.8. In Example 7.2, the stratum $Z(0,1,0)$ lives inside the closed stratum $\overline{Z}(0,0,1)$ as $0 \in \mathbb{R}_+$. (Note that for a multifaceted manifold X, the inclusion of any multifacet $\partial_i X \hookrightarrow X$ also has the same local model, but we will treat those differently; in the stratified spaces X that we will construct, the points with the local model $Z(0,1,0) \in \overline{Z}(0,0,1)$ will be part of the special boundary, while points in the multifacets will be part of the ordinary boundary.)

Example 7.9. In Example 7.3, the local model of $Z(0,2,0;(1,1))$ inside Z_2 is same as $\mathbb{R} \times \{0\}$ inside $\mathbb{R} \times Z_1 \times Z_1$, which is a stratified space with four 3-dimensional strata, four 2-dimensional strata, and a unique 1-dimensional stratum:

$\mathbb{R}\times$ 

The local model of $Z(0,2,0;(1,1))$ inside $\overline{Z}(0,0,2)$ is same as $\mathbb{R}\times\{0\}$ inside $\mathbb{R}\times\overline{Z}(0,0,1)\times\overline{Z}(0,1,0) \cong \mathbb{R} \times \mathbb{R}^2_+$, which is one of the four closed 3-dimensional strata of $\mathbb{R} \times Z_1 \times Z_1$. (Note once again, codimension-2 boundary inside a multifaceted manifold has the same local model, but will be treated differently.) Finally, the local model of $Z(0,2,0;(1,1))$ inside $\overline{Z}(0,1,1)$ is same as $\mathbb{R} \times \{0\}$ inside $\mathbb{R} \times \big(Z(0,1,0) \times \overline{Z}(0,0,1) \cup \overline{Z}(0,0,1) \times Z(0,1,0)\big) \cong \mathbb{R} \times \{(x,y) \in \mathbb{R}^2_+ \mid xy = 0\}$, which is the union of two of the four closed 2-dimensional strata of $\mathbb{R} \times Z_1 \times Z_1$.

We can now prove the following.

Proposition 7.10. *The stratification of $Z_N \cong \mathbb{R}^{2N-1}$ described in Section 7.2 is a Whitney stratification.*

Proof. Given the local models from Equation (7.8) and the fact that products of Whitney stratifications are Whitney (cf. Proposition 6.7), it suffices to consider the origin $0 \in Z_N$, and show that all the bigger strata are Whitney regular over it.

Consider the projection

$$\pi\colon \mathbb{C}^N \to Z_N, \quad \pi(z_1,\dots,z_N) = (s_1,\dots,s_N)$$

where s_i are the elementary symmetric polynomials from (7.3). Any stratum $Y \subset Z_N$ is the image under π of a subset of $\mathbb{C}^N$ given by linear equalities and inequalities. Since π is a polynomial mapping, Proposition 6.5 implies that Y is semialgebraic. Proposition 6.6 shows that the bad set $B(\{0\}, Y) \subset \{0\}$ is empty, and therefore Y is Whitney regular over the origin. ☐

7.5. **More general local models.** We now complete the description of the local models that will appear in the stratified spaces in this paper. More generally than Z_N, let $\vec{N} = (N_2,\dots,N_n) \in \mathbb{N}^{n-1}$, and consider the space

$$Z_{\vec{N}} := \left(\mathrm{Sym}^{N_2}(\mathbb{C}) \times \cdots \times \mathrm{Sym}^{N_n}(\mathbb{C})\right)/\mathbb{R}, \tag{7.9}$$

where we divided by the diagonal action of $\mathbb{R}$. If $\vec{N} = 0$, this is a point; otherwise, this is homeomorphic to $\mathbb{R}^{2|\vec{N}|-1}$, and admits a decomposition into strata

$$Z(\vec{p}^{\,-}, \vec{p}^{\,0}, \vec{p}^{\,+}; \vec{\lambda})$$

where $\vec{p}^{\,*} = (p^*_2,\dots,p^*_n)$ for $* \in \{-,0,+\}$ and $\vec{\lambda} = (\lambda_2,\dots,\lambda_n)$ satisfy

$$p_i^- + p_i^0 + p_i^+ = N_i, \;\; \lambda_i \in \mathrm{Part}(p_i^0).$$

Specifically, $Z(\vec{p}^{\,-},\vec{p}^{\,0},\vec{p}^{\,+};\vec{\lambda})$ is given by asking the imaginary parts of the coordinates in each $\mathrm{Sym}^{N_i}(\mathbb{C})$ to consist of p_i^- negative numbers, p_i^0 zeros (with the corresponding real parts decomposed according to the partition λ_i), and p_i^+ positive numbers.

Definition 7.11. The *standard frame* for the normal bundle to a stratum $Z(\vec{p}^{\,-},\vec{p}^{\,0},\vec{p}^{\,+};\vec{\lambda}) \subset Z_{\vec{N}}$ is obtained by concatenating the standard frames for each $Z(p_i^-,p_i^0,p_i^+;\lambda_i) \subset Z_{N_i}$ described in Definition 7.5.

Even more generally, for the local models in Section 8 we will consider products of the ones considered above, as well as half-intervals $[0,\infty)$ that account for the usual (non-special) boundaries of $\langle n\rangle$-manifolds, and $\mathbb{R}$ factors that just correspond to some tangent directions inside the stratum. The most general model is of the form

$$\mathbb{R}^a \times \mathbb{R}_+^{r-1} \times Z_{\vec{N}^1} \times Z_{\vec{N}^2} \times \cdots \times Z_{\vec{N}^r}, \tag{7.10}$$

with the induced product stratification from its factors—where $\mathbb{R}$ has a single stratum, and $\mathbb{R}_+$ has two strata: $\{0\}$ and $(0,\infty)$. The local models will be based on the strata of this space inside the closures of bigger strata.

Note that we can define tailored local coordinates around any point in (7.10), in a manner similar to what we did for Z_N in Section 7.3.

Definition 7.12. The *standard frame* for the normal bundle to a stratum inside the space (7.10) is obtained by concatenating the standard frames for each of the factors, where for $\{0\} \subset \mathbb{R}^+$ we use the standard unit vector, and for the strata in each $Z_{\vec{N}^i}$ we use the frames from Definition 7.11.

Definition 7.13. Given strata

$$Y, X \subset \mathbb{R}^a \times \mathbb{R}_+^{r-1} \times Z_{\vec{N}^1} \times Z_{\vec{N}^2} \times \cdots \times Z_{\vec{N}^r}$$

with $Y \le X$, we let the *local model for Y in the normal directions inside X*, denoted $\mathcal{L}(Y;X)$, be the intersection of X with a small ball in the linear subspace spanned by the standard frame for the normal bundle to Y (in tailored local coordinates around any point of Y). The union of $\mathcal{L}(Y;X)$ over all $X \ge Y$ is denoted $\mathcal{L}(Y)$.

Proposition 7.14. *The stratification from Equation (7.10) is a Whitney stratification.*

Proof. Since the Whitney condition is local, observe that a stratification of a space of the form $V/\mathbb{R}$ (where $\mathbb{R}$ acts freely on V) is Whitney if and only if its pullback to V is Whitney. In Proposition 7.10 we established that the stratification of $Z_N = \mathrm{Sym}^n(\mathbb{C})/\mathbb{R}$ is Whitney. Furthermore, Proposition 6.7 says that the product of Whitney stratifications is Whitney. Combining these facts, we get the conclusion. □

8. Moduli spaces

Let us recall some notation from Section 2.1. Let $O_1,\dots,O_n$ and $X_1,\dots,X_n$ be the markings on the grid. We only consider (positive) domains $D \in \mathscr{D}^+(x,y)$ that do not go over the marking O_1; let $\mathbb{O}(D) \in \mathbb{Z}^{n-1}$ be the vector that records the coefficients of D at the remaining O-markings. For each j, let H_j, respectively V_j, be the horizontal row, respectively vertical column, that contains O_j. The set of periodic domains $\mathscr{P}$ can be identified with $\mathscr{D}(x,x)$ for any x, and we have $\mathscr{P} = \mathbb{Z}\langle H_2,\dots,H_n,V_2,\dots,V_n\rangle$.

Let $\mathscr{T} = \mathscr{T}(\mathbb{G})$ be the set of triples $(D,\vec{N},\vec{\lambda})$ where $D \in \mathscr{D}^+(x,y)$, and

$$\vec{N} = (N_2,\dots,N_n) \in \mathbb{N}^{n-1},$$

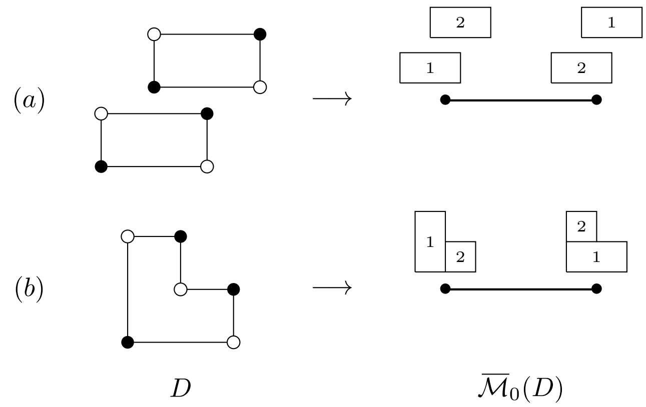


FIGURE 8. Domains of index two on the grid, and the associated moduli spaces $\overline{\mathcal{M}}_0(D)$. On the left hand side, the black dots are part of the initial point x in each domain, and the white dots part of the final point y. The ends of the moduli space correspond to different decompositions $D = D^1 * D^2$. In each picture we indicate D^i with the respective digit $i \in \{1, 2\}$.

$$\vec{\lambda} = (\lambda_2, \dots, \lambda_n), \ \lambda_j \in \mathrm{Part}(N_j).$$

(These are the generators of the chain complex $CDP_*(\mathbb{G})$ from Section 4.2.) For each $(D, \vec{N}, \vec{\lambda}) \in \mathscr{T}$, we will construct a stratified space

$$\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$$

in Section 12. This will come equipped with an embedding in a Euclidean space, be Whitney stratified and hence (by Theorem 6.16) Thom-Mather stratified, and the local models for the stratification will be those considered in Section 7. In this section, we will present some model spaces that could *potentially* play the role of $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$, in a few simple cases, as illustrated in Figures 8–16.

As alluded to in the beginning of Section 4.2, the space $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ will be a model for the compactified moduli space of pseudo-holomorphic strips in $\mathrm{Sym}^n(T^2)$ relative to $\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_n, \mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_n$, modulo translation by $\mathbb{R}$, such that:

- the strips have domain D;
- each strip is equipped with $|\vec{N}| = N_2 + \cdots + N_n$ marked points on the α and β-boundaries, combined into groups of N_j points, $j = 2, \dots, n$. The N_j points in the j^{th} group are meant to be the places where a holomorphic α or β disk-degeneration with domain H_j or V_j has bubbled off;
- for each $j = 2, \dots, n$, the N_j points in the j^{th} group are partitioned according to λ_j. Points in the same part of λ_j are supposed to be at the same height on the boundary of the strip.

See Figure 3.

There is a special case that we will not discuss in this paper, namely when D is trivial (that is, equal to c_x for some $x \in \mathbb{S}$) and $\vec{N} = \vec{0}$. (In that case, the moduli space should be a point divided by a trivial $\mathbb{R}$ action.) From now on we will always assume that at least one of D and $\vec{N}$ is nonzero. Then, the dimension k of $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ will be given by

$$k = \mu(D) - 1 + |\vec{\lambda}| = \mathrm{gr}(D, \vec{N}, \vec{\lambda}) - 1 \tag{8.1}$$

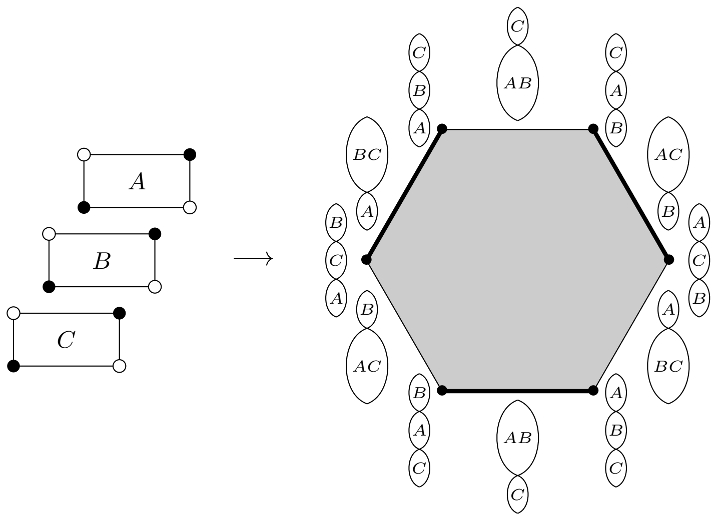


FIGURE 9. A domain of index three on the grid, and the associated moduli spaces $\overline{\mathcal{M}}_0(D)$. For each edge and vertex on the boundary we show the corresponding decomposition $D = D^1 * D^2$ or $D = D^1 * D^2 * D^3$ by a picture of the trajectory breaking. (Holomorphic strips from x to y are drawn vertically, with x at top and y at bottom (and α on the left and β on the right)—for example, the bottom edge corresponds to $(A + B) * C$.) The multifacet $\partial_1\overline{\mathcal{M}}_0(D)$ is made of the thin edges, and the multifacet $\partial_2\overline{\mathcal{M}}_0(D)$ is made of the thick edges.

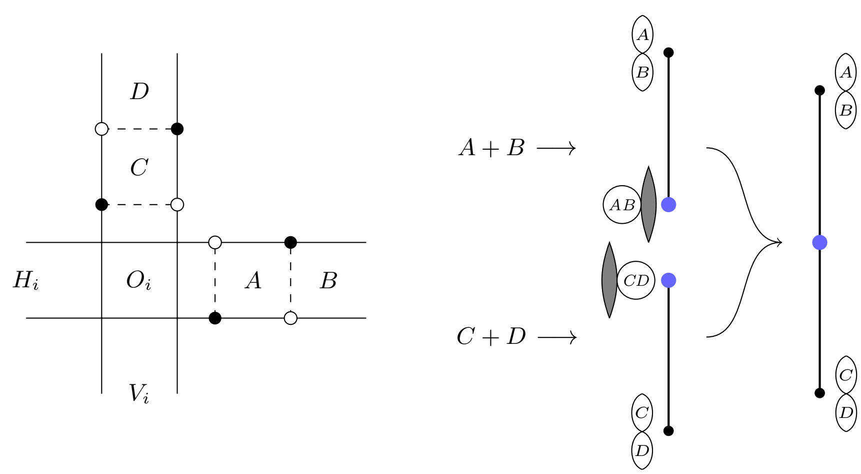


FIGURE 10. Gluing the moduli spaces for the row and the column that contain the same marking O_i. The special boundary points are shown in blue. These blue points are associated to a configuration made of a trivial strip (shown in gray) and a boundary degeneration.

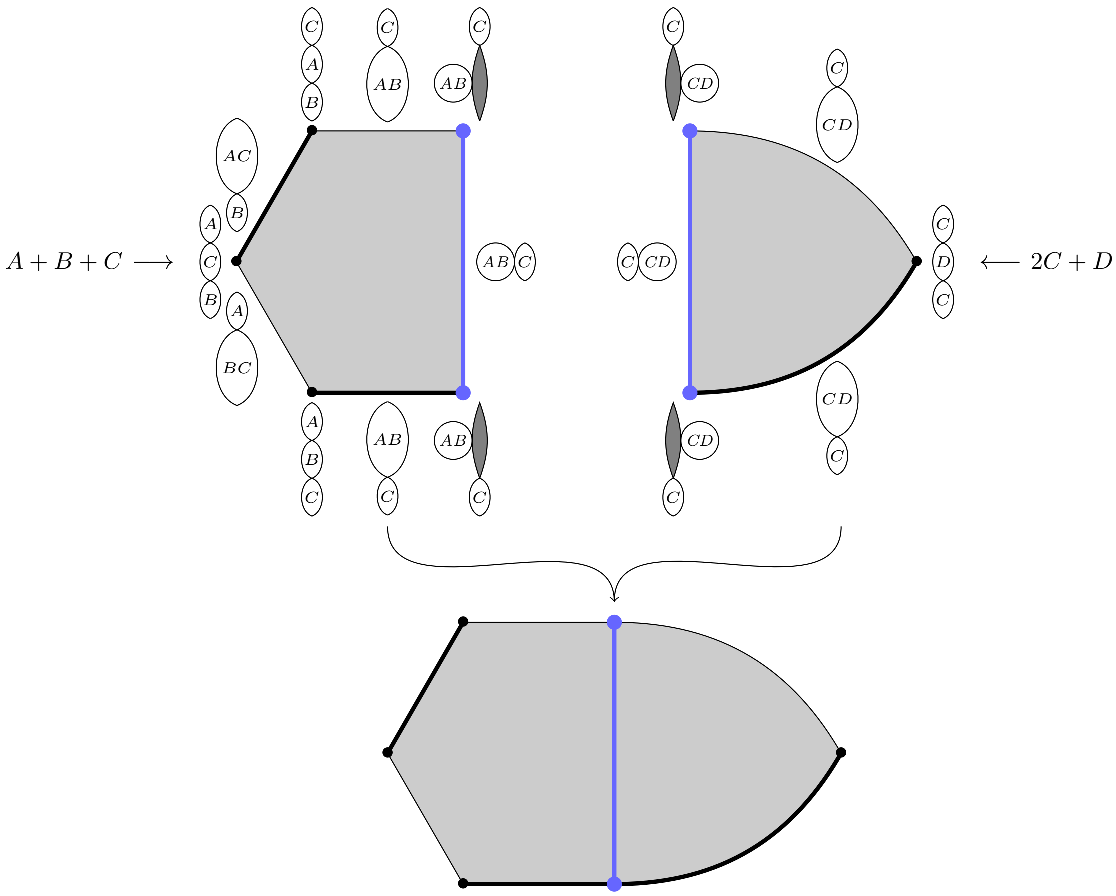


FIGURE 11. Reusing the rectangles A, B, C, D from Figure 10, we glue the moduli spaces $\overline{\mathcal{M}}_0(A+B+C)$ and $\overline{\mathcal{M}}_0(2C+D)$ along their special boundaries (the blue edges) to get a $\langle 2\rangle$-manifold, a 4-gon. As in Figure 9, the thin edges represent $\partial_1\overline{\mathcal{M}}_0$, and the thick edges represent $\partial_2\overline{\mathcal{M}}_0$.

where $\mathrm{gr}(D, \vec{N}, \vec{\lambda})$ is the homological grading of $(D, \vec{N}, \vec{\lambda})$, as an element of the chain complex CDP_*, from Equation (4.4).

As we shall see in later sections, the strata of $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ will be products of lower-dimensional moduli spaces, corresponding to trajectory breaking or bubbling off further α- and β-degenerations. There will be a single codimension-zero stratum in $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$, denoted $\mathcal{M}_{\vec{N},\vec{\lambda}}(D)$. The strata that correspond to some bubbles will comprise what we call the *special boundary* of $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$.

For simplicity, when $\vec{N} = \vec{0}$, we will write

$$\overline{\mathcal{M}}_0(D) \coloneqq \overline{\mathcal{M}}_{\vec{0},\vec{0}}(D).$$

Let us put an equivalence relation on domains by

$$D \sim D' \iff (D - D' \in \mathscr{P} \text{ and } \mathbb{O}(D) = \mathbb{O}(D')) \iff D - D' \in \mathbb{Z}\langle H_2 - V_2, \dots, H_n - V_n\rangle. \tag{8.2}$$

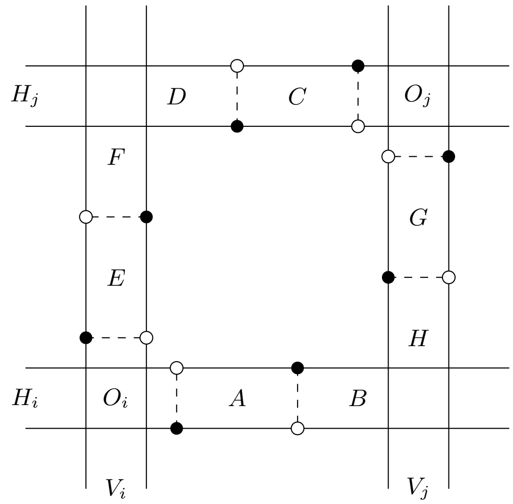


FIGURE 12. Two rows and two columns on the grid.

Note that an equivalence class of domains is specified by the initial and final points of the domain (call them x and y), as well as the vector $\mathbb{O}(D) = (m_2, \dots, m_n)$. Our construction will ensure that the moduli spaces $\overline{\mathcal{M}}_0(D)$, over all D in the same equivalence class, will glue together along their special boundaries, to produce a single $\langle k \rangle$-manifold, which we will denote

$$\overline{\mathcal{M}}([D]) = \overline{\mathcal{M}}(x, U_2^{m_2} \dots U_n^{m_n} y). \tag{8.3}$$

The construction of the stratified spaces $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ will be given in Sections 12. For now, to help the reader get some intuition, we will discuss a few simple cases, and present some examples of spaces that could *potentially* play the role of $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ in those cases. We emphasize that these spaces are not actually what the later constructions will produce. The final constructions will be inductive and hard to make explicit. Rather, the spaces we describe in the examples below are merely some explicit spaces that satisfy the formal properties of $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$. Specifically, they have the right dimension, their strata have the correct local model and are indexed on the different possibilities for trajectory breaking and bubbles, and the spaces corresponding to domains in the same equivalence class can be glued together to form $\langle k \rangle$-manifolds.

Example 8.1. Suppose $D = c_x \in \mathscr{D}^+(x,x)$ is the trivial domain. Then,

$$\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(c_x) = \overline{\Big(\prod_j \mathrm{Sym}^{\ell(\lambda_j)}(\mathbb{R})\Big)/\mathbb{R}},$$

where the compactification is induced from the compactification of $\mathbb{R}$ by $\overline{\mathbb{R}} = \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$.

Example 8.2. Suppose $D \in \mathscr{R}(x,y)$ is a rectangle. Then,

$$\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D) = \overline{\prod_j \mathrm{Sym}^{\ell(\lambda_j)}(\mathbb{R})},$$

with the compactification again induced from the same compactification of $\mathbb{R}$ by $\overline{\mathbb{R}}$. In particular, if $\vec{N} = 0$, $\overline{\mathcal{M}}_0(D)$ is a point.

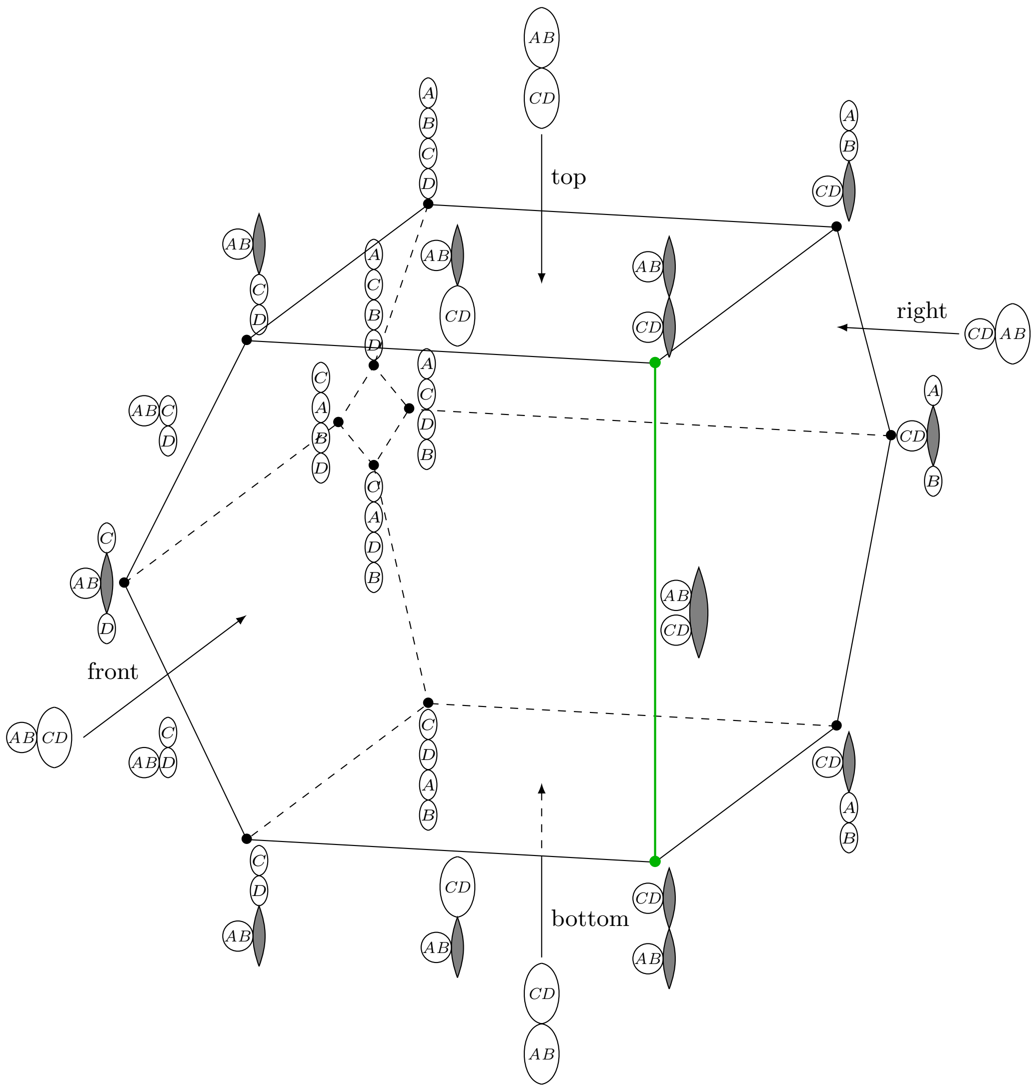


FIGURE 13. For the picture in Figure 12, we show the moduli space for $H_i + H_j = A + B + C + D$. This is a polyhedron with 9 facets; two of these (the front and the right facet) form the special boundary. We show the configurations corresponding to each vertex, and to some of the facets (the top, bottom, right, and front one). We also show the configurations for the five edges along the front facet. (In particular, note that the green edge corresponds to two disk degenerations.) The configurations that correspond to the remaining edges and facets can be easily deduced.

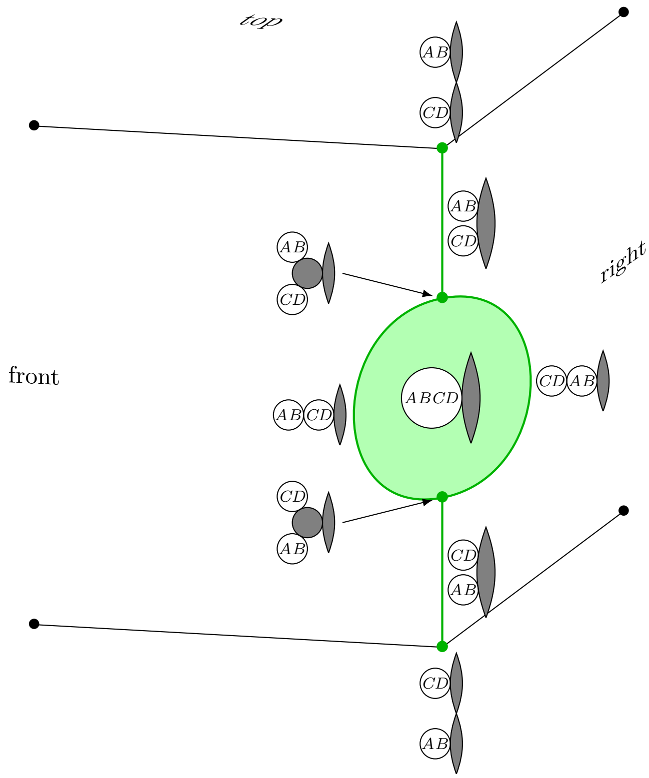


FIGURE 14. If we consider the actual Gromov compactification of the moduli space, then the green edge from Figure 13 gets replaced by more complicated spaces of bubble trees, as shown here. We have shown the labels of the green vertices, the green edges, and the green facet. (The other labels are same as in Figure 13.)

Example 8.3. Let D be a positive domain of index two on the grid, that is, either: (a) the union of two rectangles or (b) an L-shaped hexagon, as discussed in Item (G-13e) in Section 2.1. Then $\overline{\mathcal{M}}_0(D)$ is an interval, which can be viewed as a 1-dimensional $\langle 1\rangle$-manifold. The two ends correspond to the different ways of splitting D into two domains of index one (trajectory breaking). See Figure 8.

Example 8.4. More generally, suppose D is a positive domain of index $k+1$ that does not contain any horizontal annulus H_i or any vertical annulus V_j, so that α- and β-degenerations are impossible. Then $\overline{\mathcal{M}}_0(D)$ is a k-dimensional $\langle k\rangle$-manifold, with the boundary corresponding to trajectory breaking. The i-colored multifacet $\partial_i(\overline{\mathcal{M}}_0(D))$ (for $i=1,\dots,k$) corresponds to splittings of D the form $D^1 * D^2$, where $\mu(D^1)=i$ and $\mu(D^2)=k+1-i$. See Figure 9 for a picture of $\overline{\mathcal{M}}_0(D)$ for an index three domain. In general, to an index $k+1$ domain made of $k+1$ disjoint rectangles one can associate the k-dimensional permutohedron (cf. Example 5.2). Other types of domains yield other $\langle k\rangle$-manifolds.

Example 8.5. When H_i is a full row and $\vec{N}=0$, we let $\overline{\mathcal{M}}_0(H_i)$ be an interval, where one end corresponds to the decomposition into two rectangles and the other end is the special boundary, corresponding to an α-degeneration. For the column V_i that contains the same O_i marking as

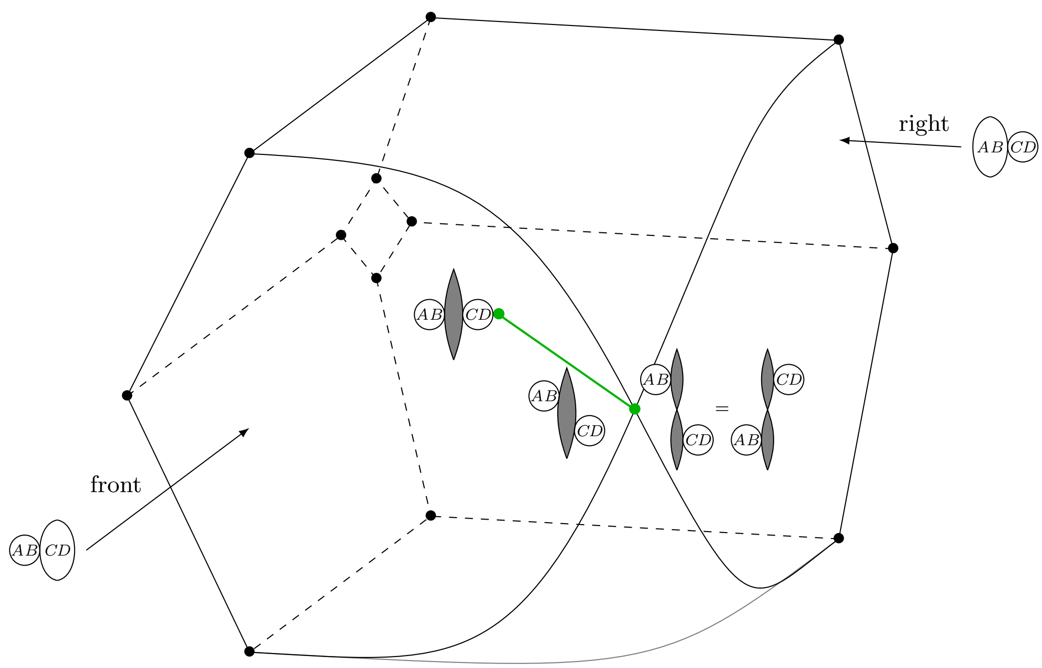


FIGURE 15. The moduli space for $H_i + V_i = A + B + C + D$, where H_i and V_i are as in Figure 10. This is obtained from the polyhedron in Figure 13 by folding the green edge in half. (This can be visualized by pushing the midpoint of the green edge of Figure 13 towards the interior until it folds into half.) The front, bottom, top and right facets from Figure 13 now meet at a single point (the right green dot); moreover, the two edges on the top facet and the two edges on the bottom facet enter this vertex from exactly opposite directions. For simplicity, we only show the configurations for the green edge, its two endpoints, and the two facets that form the special boundary (the front and the right facet, which meet along the green edge from exactly opposite directions). The other labels are just as in Figure 13, except that the CD disk degenerations are now to the right of the strips. Note that the moduli space shown here is not a convex polyhedron, but rather a stratified space, where the local picture near the left green dot is the Whitney umbrella from Figure 7.

H_i, $\overline{\mathcal{M}}_0(V_i)$ is another interval, also with one special boundary point, this time corresponding to a β-degeneration. Gluing $\overline{\mathcal{M}}_0(H_i)$ to $\overline{\mathcal{M}}_0(V_i)$ along their special boundaries yields the $\langle 1\rangle$-manifold $\overline{\mathcal{M}}([H_i]) = \overline{\mathcal{M}}([V_i])$. See Figure 10.

Example 8.6. Figure 11 shows the spaces $\overline{\mathcal{M}}_0$ for two domains of index three: the column $V_i = C+D$ plus a rectangle C contained it, and the row $H_i = A+B$ plus the disjoint rectangle C. These domains would be glued together to produce the $\langle 2\rangle$-manifold $\overline{\mathcal{M}}([A+B+C]) = \overline{\mathcal{M}}([2C+D])$.

Example 8.7. Suppose we have rows $H_i = A+B$, $H_j = C+D$, as well as columns $V_i = E+F$, $V_j = G+H$ as in Figure 12. Then, the moduli space $\overline{\mathcal{M}}_0(H_i+H_j)$ is shown in Figure 13, with special boundary consisting of the front and the right facet; those for the domains H_i+V_j, V_i+H_j

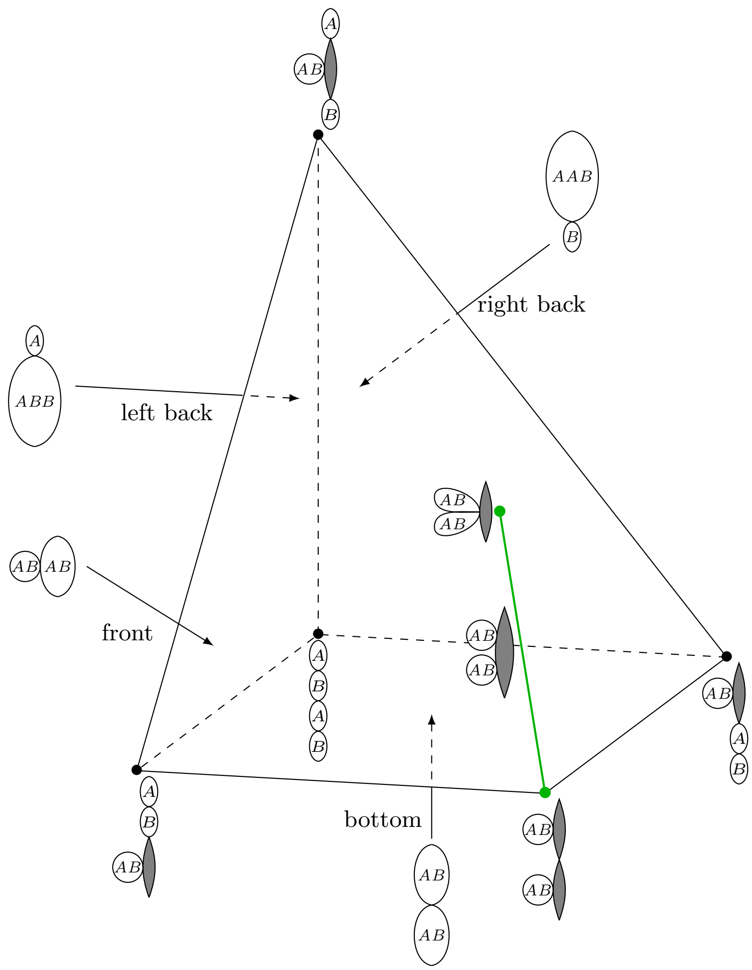


FIGURE 16. The moduli space for $2H_i = 2A + 2B$, where H_i is as in Figure 10. This is obtained from a convex pyramid (with a quadrilateral base) by smoothing along the top half of the front edge, and pulling the midpoint of that edge (the top green dot) outwards, so that the local model of the top green dot inside the space is like $Z(0, 2, 0; (2)) \subset \overline{Z}(2, 0, 0)$ from Figure 7. We labeled the configurations corresponding to each vertex and facet, as well as that for the green edge. The labels on the other edges can be easily deduced.

and $V_i + V_j$ are very similar. These four spaces glue together along their special boundaries to form the $\langle 3\rangle$-manifold $\overline{\mathcal{M}}_0([H_i + H_j])$, which is a three-dimensional permutohedron. Note that in this gluing, the special edge drawn in green in Figure 13 is common to all four polyhedra.

Remark 8.8. In symplectic geometry we encounter moduli spaces of bubble trees that we do not consider here. For example, in Figure 13 the green edge corresponds to two disk degenerations, and as we move along the edge we change the relative heights where these two degenerations take place. In particular, there is a point in the middle that corresponds to the two degenerations happening at the same height. If we were to actually consider the Gromov compactification from symplectic geometry, instead of that point we would have a whole new (two-dimensional) facet, corresponding to degenerating an index four disk with domain $H_i + H_j = A + B + C + D$, as in Figure 14. Thus, our moduli space compactifications are different from the usual compactifications in symplectic geometry—ours is a quotient of the usual one, while the usual one is a blowup of ours. We do not delve into this issue further since in this paper we do not construct the actual moduli spaces from holomorphic geometry, but rather construct their models inductively by obstruction theory.

Example 8.9. Let us return to the domains from Figure 10, where $H_i = A+B$ is a row and $V_i = C+D$ is a column. The space $\overline{\mathcal{M}}_0$ for the domain $H_i + V_i = A + B + C + D$ made of a row and a column is shown in Figure 15. In fact, it is almost the same polyhedron as in Figure 13, except that the green edge is folded in half. The folding is due to the fact that since H_i and V_i go through the same O_i marking, we want to identify the AB and CD disk degenerations. Thus, the point on the green edge where the AB degeneration is at a certain distance up from the CD degeneration, is identified with the point where the CD degeneration is on top of AB, at the same distance.

To be more precise, with the notation from Example 8.1, the green line in Figure 13 is the space $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(0)$ with $\vec{N}$ being the vector with 1's in positions i and j (and 0 otherwise), and $\vec{\lambda}$ the unique possible vector of partitions. This space is the compactification of

$$(\mathrm{Sym}^1(\mathbb{R}) \times \mathrm{Sym}^1(\mathbb{R}))/\mathbb{R},$$

which is just $\overline{\mathbb{R}} = \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$.

On the other hand, the (folded) green line in Figure 15 is $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(0)$ with $\vec{N}$ being the vector with a single 2 in position i, and $\vec{\lambda}$ consisting of trivial partitions except for $\lambda_i = (1, 1)$. This is the compactification of

$$\mathrm{Sym}^2(\mathbb{R})/\mathbb{R}.$$

In particular, there is a special point (the left green dot in Figure 15) where the AB and CD degenerations happen at the same height. The local model of special point inside the space $\overline{\mathcal{M}}_0(H_i + V_i)$ is

$$Z(0, 2, 0; (2)) \subset \overline{Z}(1, 0, 1)$$

from Figure 7. The green line corresponds to the thickened line in Figure 7, and the front and right facets in Figure 15 meet along the green line, forming a Whitney umbrella.

Example 8.10. Reusing the same configuration from Figure 10, we now consider the domain $2H_i = 2A + 2B$ (a row with multiplicity two). The corresponding space $\overline{\mathcal{M}}_0(2H_i)$ is pictured in Figure 16. This is glued with a similar space $\overline{\mathcal{M}}_0(2V_i)$, as well as with the space $\overline{\mathcal{M}}_0(H_i+V_i)$ from Example 8.9, to yield a single $\langle 3\rangle$-manifold $\overline{\mathcal{M}}_0([2H_i])$. Around the top green dot, the gluing is modeled on the Whitney umbrella from Figure 7, with $\overline{\mathcal{M}}_0(2H_i)$, $\overline{\mathcal{M}}_0(H_i + V_i)$ and $\overline{\mathcal{M}}_0(2V_i)$ playing the roles of $Z(2, 0, 0)$, $Z(1, 0, 1)$ and $Z(0, 0, 2)$, respectively.

9. The stratification

We now describe the intended stratification of the spaces $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$, where $(D, \vec{N}, \vec{\lambda}) \in \mathscr{T}$. We will ensure that these spaces have a unique codimension zero stratum, which we denote $\mathcal{M}_{\vec{N},\vec{\lambda}}(D)$, and we let $\partial\mathcal{M}_{\vec{N},\vec{\lambda}}(D)$ denote $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D) \setminus \mathcal{M}_{\vec{N},\vec{\lambda}}(D)$. (These spaces are not topological manifolds in general, and so this is not the boundary in any usual sense.)

9.1. **Enumeration of strata.** We ask that $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ has the following strata:

$$\mathcal{M}_{\vec{N}^1+\mathbb{O}(E^1)+\mathbb{O}(F^1),\vec{\lambda}^1}(D^1)\times\cdots\times\mathcal{M}_{\vec{N}^r+\mathbb{O}(E^r)+\mathbb{O}(F^r),\vec{\lambda}^r}(D^r), \tag{9.1}$$

with $r\geq 1$ and generators $x=w_0,w_1,\dots,w_{r-1},w_r=y$, such that for each $1\leq i\leq r$,

$$D^i\in\mathscr{D}^+(w_{i-1},w_i),$$

$$E^i=\sum_{j=2}^n O_j(E^i)H_j\in\mathscr{P}^+\text{ is a sum of rows,}$$

$$F^i=\sum_{j=2}^n O_j(F^i)V_j\in\mathscr{P}^+\text{ is a sum of columns,}$$

satisfying

$$\sum_i(D^i+E^i+F^i)=D.$$

Further,

$$\vec{N}^i=(N_2^i,\dots,N_n^i)\in\mathbb{N}^{n-1},\quad\sum_i\vec{N}^i=\vec{N},$$

and

$$\vec{\lambda}^i=(\lambda_2^i,\dots,\lambda_n^i),\quad\lambda_j^i\in\mathrm{Part}(N_j^i+O_j(E^i)+O_j(F^i))$$

are such that there exist some other partitions

$$\vec{\eta}^i=(\eta_2^i,\dots,\eta_n^i),\quad\eta_j^i\in\mathrm{Part}(N_j^i) \tag{9.2}$$

with $\eta_j^i\geq\lambda_j^i$ (in the notation of Section 7) and

$$\eta_j^1*\cdots*\eta_j^r=\lambda_j,\ j=2,\dots,n.$$

Here, $*$ is the concatenation of partitions defined in Equation (4.2). Note that, if $\vec{\eta}^i$ exist, then they are unique. This is because an ordered partition can be uniquely (if at all) decomposed as a concatenation of partitions of specified sizes.

A few explanations are in order. In the description of the strata, the D^i's are the pieces in the trajectory breaking, the E^i's correspond to α-boundary degenerations, and the F^i's to β-boundary degenerations. The points where the boundary degenerations through O_j are attached were originally partitioned according to λ_j. When the trajectory breaks into r pieces, these points get split into r groups, where the i^{th} group is partitioned according to η_j^i. Since we also pick up extra boundary degenerations from the E^i and F^i, we add more points. We could also join some of the parts, to make the partition η_j^i coarser, since this is what happens in lower dimensional strata; compare Equations (7.2) and (7.5). The result of this process is the partition $\lambda_j^i\leq\eta_j^i$.

The strata of $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ are required to satisfy the following coherence relations with respect to their closures. Given a stratum as in Equation (9.1), its closure in $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ should be the product of the closures of its factors:

$$\overline{\mathcal{M}}_{\vec{N}^1+\mathbb{O}(E^1)+\mathbb{O}(F^1),\vec{\lambda}^1}(D^1)\times\cdots\times\overline{\mathcal{M}}_{\vec{N}^r+\mathbb{O}(E^r)+\mathbb{O}(F^r),\vec{\lambda}^r}(D^r). \tag{9.3}$$

Further, if for $i=1,\dots,r$ we have strata

$$\prod_{k=1}^{m_i}\mathcal{M}_{\vec{N}^{i,k}+\mathbb{O}(E^{i,k})+\mathbb{O}(F^{i,k}),\vec{\lambda}^{i,k}}(D^{i,k})\subset\overline{\mathcal{M}}_{\vec{N}^i+\mathbb{O}(E^i)+\mathbb{O}(F^i),\vec{\lambda}^i}(D^i)$$

we ask that the inclusion of the product stratum

$$\prod_{k=1}^{m_1} \mathcal{M}_{\vec{N}^{1,k}+\mathbb{O}(E^{1,k})+\mathbb{O}(F^{1,k}),\vec{\lambda}^{1,k}}(D^{1,k}) \times \cdots \times \prod_{k=1}^{m_r} \mathcal{M}_{\vec{N}^{r,k}+\mathbb{O}(E^{r,k})+\mathbb{O}(F^{r,k}),\vec{\lambda}^{r,k}}(D^{r,k})$$

into $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ factors through Equation (9.3).

Let us also check that Equation (9.3) ensures that $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ has a unique codimension zero stratum. Using the dimension formula (8.1), the codimension of the stratum described in Equation (9.1) is

$$r-1+\sum_{i=1}^{r}\mu(E^i)+\sum_{i=1}^{r}\mu(F^i)+\sum_{j=2}^{n}\Big(\ell(\lambda_j)-\sum_{i=1}^{r}\ell(\lambda^i_j)\Big). \tag{9.4}$$

We can write each of the last summands as

$$\ell(\lambda_j)-\sum_{i=1}^{r}\ell(\lambda^i_j)=\sum_{i=1}^{r}(\ell(\eta^i_j)-\ell(\lambda^i_j)).$$

Using the co-length inequality (4.3), we have

$$\ell(\eta^i_j)-\ell(\lambda^i_j)\geq -O_j(E^i)-O_j(F^i).$$

Since $\mu(E^i)=2\sum_j O_j(E^i)$ and $\mu(F^i)=2\sum_j O_j(F^i)$, we deduce that the codimension given in Equation (9.4) is at least

$$r-1+\sum_{i=1}^{r}\sum_{j=2}^{n}\big(O_j(E^i)+O_j(F^i)\big)\geq r-1.$$

In particular, $\mathcal{M}_{\vec{N},\vec{\lambda}}(D)$ appears as the unique codimension zero stratum, with $r=1$, $D^1=D$, $E^1=F^1=0, \vec{\lambda}^1=\vec{\eta}^1=\vec{\lambda}$.

Example 9.1. If we consider the domain $2H_i=2A+2B$ from Example 8.10, the moduli space $\overline{\mathcal{M}}_0(2H_i)$ has the following (open) strata:

- Dimension 3: $\mathcal{M}_0(2H_i)$;
- Dimension 2: $\mathcal{M}_0(A)\times\mathcal{M}_0(A+2B)$, $\mathcal{M}_0(2A+B)\times\mathcal{M}_0(B)$, $\mathcal{M}_0(H_i)\times\mathcal{M}_0(H_i)$, $\mathcal{M}_{\mathbb{O}(H_i),(1)_i}(H_i)$;
- Dimension 1: $\mathcal{M}_0(A)\times\mathcal{M}_0(B)\times\mathcal{M}_0(H_i)$, $\mathcal{M}_0(A)\times\mathcal{M}_0(H_i)\times\mathcal{M}_0(B)$, $\mathcal{M}_0(H_i)\times\mathcal{M}_0(A)\times\mathcal{M}_0(B)$, $\mathcal{M}_0(H_i)\times\mathcal{M}_{\mathbb{O}(H_i),(1)_i}(c_x)$, $\mathcal{M}_{\mathbb{O}(H_i),(1)_i}(c_x)\times\mathcal{M}_0(H_i)$, $\mathcal{M}_0(A)\times\mathcal{M}_{\mathbb{O}(H_i),(1)_i}(B)$, $\mathcal{M}_{\mathbb{O}(H_i),(1)_i}(A)\times\mathcal{M}_0(B)$, $\mathcal{M}_{\mathbb{O}(2H_i),(1,1)_i}(c_x)$;
- Dimension 0: $\mathcal{M}_0(A)\times\mathcal{M}_0(B)\times\mathcal{M}_0(A)\times\mathcal{M}_0(B)$, $\mathcal{M}_0(A)\times\mathcal{M}_0(B)\times\mathcal{M}_{\mathbb{O}(H_i),(1)_i}(c_x)$, $\mathcal{M}_0(A)\times\mathcal{M}_{\mathbb{O}(H_i),(1)_i}(c_y)\times\mathcal{M}_0(B)$, $\mathcal{M}_{\mathbb{O}(H_i),(1)_i}(c_x)\times\mathcal{M}_0(A)\times\mathcal{M}_0(B)$, $\mathcal{M}_{\mathbb{O}(H_i),(1)_i}(c_x)\times\mathcal{M}_{\mathbb{O}(H_i),(1)_i}(c_x)$, $\mathcal{M}_{\mathbb{O}(2H_i),(2)_i}(c_x)$.

See also Figure 16.

Remark 9.2. In the above example, we denoted by $(\lambda)_i$ the vector consisting of a partition λ in position i, and trivial partitions elsewhere.

Using this stratification on the moduli spaces $\overline{\mathcal{M}}(D,\vec{N},\vec{\lambda})$, we get the following partial order on the set $\mathscr{T}(\mathbb{G})$. Declare

$$(D',\vec{N}',\vec{\lambda}')\leq(D,\vec{N},\vec{\lambda}) \tag{9.5}$$

if $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ has an open stratum $\prod_i \mathcal{M}_{\vec{N}^i+\mathbb{O}(\mathbb{E}^i)+\mathbb{O}(F^i),\vec{\lambda}^i}(D^i)$ with $(D',\vec{N}',\vec{\lambda}')=(D^i,\vec{N}^i+\mathbb{O}(\mathbb{E}^i)+\mathbb{O}(F^i),\vec{\lambda}^i)$ for some i. This is easily seen to be a partial order: for anti-reflexivity, note that if $(D',\vec{N}',\vec{\lambda}')<(D,\vec{N},\vec{\lambda})$, then $\mathrm{gr}(D',\vec{N}',\vec{\lambda}')<\mathrm{gr}(D,\vec{N},\vec{\lambda})$. (This partial order is related to the chain complex CDP_* from Section 4.2. If $(D',\vec{N}',\vec{\lambda}')$ appears in $\delta(D,\vec{N},\vec{\lambda})$ in any of the terms $\delta^{\mathrm{I}},\delta^{\mathrm{II}},\delta^{\mathrm{III}},\delta^{\mathrm{IV}}$ (from Equation (4.6)–(4.9)), then $(D',\vec{N}',\vec{\lambda}')<(D,\vec{N},\vec{\lambda})$ with $\mathrm{gr}(D',\vec{N}',\vec{\lambda}')=\mathrm{gr}(D,\vec{N},\vec{\lambda})-1$; indeed, the partial order from Equation (9.5) is generated by these atomic relations.)

9.2. **Codimension one strata.** For future reference, let us also describe the codimension-one strata of $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$. Using Formula (9.4), a similar analysis shows that this consists of strata of three possible types, corresponding to trajectory breaking, boundary degeneration (Type II), and coarsening of partitions (Type III); in order to stress the correspondence with the different types of terms in the differential δ on CDP_* (cf. Remark 9.3), we have subdivided trajectory breaking into two further subtypes (Type I and IV).

Type I codimension-one strata correspond to $r=2$ and pure trajectory breaking into two non-constant domains, with no additional disk degenerations ($E^i=F^i=0$), and no coarsening of the partitions:

$$\mathcal{M}_{\vec{N}^1,\vec{\lambda}^1}(D^1)\times\mathcal{M}_{\vec{N}^2,\vec{\lambda}^1}(D^2),$$

where

- w is an intermediate generator,
- $D^1\in\mathscr{D}^+(x,w)$ and $D^2\in\mathscr{D}^+(w,y)$ are such that $D^1+D^2=D$, and neither D^1 nor D^2 is trivial,
- $\vec{N}^1,\vec{N}^2\in\mathbb{N}^{n-1}$ are such that $\vec{N}^1+\vec{N}^2=\vec{N}$,
- $\vec{\lambda}^i=(\lambda_2^i,\dots,\lambda_n^i), i=1,2$, are vectors of partitions such that $\lambda_j^1*\lambda_j^2=\lambda_j$ for all $j=2,\dots,n$.

Note that, among these strata, the terms where one of the two factors is zero-dimensional are when either D^1 or D^2 is a rectangle, and the partition corresponding to that rectangle is empty:

$$\mathcal{M}_{\vec{N},\vec{\lambda}}(D^1)\times\mathcal{M}_0(R),\ \text{ with } D^1\in\mathscr{D}^+(x,w), R\in\mathscr{R}(w,y), D^1+R=D,$$
$$\mathcal{M}_0(R)\times\mathcal{M}_{\vec{N},\vec{\lambda}}(D^2),\ \text{ with } R\in\mathscr{R}(x,w), D^2\in\mathscr{D}^+(w,y), R+D^2=D.$$

Type II codimension-one strata correspond to no trajectory breaking ($r=1$) and a single boundary degeneration (with domain row H_j or column V_j for some $2\leq j\leq n$), and no coarsening of the partitions:

$$\mathcal{M}_{\vec{N}+\vec{e}_j,\vec{\lambda}'}(D^1), \text{ with } D^1+H_j=D,$$
$$\mathcal{M}_{\vec{N}+\vec{e}_j,\vec{\lambda}'}(D^1), \text{ with } D^1+V_j=D,$$

where

$$\vec{\lambda}'=(\lambda_2',\dots,\lambda_n')$$

is such that $\lambda_j'\in\mathrm{UE}(\lambda_j)$, and $\lambda_s'=\lambda_s$ for all $s\neq j$. Here, $\mathrm{UE}(\lambda_j)$ is the set of unit enlargements of λ_j (cf. Definition 4.2).

Type III codimension-one strata correspond to no trajectory breaking ($r=1$) and no boundary degenerations, but rather an elementary coarsening of a partition λ_j (for some j):

$$\mathcal{M}_{\vec{N},\vec{\lambda}'}(D),\ \text{ with } \vec{\lambda}'=(\lambda_2',\dots,\lambda_n'),$$

where $\lambda_j'\in\mathrm{EC}(\lambda_j)$, and $\lambda_s'=\lambda_s$ for all $s\neq j$. Here, $\mathrm{EC}(\lambda_j)$ is the set of elementary coarsenings (cf. Definition 4.1).

Type IV codimension-one strata correspond to pure trajectory breaking ($r=2$) where one of the domains is constant, but with no additional disk degenerations ($E^i = F^i = 0$), and no coarsening of the partitions:

$$\mathcal{M}_{\vec{N}^1,\vec{\lambda}^1}(c_x) \times \mathcal{M}_{\vec{N}^2,\vec{\lambda}^1}(D),$$
$$\mathcal{M}_{\vec{N}^1,\vec{\lambda}^1}(D) \times \mathcal{M}_{\vec{N}^2,\vec{\lambda}^1}(c_y),$$

where

- $\vec{N}^1, \vec{N}^2 \in \mathbb{N}^{n-1}$ are such that $\vec{N}^1 + \vec{N}^2 = \vec{N}$,
- $\vec{\lambda}^i = (\lambda_2^i, \dots, \lambda_n^i), i = 1, 2$, are vectors of partitions such that $\lambda_j^1 * \lambda_j^2 = \lambda_j$ for all $j = 2, \dots, n$.

Note that, among these strata, the terms where one of the two factors is zero-dimensional are when the vector of partitions corresponding to the constant domain consists of empty partitions and a single length-one partition:

$$\mathcal{M}_{N^1\vec{e}_j,(N^1)_j}(c_x) \times \mathcal{M}_{\vec{N}^2,\vec{\lambda}^2}(D),$$
$$\mathcal{M}_{\vec{N}^1,\vec{\lambda}^1}(D) \times \mathcal{M}_{N^2\vec{e}_j,(N^2)_j}(c_y).$$

Here, N^1, N^2 (when written without the vector symbols) are natural numbers, and $(N^1)_j, (N^2)_j$ denote vectors of partitions as in Remark 9.2.

Remark 9.3. The different types of strata correspond to different kinds of terms in the differential δ on the complex CDP_*; cf. Section 4.2. Type I strata, where one of the factors is zero dimensional, correspond to terms of δ of Type I; Type II corresponds to Type II; Type III to Type III; and Type IV, where one of the factors is zero dimensional, to Type IV.

Definition 9.4. Let $X = \overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ be of dimension k. Define its *thick dimension* by the equation

$$\operatorname{tdim} \overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D) = \mu(D) - 1 + 2|\vec{N}|. \tag{9.6}$$

(This definition is justified in the next section.) Let $l = \operatorname{tdim} \overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$. By analogy with the notation for $\langle n \rangle$-manifolds in Section 5, for $i = 1, \dots, l$, we let $\partial_i X$ be the closure of the union of all codimension-one strata of Types I and IV of the form

$$\mathcal{M}_{\vec{N}^1,\vec{\lambda}^1}(D^1) \times \mathcal{M}_{\vec{N}^2,\vec{\lambda}^1}(D^2)$$

with

$$\operatorname{tdim} \mathcal{M}_{\vec{N}^1,\vec{\lambda}^1}(D^1) = i - 1.$$

We also let the *special boundary* of X, denoted $\partial_s X$, be the closure of the union of all codimension-one strata of Types II and III.

It is easy to see that every higher codimension stratum is contained in the closure of a codimension-one stratum. Therefore, altogether, the boundary of X is

$$\partial X = (\partial_1 X \cup \cdots \cup \partial_l X) \cup \partial_s X.$$

9.3. **Local models.** Let us describe the local models for how the strata from Equation (9.1) should live inside the moduli spaces $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$. Note that every $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ is itself a stratum of a space of the form

$$\overline{\mathcal{M}}_0(\widetilde{D}),$$

with $\widetilde{D} = D + \widetilde{E} + \widetilde{F}, \widetilde{E} \in \mathscr{P}^+$ a sum of rows, $\widetilde{F} \in \mathscr{P}^+$ a sum of columns, $\mathbb{O}(\widetilde{E}) + \mathbb{O}(\widetilde{F}) = \vec{N}$.

There are several possible choices of such $\widetilde{D}$, depending on the choice of $\widetilde{E}, \widetilde{F}$; indeed, if $\vec{N} = (N_2, \dots, N_n)$, there are exactly $(N_2+1)(N_3+1)\cdots(N_n+1)$ such choices. (For example, for the green line $\mathcal{M}_{2\vec{e}_i,(1,1)_i}(c_x)$ in the Whitney umbrella from Figures 15 and 16 we have three such choices, $2H_i, H_i+V_i, 2V_i$.)

The dimension of each $\overline{\mathcal{M}}_0(\widetilde{D})$ is given by the thick dimension $l = \operatorname{tdim} \overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$. The different possible $\widetilde{D}$ are in the same equivalence class $[\widetilde{D}]$. As mentioned in Equation (8.3), we will glue these $\overline{\mathcal{M}}_0(\widetilde{D})$ along their special boundaries to form an l-dimensional $\langle l\rangle$-manifold

$$\overline{\mathcal{M}}([\widetilde{D}]) = \bigcup \overline{\mathcal{M}}_0(\widetilde{D}),$$

with $\partial_i\overline{\mathcal{M}}([\widetilde{D}]) = \bigcup \partial_i\overline{\mathcal{M}}_0(\widetilde{D})$, for each $i = 1, 2, \dots, l$. That is, after gluing, $\overline{\mathcal{M}}([\widetilde{D}])$ no longer has any special boundary, and ∂_i corresponds precisely to trajectory breaking into $(i-1)$ and $(l-i)$-dimensional pieces:

$$\partial_i\overline{\mathcal{M}}([\widetilde{D}]) = \coprod_{\substack{[\widetilde{D}^1]+[\widetilde{D}^2]=[\widetilde{D}] \\ \operatorname{tdim}\mathcal{M}([\widetilde{D}^1])=i-1}} \overline{\mathcal{M}}([\widetilde{D}^1]) \times \overline{\mathcal{M}}([\widetilde{D}^2]).$$

Recall from Section 6.2 that if we specify a tubular neighborhood T_X of a stratum X inside a stratified space, the tubular neighborhoods $\overline{T}_{X,Y}$ of X inside other closed strata $\overline{Y}$ are just given by intersecting T_X with $\overline{Y}$. Thus, to understand the local model of a stratum inside $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$, it suffices to consider its local model inside the bigger space $\overline{\mathcal{M}}_0(\widetilde{D})$.

Using the notation of Section 7.5, we ask that the local model in the normal directions for the stratum

$$X = \mathcal{M}_{\vec{N}^1+\mathbb{O}(E^1)+\mathbb{O}(F^1),\vec{\lambda}^1}(D^1) \times \cdots \times \mathcal{M}_{\vec{N}^r+\mathbb{O}(E^r)+\mathbb{O}(F^r),\vec{\lambda}^r}(D^r)$$

inside $\overline{\mathcal{M}}_0(\widetilde{D})$ is the same as the local model for

$$\mathbb{R}^a \times \{0\} \times Z(0, \vec{N}^1 + \mathbb{O}(E^1) + \mathbb{O}(F^1), 0; \vec{\lambda}^1) \times \cdots \times Z(0, \vec{N}^r + \mathbb{O}(E^r) + \mathbb{O}(F^r), 0; \vec{\lambda}^r)$$

inside the union of

$$\mathbb{R}^a \times \mathbb{R}_+^{r-1} \times \overline{Z}(\mathbb{O}(E^1) + \mathbb{O}(\widetilde{E}^1), 0, \mathbb{O}(F^1) + \mathbb{O}(\widetilde{F}^1)) \times \cdots \times \overline{Z}(\mathbb{O}(E^r) + \mathbb{O}(\widetilde{E}^r), 0, \mathbb{O}(F^r) + \mathbb{O}(\widetilde{F}^r)),$$

over all possible choices of $\widetilde{E}^i \in \mathscr{P}^+$ a sum of rows and $\widetilde{F}^i \in \mathscr{P}^+$ a sum of columns satisfying

$$\mathbb{O}(\widetilde{E}^i) + \mathbb{O}(\widetilde{F}^i) = \vec{N}^i, \quad \sum_i \widetilde{E}^i = \widetilde{E}, \quad \sum_i \widetilde{F}^i = \widetilde{F}.$$

Here a is given by

$$\begin{aligned} a &= \dim X - \sum_i \dim Z(0, \vec{N}^i + \mathbb{O}(E^i) + \mathbb{O}(F^i), 0; \vec{\lambda}^i) \\ &= \sum_i \mu(D^i) + \#\{i \mid \vec{N}^i + \mathbb{O}(E^i) + \mathbb{O}(F^i) = 0\}. \end{aligned}$$

Since $\overline{\mathcal{M}}([\widetilde{D}]) = \bigcup \overline{\mathcal{M}}_0(\widetilde{D})$, we will require that the local model in the normal directions for the stratum X in $\overline{\mathcal{M}}([\widetilde{D}])$ is the union of the above local models, which is same as the local model for

$$\mathbb{R}^a \times \{0\} \times Z(0, \vec{N}^1 + \mathbb{O}(E^1) + \mathbb{O}(F^1), 0; \vec{\lambda}^1) \times \cdots \times Z(0, \vec{N}^r + \mathbb{O}(E^r) + \mathbb{O}(F^r), 0; \vec{\lambda}^r)$$

inside

$$\mathbb{R}^a \times \mathbb{R}_+^{r-1} \times Z_{\vec{N}^1+\mathbb{O}(E^1)+\mathbb{O}(F^1)} \times \cdots \times Z_{\vec{N}^1+\mathbb{O}(E^1)+\mathbb{O}(F^1)}.$$

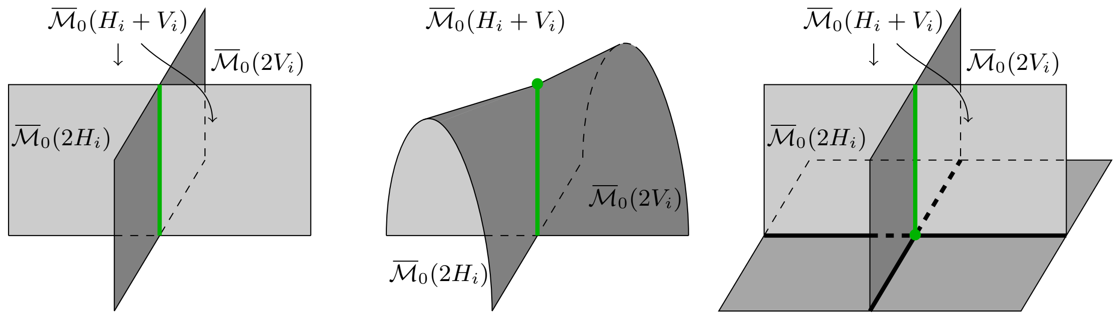


FIGURE 17. The local models of $\mathcal{M}_{2\vec{e}_i,(1,1)_i}(c_x)$ (left), $\mathcal{M}_{2\vec{e}_i,(2)_i}(c_x)$ (middle), and $\mathcal{M}_{\vec{e}_i,(1)_i}(c_x) \times \mathcal{M}_{\vec{e}_i,(1)_i}(c_x)$ (right) inside $\overline{\mathcal{M}}([2H_i])$, from Example 9.5.

We will come back to these local models in Section 10, when we will describe neat embeddings of our moduli spaces. In the meantime, we present the following illuminating example.

Example 9.5. Continuing from Examples 8.9 and 8.10 (Figures 15 and 16), the local model of the green edge $\mathcal{M}_{2\vec{e}_i,(1,1)_i}(c_x)$ inside $\overline{\mathcal{M}}([2H_i])$ is same as the local model of $Z(0, 2\vec{e}_i, 0; (1,1)_i)$ inside $Z_{2\vec{e}_i}$, which in turn, is same as the local model of $\mathbb{R} \times \{0\}$ inside $\mathbb{R} \times Z_1 \times Z_1$ (cf. Example 7.9); the latter space has four codimension zero closed strata, of which $\mathbb{R} \times \overline{Z}(1,0,0) \times \overline{Z}(1,0,0)$ is the local model inside $\overline{\mathcal{M}}_0(2H_i)$, $\mathbb{R} \times \overline{Z}(0,0,1) \times \overline{Z}(0,0,1)$ is the local model inside $\overline{\mathcal{M}}_0(2V_i)$, and $\mathbb{R} \times \big((\overline{Z}(1,0,0) \times \overline{Z}(0,0,1)) \cup (\overline{Z}(0,0,1) \times \overline{Z}(1,0,0))\big)$ is the local model inside $\overline{\mathcal{M}}_0(H_i+V_i)$.

The local model of the green vertex $\mathcal{M}_{2\vec{e}_i,(2)_i}(c_x)$ inside $\overline{\mathcal{M}}([2H_i])$ is same as the local model of the point $Z(0, 2\vec{e}_i, 0; (2)_i)$ inside $Z_{2\vec{e}_i}$, and its three codimension zero closed strata $Z(2\vec{e}_i, 0, 0)$, $Z(\vec{e}_i, 0, \vec{e}_i)$, $Z(0, 0, 2\vec{e}_i)$ are the local models inside $\overline{\mathcal{M}}_0(2H_i)$, $\overline{\mathcal{M}}_0(H_i+V_i)$, $\overline{\mathcal{M}}_0(2V_i)$, respectively.

Finally, the local model of the other green vertex $\mathcal{M}_{\vec{e}_i,(1)_i}(c_x) \times \mathcal{M}_{\vec{e}_i,(1)_i}(c_x)$ inside $\overline{\mathcal{M}}([2H_i])$ is same as the local model of the point $\{0\} \times Z(0, \vec{e}_i, 0; (1)_i) \times Z(0, \vec{e}_i, 0; (1)_i)$ inside $\mathbb{R}_+ \times Z_{\vec{e}_i} \times Z_{\vec{e}_i}$; the latter space also has four codimension zero closed strata, of which $\mathbb{R}_+ \times \overline{Z}(\vec{e}_i,0,0) \times \overline{Z}(\vec{e}_i,0,0)$ is the local model inside $\overline{\mathcal{M}}_0(2H_i)$, $\mathbb{R}_+ \times \overline{Z}(0,0,\vec{e}_i) \times \overline{Z}(0,0,\vec{e}_i)$ is the local model inside $\overline{\mathcal{M}}_0(2V_i)$, and $\mathbb{R}_+ \times \big((\overline{Z}(\vec{e}_i,0,0) \times \overline{Z}(0,0,\vec{e}_i)) \cup (\overline{Z}(0,0,\vec{e}_i) \times \overline{Z}(\vec{e}_i,0,0))\big)$ is the local model inside $\overline{\mathcal{M}}_0(H_i+V_i)$.

See Figure 17.

10. EMBEDDINGS AND FRAMINGS

The moduli spaces $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ will come equipped with suitable embeddings in

$$\mathbb{E}^d_l \coloneqq \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_+ \times \cdots \times \mathbb{R}_+ \times \mathbb{R}^d \;\cong\; \mathbb{R}^l_+ \times \mathbb{R}^{d(l+1)}$$

and they will also be framed. Here, d is some sufficiently large integer depending only on the grid $\mathbb{G}$, whereas $l = \operatorname{tdim} \overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ is the thick dimension given by the Formula (9.6); for convenience, we will henceforth also assume that d is even, so that the above isomorphism is orientation-preserving. Note that $\mathbb{E}^d_l$ is $\mathbb{E}(l, d(l+1))$ in the notation of Section 5.2.

10.1. **Neat embeddings of stratified spaces.** In this section we will describe the required properties for the embedding of $X = \overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D) \hookrightarrow \mathbb{E}^d_l$ that we plan to construct. By analogy with Section 5.2, an embedding with these properties will be called *neat*. We assume that the strata of X are as described in Section 9.

Definition 10.1. Let $X = \overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ with thick dimension $\operatorname{tdim} X = l$. A *neat embedding* of X consists of the following data:

(NE-1) An equivalence class of a subspace $U = U_{\vec{N},\vec{\lambda}}(D) \subset \mathbb{E}^d_l$, called the *thickening* of X, such that U is an l-dimensional $\langle l\rangle$-manifold and the inclusion $U \hookrightarrow \mathbb{E}^d_l$ is a neat embedding; the equivalence relation is described below.

(NE-2) A topological embedding $\iota = \iota_X \colon X \hookrightarrow U$—which is a smooth embedding when restricted to each open stratum—satisying $\iota^{-1}(\partial_i U) = \partial_i X$ for all $i = 1, \dots, l$.

(NE-3) A stratification of U with $\iota(X)$ as a closed stratum, and an identification of the local model of each open stratum of X of the form

$$Y = \mathcal{M}_{\vec{N}^1+\mathbb{O}(E^1)+\mathbb{O}(F^1),\vec{\lambda}^1}(D^1) \times \cdots \times \mathcal{M}_{\vec{N}^r+\mathbb{O}(E^r)+\mathbb{O}(F^r),\vec{\lambda}^r}(D^r)$$

inside U, with the local model of

$$A_Y = \mathbb{R}^{a_Y} \times \{0\} \times Z(0, \vec{N}^1 + \mathbb{O}(E^1) + \mathbb{O}(F^1), 0; \vec{\lambda}^1) \times \cdots \times Z(0, \vec{N}^r + \mathbb{O}(E^r) + \mathbb{O}(F^r), 0; \vec{\lambda}^r)$$

inside

$$B_Y = \mathbb{R}^{a_Y} \times \mathbb{R}^{r-1}_+ \times Z_{\vec{N}^1+\mathbb{O}(E^1)+\mathbb{O}(F^1)} \times \cdots \times Z_{\vec{N}^1+\mathbb{O}(E^1)+\mathbb{O}(F^1)}.$$

(Here $a_Y = \dim Y - \sum_i \dim Z(0, \vec{N}^1 + \mathbb{O}(E^1) + \mathbb{O}(F^1), 0; \vec{\lambda}^1)$.) In more detail, for any point $y \in Y$, let f_y be the isomorphism from the local model of $\iota(Y)$ inside U to the local model of A_Y inside B_Y; then this identification may be recorded by just remembering the pullback (under f_y) of the standard frame of A_Y inside B_Y. This framing of the normal bundle of Y inside U is called the *internal framing*. Note that the internal framings are defined separately on the open strata, since different strata have different dimensions, and hence a different number of vectors in their internal framings.

(NE-4) The equivalence relation on these subspaces U is generated by the following atomic relation: $U \sim U'$ if $U \subset U'$, the embedding $X \hookrightarrow U'$ is induced from the embedding $X \hookrightarrow U$, the stratification of U is induced from the stratification of U', and the local models of each open stratum Y inside U and U' are identified.

Remark 10.2. Definition 10.1 is inspired from the local models presented in Section 9.3. The thickening U corresponds to a "tubular" neighborhood of X in the larger space $\overline{\mathcal{M}}([\widetilde{D}])$, where $\widetilde{D} = D + \widetilde{E} + \widetilde{F}$ is as in Section 9.3. For each open stratum Y, the local model of Y in U is identified with that of A_Y inside B_Y, which is the same as the local model of Y inside $\overline{\mathcal{M}}([\widetilde{D}])$.

We will also require these neat embeddings to satisfy various compatibility relations.

Definition 10.3. Let S be a downward closed subset of $\mathscr{T}$, with respect to the partial order from Equation (9.5). A *coherent neat embedding* of S consists of neat embeddings of each $(D, \vec{N}, \vec{\lambda}) \in S$, satisfying the following compatibility conditions. Fix $(D, \vec{N}, \vec{\lambda}) \in S$, and let $X = \overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$, $l = \operatorname{tdim} X$, and $U = U_{\vec{N},\vec{\lambda}}(D)$. For any open stratum Y of of the form

$$Y = \mathcal{M}_{\vec{N}^1+\mathbb{O}(E^1)+\mathbb{O}(F^1),\vec{\lambda}^1}(D^1) \times \cdots \times \mathcal{M}_{\vec{N}^r+\mathbb{O}(E^r)+\mathbb{O}(F^r),\vec{\lambda}^r}(D^r),$$

set $Y_i = \mathcal{M}_{\vec{N}^i+\mathbb{O}(E^i)+\mathbb{O}(F^i),\vec{\lambda}^i}(D^i)$, $l_i = \operatorname{tdim} Y_i$, and $U_i = U_{\vec{N}^i+\mathbb{O}(E^i)+\mathbb{O}(F^i),\vec{\lambda}^i}(D^i)$. Then

- There exists thickening V_i of $\overline{Y}_i$, equivalent to U_i, and an open subset V of U containing $\overline{Y}$, such that

$$V_1 \times [0, \epsilon) \times V_2 \times [0, \epsilon) \times \cdots \times [0, \epsilon) \times V_r = V \tag{10.1}$$

(for some small $\epsilon > 0$), as subsets of $\mathbb{E}^d_{l_1} \times \mathbb{R}_+ \times \mathbb{E}^d_{l_2} \times \mathbb{R}_+ \times \cdots \times \mathbb{R}_+ \times \mathbb{E}^d_{l_r} = \mathbb{E}^d_l$; moreover, the stratification of V induced from that on U agrees with the product stratification of $V_1 \times [0,\epsilon) \times \cdots \times [0,\epsilon) \times V_r$.

- The embedding ι_X, restricted to $\overline{Y}$, is the product embedding $\left(\iota_{\overline{Y}_1}, \dots, \iota_{\overline{Y}_r}\right)$, composed with the inclusion
$$\mathbb{E}^d_{l_1} \times \mathbb{E}^d_{l_2} \times \cdots \times \mathbb{E}^d_{l_r} \cong \mathbb{E}^d_{l_1} \times \{0\} \times \mathbb{E}^d_{l_2} \times \{0\} \times \cdots \times \{0\} \times \mathbb{E}^d_{l_r} \hookrightarrow \mathbb{E}^d_l.$$
- For any point $y = (y_1, \dots, y_r)$ in Y, let f_y be the identification of the local model of Y inside V with that of A_Y inside B_Y from Item (NE-3) of Definition 10.1.
  - The restriction of f_y to the local model of Y inside X should be an identification with the local model of A_Y inside
$$\mathbb{R}^{a_Y} \times \mathbb{R}^{r-1}_+ \times \prod_i \overline{Z}(\mathbb{O}(E^i), \vec{N}^i, \mathbb{O}(F^i); \vec{\eta}^i).$$
(Here $\vec{\eta}^i$ is the vector of partitions from Equation (9.2).)
  - Identifying V with $V_1 \times [0,\epsilon) \times \cdots \times [0,\epsilon) \times V_r$, the restriction of f_y to the local model of Y inside $V_1 \times \{0\} \times \cdots \times \{0\} \times V_r$ should be an identification with the local model of A_Y inside
$$\mathbb{R}^{a_Y} \times \{0\} \times \prod_i Z_{\vec{N}^i + \mathbb{O}(E^i) + \mathbb{O}(F^i)};$$
moreover, this identification is required to agree with the product identification of f_{y_i} from the local model of Y_i inside V_i with that of A_{Y_i} inside B_{Y_i}. In other words, the internal framing of Y inside V is obtained by concatenating the $(r-1)$ standard unit vectors in the $[0,\epsilon)$ factors of $V_i \times [0,\epsilon) \times \cdots \times [0,\epsilon) \times V_r$ and the internal framings of Y_i inside V_i.

In addition to internal framings (which is part of the data of neat embeddings), we also have a notion of external framings for neat embeddings.

Definition 10.4. Suppose we have a neat embedding of $X = \overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ into $\mathbb{E}^d_l$, with associated thickening U. An *external framing* of X is a framing of the $d(l+1)$-dimensional normal bundle to U in $\mathbb{E}^d_l$; in other words, a smoothly varying, ordered basis for a complement of TU in $T\mathbb{E}^d_l$ (see Convention 7.4).

Once again, we have a notion of compatibility of external framings.

Definition 10.5. Let S be a downward closed subset of $\mathscr{T}$, along with a coherent neat embedding. A *coherent external framing* of S consists of external framings of $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ for each $(D, \vec{N}, \vec{\lambda}) \in S$ satisfying the following. Fix $(D, \vec{N}, \vec{\lambda}) \in S$, and let $X = \overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ and $U = U_{\vec{N},\vec{\lambda}}(D)$. For any open stratum $Y = \prod_i \mathcal{M}_{\vec{N}^i + \mathbb{O}(E^i) + \mathbb{O}(F^i), \vec{\lambda}^i}(D^i)$ of X, let $Y_i = \mathcal{M}_{\vec{N}^i + \mathbb{O}(E^i) + \mathbb{O}(F^i), \vec{\lambda}^i}(D^i)$ and $U_i = U_{\vec{N}^i + \mathbb{O}(E^i) + \mathbb{O}(F^i), \vec{\lambda}^i}(D^i)$. Let V_i be a thickening of $\overline{Y}_i$ equivalent to U_i, and let V be the open subset of U containing $\overline{Y}$ satisfying Equation (10.1). The the framing of the normal bundle of U, restricted to V, is the product framing of the normal bundles of V_i.

Example 10.6. If $X = \mathcal{M}_0(D)$ and D does not contain any row or column (as in Example 8.4), then X is a $\langle k \rangle$-manifold, the thickening U is just X itself, and the notion of neat embedding coincides with that for $\langle k \rangle$-manifolds given in Section 5.2. For example, the hexagon moduli space from Figure 9 is a $\langle 2 \rangle$-manifold, and Figure 4 shows a neat embedding of that hexagon. The procedure we will use in Section 12 to construct such a neat embedding with external framings will be as follows. We start by choosing embeddings of the zero-dimensional moduli spaces $\mathcal{M}_0(A)$, $\mathcal{M}_0(B)$,

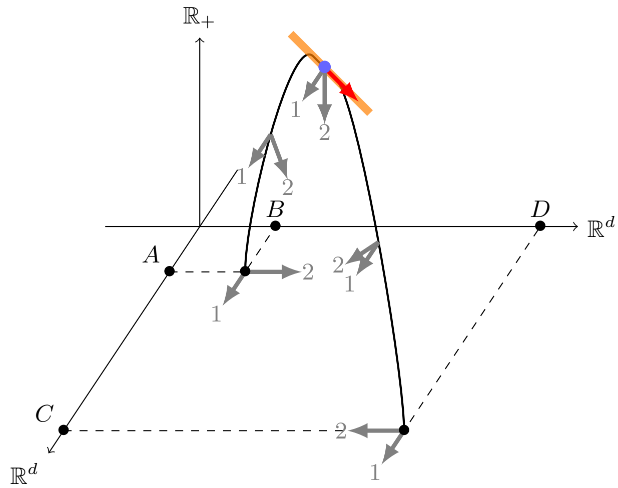


FIGURE 18. Neat embeddings for the moduli spaces in Figure 10. The orange interval is the thickening of the blue dot, with the internal frame shown by the red arrow. The gray arrows (ordered 1 and 2) indicate the external framings.

and $\mathcal{M}_0(C)$ in $\mathbb{R}^d$ along with framings of their normal bundles. The six black dots on the $\mathbb{R}^{3d}$ line are products of these moduli spaces (with product normal framings), corresponding to permutations in the order in which they appear as vertices in Figure 9:

$$\mathcal{M}_0(A)\times\mathcal{M}_0(B)\times\mathcal{M}_0(C),\quad \mathcal{M}_0(A)\times\mathcal{M}_0(C)\times\mathcal{M}_0(B),\dots$$

We will then construct neat embeddings of the one-dimensional moduli spaces $\mathcal{M}_0(A+B)$, $\mathcal{M}_0(B+C)$ and $\mathcal{M}_0(A+C)$ in $\mathbb{R}_+\times\mathbb{R}^{2d}$, along with a normal framing that extends the framing on the boundary. By taking products of the zero- and one-dimensional moduli spaces we get neat embeddings (with product normal framings) of the edges of the hexagon. Finally, we extend to an embedding of the entire hexagon, and frame its normal bundle. (This procedure automatically ensures that the neat embeddings and external framings are coherent.)

Example 10.7. In Figure 18, we show neat embeddings for the moduli spaces from Figure 10, corresponding to the row $H_i = A + B$ and the column $V_i = C + D$, along with their external framings. The two moduli spaces are glued at the blue point $\overline{\mathcal{M}}_{1,(1)}(c_x)$ where we also specify a thickening of that point (the orange interval) and the internal frame at that point (the red vector). The two halves of the orange interval (or an equivalent smaller thickening) are the tubular neighborhoods of the blue point inside the two moduli spaces.

Example 10.8. In Figure 19 we show a neat embedding for the moduli space $\overline{\mathcal{M}}_0(2C + D)$ from Figure 11, together with an external framing. We also show the thickening of the special (blue) boundary. A tubular neighborhood of this blue edge inside the moduli space $\overline{\mathcal{M}}_0(2C+D)$ is half of this thickening (or an equivalent smaller one). The other half of the thickening would be a tubular neighborhood of the blue edge inside the moduli space $\overline{\mathcal{M}}_0(A + B + C)$ from Figure 11.

Example 10.9. In Figure 20 we show a neat embedding for the front facet of Figure 16, that is, the moduli space for the domain AB with the disk AB attached. The thickening of the green edge,

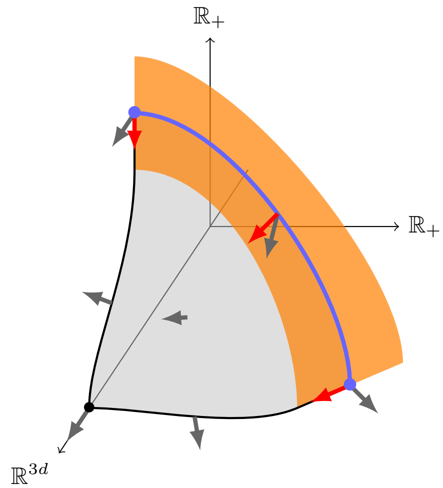


FIGURE 19. A neat embedding for the triangular moduli space from the right hand side of Figure 11. The gray arrows indicate the external framing. The thickening of the special (blue) boundary is shown in orange, with internal framing indicated by the red arrows.

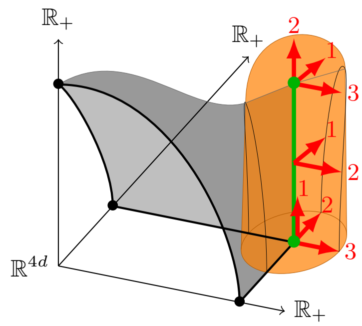


FIGURE 20. A neat embedding for the front facet of the moduli space shown in Figure 16. The thickening of the green edge is shown in orange, along with its stratification (which is identical to the stratification from Figure 17). The internal frames along the green edge are shown by the (numbered) red arrows.

along with its internal frames, is also shown. (The internal frames have two or three vectors at the 1- and 0-dimensional strata of the green edge, respectively.) The external frames cannot be shown due to severe dimension reduction.

11. THE EMBEDDED FRAMED COBORDISM GROUP

The obstruction classes that we will define during the construction will naturally live inside a group

$$\tilde{\Omega}^k_{\mathrm{fr}} = \mathrm{colim}_m \, \tilde{\Omega}^k_{\mathrm{fr},m},$$

which we call the *embedded framed cobordism group.*

Recall that the usual framed cobordism group

$$\Omega^k_{\mathrm{fr}} = \mathrm{colim}_m \, \Omega^k_{\mathrm{fr},m}$$

is defined as follows: the elements of $\Omega^k_{\mathrm{fr},m}$ are the equivalence classes of closed k-dimensional manifolds M embedded in $\mathbb{R}^m$, together with a framing of the normal bundle; the equivalence relation is given by framed cobordisms in $\mathbb{R}^m \times [0,1]$; and the group structure is $[M_1] + [M_2] = [M \amalg M_2']$, where M_2' is a sufficiently large translation of M_2. There is a natural map $\sigma\colon \Omega^k_{\mathrm{fr},m} \to \Omega^k_{\mathrm{fr},m+1}$, and Ω^k_{fr} is the colimit.

The group $\tilde{\Omega}^k_{\mathrm{fr},m}$ is defined similarly to $\Omega^k_{\mathrm{fr},m}$, except we require the framed cobordisms to also be embedded in $\mathbb{R}^m$. More precisely:

(Ω-1) The elements of $\tilde{\Omega}^k_{\mathrm{fr},m}$ are the equivalence classes of closed k-dimensional manifolds M embedded in $\mathbb{R}^m$, together with a vector field $\vec{v}$ (in $\mathbb{R}^m$) along M which is everywhere transverse to TM, and a framing of an $(m-k-1)$-dimensional complement of $TM \oplus \langle \vec{v} \rangle$. We assume $m \geq 2k+3$. (Also, we will always follow Convention 7.4: framings are not necessarily orthonormal, and complements are not necessarily orthogonal.)

(Ω-2) The equivalence relation stipulates $(M_1, \vec{v}_1) \sim (M_2, \vec{v}_2)$ if there is an embedded framed cobordism in $\mathbb{R}^m$ from M_1 to M_2', which starts in the direction of $\vec{v}_1$ and ends in the direction of $-\vec{v}_2$. Here, M_2' is a translation of M_2 in a generic direction so that $M_1 \cap M_2' = \varnothing$. We call a direction $\vec{e} \in S^{m-1}$ *generic for* $(M, \vec{v})$ if the projection $\pi\colon \mathbb{R}^m \to \mathbb{R}^{m-1}$ to the hyperplane perpendicular to $\vec{e}$ sends M diffeomorphically unto an embedded submanifold of $\mathbb{R}^{m-1}$ and $\vec{v}$ to a vector field in $\mathbb{R}^{m-1}$ along $\pi(M)$ which is everywhere transverse to the tangent space of $\pi(M)$. (A standard application of Sard's lemma shows that if $m \geq 2k+2$, then non-generic directions constitute a measure zero subset of S^{m-1}.)

(Ω-3) The group structure on $\tilde{\Omega}^k_{\mathrm{fr}}$ is given by $[(M_1, \vec{v}_1)] + [(M_2, \vec{v}_2)] = [(M_1, \vec{v}_1) \amalg (M_2', \vec{v}_2)]$, where M_2' is a translation of M_2 in a generic direction, as above.

(Ω-4) The zero element is the empty submanifold, and negation is given by reversing $\vec{v}$, that is, $-[(M, \vec{v})] = [(M, -\vec{v})]$.

The above definition deserves some justification, specifically to show that $\sim$ defines a well-defined equivalence relation, and that $(M, -\vec{v})$ is the inverse of $(M, \vec{v})$. The following lemmas are key.

Lemma 11.1. *Consider a framed $(M, \vec{v})$ as above and let $\vec{e}$ be a generic direction for $(M, \vec{v})$. Let M' denote a pushoff in the direction of $\vec{e}$. Then there is an embedded framed cobordism (as described in Item (Ω-2)) from $(M, \vec{v})$ to $(M', \vec{e})$, for some normal framing of $(M', \vec{e})$.*

Proof. By rescaling if necessary, we can assume the pushoff M' of M is by the unit vector $\vec{e}$.

Fix a smooth embedding $\gamma\colon [0,1] \to \mathbb{R}^2$ such that

$$\gamma(0) = (0,0), \quad \gamma(1) = (1,0), \quad \gamma'(0) = (0,1), \quad \gamma'(1) = (1,0)$$

and such that the image of γ is contained in the strip $S_\epsilon = [0,1] \times [0, \epsilon)$, for $\epsilon > 0$ small.

For every $p \in M$, let $V_p = \mathrm{Span}(\vec{e}, \vec{v}_p)$ and let $A_p\colon \mathbb{R}^2 \to V_p$ be the linear isomorphism that takes $(1,0)$ to $\vec{e}$ and $(0,1)$ to $\vec{v}_p$. If ϵ is sufficiently small, the genericity condition on $\vec{e}$ guarantees that the union of all $p + A_p(S_\epsilon)$ forms a smoothly embedded bundle over M, with fiber S_ϵ. Then, the map

$$f\colon [0,1] \times M \to \mathbb{R}^m, \quad f(t,p) = p + A_p \circ \gamma(t)$$

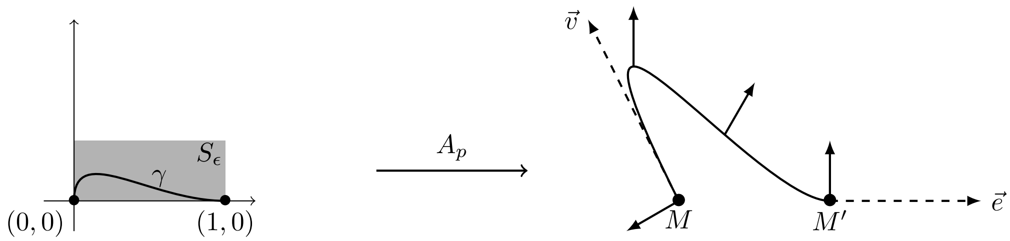


FIGURE 21. A picture illustrating the proof of Lemma 11.1; the notation is same from the lemma. The vector fields $\vec{v}$ and $\vec{e}$ (at M and M') are shown by dashed arrows, and the normal framings are shown by solid arrows.

describes a smoothly embedded cobordism S from $(M, \vec{v})$ to $(M', \vec{e})$. We can also choose a normal framing on this cobordism, which agrees with the given framing at $(M, \vec{v})$. Figure 21 illustrates the proof. □

Lemma 11.2. *As in Lemma 11.1, consider $(M, \vec{v})$, a generic direction $\vec{e}$ for $(M, \vec{v})$, and a pushoff M' in the direction of $\vec{e}$. Then there is an embedded framed cobordism from $(M, \vec{v})$ to $(M', \vec{v})$ (as described in Item (Ω-2)), with the normal framing on $(M', \vec{v})$ the same as the given normal framing on $(M, \vec{v})$.*

Proof. Let N be another pushoff of M in the direction of $\vec{e}$; assume this pushoff is much smaller compared to the given pushoff M'. Let N' be the symmetric pushoff of M' in the direction of $-\vec{e}$. By Lemma 11.1, there is an embedded framed cobordism F from $(M, \vec{v})$ to $(N, \vec{e})$, for some normal framing of $(N, \vec{e})$. Consider the symmetric cobordism F' from $(M', -\vec{v})$ to $(N', -\vec{e})$, and view it as a cobordism from $(N', \vec{e})$ to $(M', \vec{v})$. The framing of F induces a framing of F' by symmetry; in particular, the normal framings on $(N, \vec{e})$ and $(N', \vec{e})$ agree, and the normal framings on $(M, \vec{v})$ and $(M', \vec{v})$ agree.

Simply by translating along the $\vec{e}$ direction, we get an embedded framed cobordism S from $(N, \vec{e})$ to $(N', \vec{e})$. Then the union $F \cup S \cup F'$ is a framed cobordism from $(M, \vec{v})$ to $(M', \vec{v})$ as required. □

Lemma 11.3. *Given a framed $(M, \vec{v})$ with a generic direction $\vec{e}$, let M' be a pushoff of M in the direction of $\vec{e}$, and let $(-M', -\vec{v})$ be obtained from $(M', -\vec{v})$ by changing the sign of one of the framing vectors. Then, there exists an embedded framed cobordism from $(M, \vec{v})$ to $(-M', -\vec{v})$.*

Proof. Note that when flowing M we are allowed to continuously deform its framing. Thus, without loss of generality, we can assume that one of the framing vectors of M, call it $\vec{w}$, lies in the plane spanned by $\vec{e}$ and $\vec{v}$; in fact, we can assume it to be perpendicular to $\vec{v}$ in that plane.

Let M'' be a smaller pushoff of M in the direction of $\vec{e}$, so that M'' is intermediate between M and M'. Consider the cobordism S from $(M, \vec{v})$ to $(M'', \vec{e})$ defined in Lemma 11.1, and compose it with the reverse of the cobordism from $(M', \vec{v})$ to $(M'', -\vec{e})$, provided by the same lemma. Altogether, we get a cobordism from $(M, \vec{v})$ to $(M', -\vec{v})$. Furthermore, we can choose one of the vector fields in the framing to be perpendicular to S in the plane spanned by $\vec{v}$ and $\vec{e}$, and let the other vectors stay constant. Following the framing, we see that the distinguished framing vector $\vec{w}$ gets turned

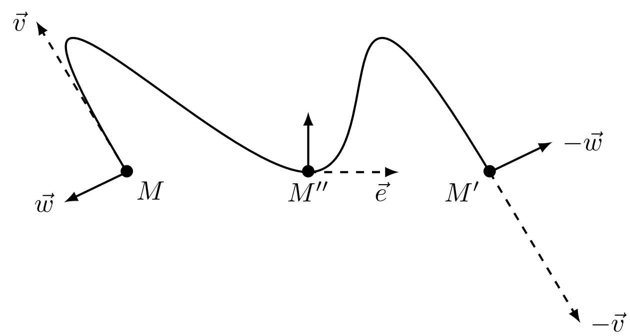


FIGURE 22. The cobordism in Lemma 11.3. Following the same convention from Figure 21, the vector fields $\vec{v}$, $-\vec{v}$, $\vec{e}$ (at M, M', M'') are drawn with dashed arrows, and the normal vector field is shown by solid arrows.

into $-\vec{w}$; see Figure 22. Thus, when taking into account the framing, the end of the cobordism is $(-M', -\vec{v})$. ☐

Armed with these lemmas, we can prove:

Proposition 11.4. *Items (Ω-1)-(Ω-4) make $\tilde{\Omega}^k_{\mathrm{fr},m}$ into a well-defined Abelian group.*

Proof. Let us start by showing that the relation $\sim$ is well-defined, i.e., it does not depend on which translation we choose in (Ω-2). Consider $(M, \vec{v})$ and $(N, \vec{w})$, and assume there is an embedded framed cobordism S from $(M, \vec{v})$ to $(N', \vec{w})$ for some generic pushoff N'. If N'' is another generic pushoff, then by Lemma 11.2, there are embedded framed cobordisms F from $(N', \vec{w})$ to $(N, \vec{w})$ and F' from $(N, \vec{w})$ to $(N'', \vec{w})$. The union $S \cup F \cup F'$ then is an immersed framed cobordism from $(M, \vec{v})$ to $(N'', \vec{w})$. However, since we assumed $m \geq 2k + 3$, by perturbing the cobordism in the interior, we may assume it is embedded.

The proof that the relation $\sim$ is transitive is similar to the above argument. The statement that $\sim$ is reflexive is same as the statement that $(M, -\vec{v})$ is the inverse of $(M, \vec{v})$, and it is Lemma 11.2. To see that $\sim$ is symmetric, note that if $(M, \vec{v}) \sim (N, \vec{w})$, by reversing the cobordism and its framing we get that $(-N, -\vec{w}) \sim (-M, -\vec{v})$; applying Lemma 11.3, we deduce that $(N, \vec{w}) \sim (M, \vec{v})$.

It is also not hard to check that the group operation $[(M_1, \vec{v}_1)] + [(M_2, \vec{v}_2)] = [(M_1, \vec{v}_1) \amalg (M_2', \vec{v}_2)]$ is well-defined, and commutative. ☐

There is a natural stabilization map

$$\sigma\colon \tilde{\Omega}^k_{\mathrm{fr},m} \to \tilde{\Omega}^k_{\mathrm{fr},m+1} \tag{11.1}$$

defined as follows. Given $(M, \vec{v})$ inside $\mathbb{R}^m$ along with a normal framing $\langle \vec{w}_1, \dots, \vec{w}_{m-k-1} \rangle$, consider it as an element of $\tilde{\Omega}^k_{\mathrm{fr},m+1}$ by considering $M \times \{0\}$ inside $\mathbb{R}^m \times \{0\} \subset \mathbb{R}^{m+1}$, using the same vector field $\vec{v}$, and using the normal framing $\langle \vec{w}_1, \dots, \vec{w}_{m-k-1}, \vec{e} \rangle$, where $\vec{e}$ is the positive unit normal vector in the new $\mathbb{R}$ direction.

We define $\tilde{\Omega}^k_{\mathrm{fr}}$ to be the colimit of the groups $\tilde{\Omega}^k_{\mathrm{fr},m}$ under the maps $\tilde{\Omega}^k_{\mathrm{fr},m} \to \tilde{\Omega}^k_{\mathrm{fr},m+1}$. It is worth comparing this new group $\tilde{\Omega}^k_{\mathrm{fr}} = \operatorname{colim}_m \tilde{\Omega}^k_{\mathrm{fr},m}$ with the usual framed cobordism group $\Omega^k_{\mathrm{fr}} = \operatorname{colim}_m \Omega^k_{\mathrm{fr},m}$.

Proposition 11.5. *The groups $\tilde{\Omega}^k_{\mathrm{fr}}$ and Ω^k_{fr} are isomorphic.*

First we need another lemma.

Lemma 11.6. *Consider a framed $(M, \vec{v})$ as in the definition of $\tilde{\Omega}^k_{\mathrm{fr},m}$, and let $\vec{w}$ be one of the vector fields in the framing of M. Let $(M, -\vec{w})$ be framed by replacing the vector field $\vec{w}$ with $\vec{v}$. Then, $(M, \vec{v})$ and $(M, -\vec{w})$ map to the same element under the stabilization map $\tilde{\Omega}^k_{\mathrm{fr},m} \to \tilde{\Omega}^k_{\mathrm{fr},m+1}$. Hence, $(M, \vec{v})$ and $(M, -\vec{w})$ represent the same element in $\tilde{\Omega}^k_{\mathrm{fr}}$.*

Proof. Let $\vec{e}$ be the new unit coordinate vector in $\mathbb{R}^{m+1}$, normal to $\mathbb{R}^m$. Under the stabilization map, we identify $M \subset \mathbb{R}^m$ with $M \times \{0\} \subset \mathbb{R}^{m+1}$, and we add $\vec{e}$ to the normal framings of $(M, \vec{v})$ and $(M, -\vec{w})$. Let M' be the pushoff of M in the direction $\vec{e}$. Note that $\vec{e}$ is a generic vector for $(M, \vec{v})$ in $\mathbb{R}^{m+1}$. Thus, it suffices to construct an embedded framed cobordism from $(M, \vec{v})$ to $(M', -\vec{w})$ in $\mathbb{R}^{m+1}$, which we do as follows.

The argument is similar to that in the proof of Lemma 11.1. Fix a smooth embedding $\gamma\colon [0,1] \to \mathbb{R}^2$ such that

$$\gamma(0) = \gamma(1) = (0,0), \quad \gamma'(0) = (1,0), \quad \gamma'(1) = (0,-1)$$

and such that the image of γ is contained in the ball $B(\epsilon)$ of radius ϵ around the origin, for $\epsilon > 0$ small. We let $\gamma^\perp$ be the normal vector field to the image of γ, obtained from γ' by a counterclockwise rotation by $90°$. For example, $\gamma^\perp(0) = (0,1)$ and $\gamma^\perp(1) = (1,0)$.

Fix also a smooth map $\zeta\colon [0,1] \to [0,1]$ with

$$\zeta(0) = 0, \quad \zeta(1) = 1, \quad \zeta'(0) = \zeta'(1) = 0 \text{ and } \zeta'(t) > 0 \text{ for } t \in (0,1).$$

For every $p \in M$, let $V_p = \operatorname{Span}(\vec{v}_p, \vec{w}_p)$ and let $A_p\colon \mathbb{R}^2 \to V_p$ be the linear isomorphism that takes $(1,0)$ to $\vec{v}_p$ and $(0,1)$ to $\vec{w}_p$. If ϵ is sufficiently small, the union of all $p + A_p(B(\epsilon))$ forms a smoothly embedded disk bundle over M in $\mathbb{R}^m$. Then, the map

$$f\colon [0,1] \times M \to \mathbb{R}^{m+1}, \quad f(t,p) = p + A_p \circ \gamma(t) + \zeta(t) \cdot \vec{e}$$

describes a smoothly embedded cobordism S from $(M, \vec{v})$ to $(M, -\vec{w})$. For the normal framing on S, we use the pushforward of $\gamma^\perp$ under A_p to interpolate between $\vec{w}$ and $\vec{v}$ as t goes from 0 to 1. We also keep $\vec{e}$ as part of the normal framing throughout the cobordism. ☐

Proof of Proposition 11.5. There is a natural map $f\colon \Omega^k_{\mathrm{fr},m} \to \tilde{\Omega}^k_{\mathrm{fr},m+1}$. Given a framed manifold $M \subset \mathbb{R}^m$, we let $f(M)$ be the same manifold, viewed inside $\mathbb{R}^m \times \{0\} \subset \mathbb{R}^{m+1}$, the vector field $\vec{v}$ be the constant positive unit vector field in the new $\mathbb{R}$ direction, and the normal framing be the original framing of M, multiplied by $(-1)^m$. It is immediate from the definitions that if $M \sim N$ in $\Omega^k_{\mathrm{fr},m}$, then $f(M) \sim f(N)$ in $\tilde{\Omega}^k_{\mathrm{fr},m+1}$.

Next, consider the following diagram:

$$\begin{array}{ccc} \Omega^k_{\mathrm{fr},m} & \xrightarrow{f} & \tilde{\Omega}^k_{\mathrm{fr},m+1} \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma} \\ \Omega^k_{\mathrm{fr},m+1} & \xrightarrow{f} & \tilde{\Omega}^k_{\mathrm{fr},m+2} \end{array} \tag{11.2}$$

where the vertical arrows are stabilization maps. We claim that the diagram (11.2) commutes after one more stabilization, i.e., $\sigma\circ\sigma\circ f = \sigma\circ f\circ\sigma$. Indeed, suppose we have a manifold $M\subset\mathbb{R}^m$ framed by the sequence of vectors $(\vec{w}_1,\dots,\vec{w}_{m-k})$. Let $\vec{e}_{m+1}$ and $\vec{e}_{m+2}$ dente the two new unit vectors when we stabilize from $\mathbb{R}^m$ to $\mathbb{R}^{m+2}$. The images of $[M,(\vec{w}_1,\dots,\vec{w}_{m-k})]\in\Omega^k_{\mathrm{fr},m}$ under the two possible compositions in (11.2) are

$$(-1)^m[(M,(\vec{w}_1,\dots,\vec{w}_{m-k},\vec{e}_{m+1})),\vec{e}_{m+2}] \quad \text{and} \quad (-1)^{m+1}[(M,(\vec{w}_1,\dots,\vec{w}_{m-k},\vec{e}_{m+2})),\vec{e}_{m+1}].$$

These become identical after one more stabilization, as proved in Lemma 11.6. From here it follows that the maps f induce a well-defined map

$$\Omega^k_{\mathrm{fr}}\to\tilde{\Omega}^k_{\mathrm{fr}}$$

on the colimits.

There is also a natural map

$$g\colon \tilde{\Omega}^k_{\mathrm{fr},m}\to\Omega^k_{\mathrm{fr},m} \tag{11.3}$$

defined as follows. Given $(M,\vec{v})$ inside $\mathbb{R}^m$ along with a normal framing $(\vec{w}_1,\dots,\vec{w}_{m-k-1})$, we map it to M with the normal framing

$$(\vec{w}_1,\dots,\vec{w}_{m-k-1},(-1)^{m+1}\vec{v}).$$

(In $\Omega^k_{\mathrm{fr},m}$, this is equivalent to the normal framing

$$((-1)^k\vec{v},\vec{w}_1,\dots,\vec{w}_{m-k-1})$$

of M.) To see that the map g is well-defined, let us consider $(M,\vec{v})$ as an element of $\tilde{\Omega}^k_{\mathrm{fr},m+1}$ as in the definition of the stabilization map (11.1). Let $M'=M\times\{1\}\subset\mathbb{R}^{m+1}$ be the unit pushoff in the new $\vec{e}$ direction. By Lemma 11.1, there is an embedded framed cobordism $S(M,\vec{v})$ from $(M\times\{0\},\vec{v})$ to $(M\times\{1\},\vec{e})$ in $\mathbb{R}^{m+1}$; indeed, the proof of the lemma shows that the cobordism lies inside $\mathbb{R}^m\times[0,1]$. Furthermore, the induced normal framing of $(M\times\{1\},\vec{e})$ in $\mathbb{R}^{m+1}$ is $(\vec{w}_1,\dots,\vec{w}_{m-k-1},-\vec{v})$. Now, if we have a framed cobordism W from $(M_1,\vec{v}_1)$ to $(M_2,\vec{v}_2)$ in $\mathbb{R}^m$, we can treat it as a cobordism inside $\mathbb{R}^m\times\{\frac{1}{2}\}$, and compose with the reverse cobordism $S(M_1,\vec{v}_1)^r$ from $(M_1,\vec{e})$ to $(M_1,\vec{v}_1)$ (viewed inside $\mathbb{R}^m\times[0,\frac{1}{2}]$) and with $S(M_2,\vec{v}_2)$ from $(M_2,\vec{v}_2)$ to $(M_2,\vec{e})$ (viewed inside $\mathbb{R}^m\times[\frac{1}{2},1]$); see Figure 23. This produces a framed cobordism in $\mathbb{R}^m\times[0,1]$ from $M_1\times\{0\}$ to $M_2\times\{1\}$, where the last framing vectors are $-\vec{v}_1$ and $-\vec{v}_2$, respectively. After multiplying the framing on this cobordism by $(-1)^m$, we get a framed cobordism from $g(M_1,\vec{v}_1)$ to $g(M_2,\vec{v}_2)$. Thus, g is well-defined. The presence of the $(-1)^{m+1}$ factor in the definition of g ensures that it commutes with the stabilization maps, producing a map $\tilde{\Omega}^k_{\mathrm{fr}}\to\Omega^k_{\mathrm{fr}}$ in the colimit.

It is immediate that the composition $g\circ f$ is the stabilization $\Omega^k_{\mathrm{fr},m}\to\Omega^k_{\mathrm{fr},m+1}$. In the other direction, $(f\circ g)(M,\vec{v})$ is equivalent in $(M,\vec{v})$ in $\tilde{\Omega}^k_{\mathrm{fr},m+1}$ using the framed cobordism $S(M,\vec{v})$ from $(M\times\{0\},\vec{v})$ to $(M\times\{1\},\vec{e})$ in $\mathbb{R}^m\times[0,1]$. It follows that the maps induced by f and g on the colimits are inverse to each other. □

12. Constructing the moduli spaces

We will construct the stratified spaces $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$, along with their embeddings and framings, inductively by dimension. We will usually denote their dimension and thick dimension by k and l; and these moduli spaces will be embedded in $\mathbb{E}^d_l$. For the reader's convenience, we first outline the procedure in Subsection 12.1, and then give more details in the following subsections.

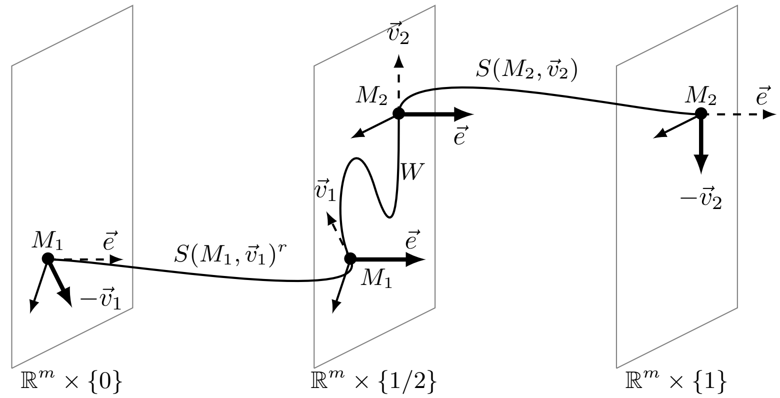


FIGURE 23. The composed cobordism from Proposition 11.5. Following earlier conventions, the vectors $\vec{e}, \vec{v}_1, \vec{v}_2, \vec{e}$ (at $M_1 \subset \mathbb{R}^m \times \{0\}$, $M_1 \subset \mathbb{R}^m \times \{1/2\}$, $M_2 \subset \mathbb{R}^m \times \{1/2\}$, $M_2 \subset \mathbb{R}^m \times \{1\}$, respectively) are drawn dashed, and the normal vectors are drawn solid, of which, the last normal vectors ($-\vec{v}_1, \vec{e}_1, \vec{e}_1, -\vec{v}_2$, respectively) are drawn thicker.

12.1. **Outline.** Let $\mathscr{T}^\dagger \subset \mathscr{T}$ be the downward closed subset consisting of all triples $(c_{x^{\mathrm{Id}}}, \vec{N}, \vec{\lambda})$ where $\vec{N}$ is just made of 0's and 1's, and let $\mathscr{T}' = \mathscr{T} \setminus \mathscr{T}^\dagger$. Recall from Proposition 4.8 that $\mathscr{T}^\dagger$ generates a subcomplex $CDP^\dagger_* \subset CDP_*$ such that the quotient complex CDP'_* is acyclic.

As a first step, we will construct the moduli spaces $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(c_{x^{\mathrm{Id}}})$, for all $(c_{x^{\mathrm{Id}}}, \vec{N}, \vec{\lambda}) \in \mathscr{T}^\dagger$. This construction is independent of what we do in this section, so we will describe it later (in Section 13).

After this, we will construct the remaining spaces $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ inductively on their dimension $k = \mathrm{gr}(D, \vec{N}, \vec{\lambda}) - 1$.

Let $\mathscr{T}'_{\leq k} = \{(D, \vec{N}, \vec{\lambda}) \in \mathscr{T}' \mid \mathrm{gr}(D, \vec{N}, \vec{\lambda}) - 1 \leq k\}$ and $\mathscr{T}'_k = \mathscr{T}'_{\leq k} \setminus \mathscr{T}'_{\leq k-1}$.

(C-1) Assume the moduli spaces $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ have already been constructed for all $(D, \vec{N}, \vec{\lambda}) \in \mathscr{T}^\dagger \cup \mathscr{T}'_{\leq k-1}$, along with coherent neat embeddings and coherent external framings.

(C-2) For any $(D, \vec{N}, \vec{\lambda}) \in \mathscr{T}'_k$, its $(k-1)$-dimensional boundary $\partial\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ has already been constructed. This has a stratified thickening, and we use it to construct an open neighborhood V of $\partial\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ inside $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ (with a coherent neat embedding and external framing).

(C-3) The neighborhood V is a Whitney stratified space with compact boundary $\partial V = \partial\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$. By smoothing a smaller open neighborhood V'', we construct a new open neighborhood V' of $\partial\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ in $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$, such that $\overline{V'} \setminus V'$ is a smooth $(k-1)$-dimensional manifold, denoted $\partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$.

(C-4) Frame $\partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ using the internal and external framings of V. We also equip $\partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ with a vector field $\vec{v}$, which is the outer normal to $\overline{V}'$. Thus, we obtain an element $[\partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)]$ in the framed cobordism group $\tilde{\Omega}^{k-1}_{\mathrm{fr}}$.

(C-5) This element $[\partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)]$ is the obstruction to filling in $\partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}$ to construct $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ (with coherent neat embedding and coherent external framing). Putting together all $(D, \vec{N}, \vec{\lambda})$'s, we have an obstruction class in the form of a cochain

$$\mathfrak{o}_k \in \mathrm{Hom}(CDP'_{k+1}, \tilde{\Omega}^{k-1}_{\mathrm{fr}}), \qquad \mathfrak{o}_k(D, \vec{N}, \vec{\lambda}) = [\partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)].$$

(C-6) We prove that $\mathfrak{o}_k$ is a cocycle.

(C-7) Since CDP' is acyclic, it follows that $\mathfrak{o}_k$ is a coboundary of some element $\mathfrak{b} \in \mathrm{Hom}(CDP'_k, \tilde{\Omega}^{k-1}_{\mathrm{fr}})$. Change the $(k-1)$-dimensional spaces by $-\mathfrak{b}$. (Note that we don't change any lower dimensional spaces.) After this change, the new $(k-1)$-dimensional boundaries $\partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ are framed null-cobordant.

(C-8) Fill in each $\partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ arbitrarily to obtain the desired moduli space $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$, embedded in $\mathbb{E}^d_l$, with a normal framing.

(C-9) We split the normal framings to the moduli spaces $\mathcal{M}_{\vec{N},\vec{\lambda}}(D)$ into internal and external framings, and construct thickenings using the internal framings.

(C-10) This finishes the induction step. Now $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ have been constructed for all $(D,\vec{N},\vec{\lambda}) \in \mathscr{T}^\dagger \cup \mathscr{T}'_{\leq k}$, along with coherent neat embeddings and coherent external framings. (The next step of the induction—when we are constructing the $(k+1)$-dimensional moduli spaces—might require modifying the just-constructed k-dimensional spaces, but none of the smaller dimensions.)

12.2. **The base case.** Let us recall the formulas (8.1) and (9.6) for the dimension k and the thick dimension l of the moduli spaces $\mathcal{M}_{\vec{N},\vec{\lambda}}(D)$:

$$k = \mu(D) - 1 + \sum \ell(\lambda_j), \tag{12.1}$$

$$l = \mu(D) - 1 + 2\sum N_j. \tag{12.2}$$

The base case in the induction corresponds to moduli spaces with $k = 0$. (For the base case we only need to do Steps (C-8) and (C-9) from the outline.) From the above formula we see that there are two kinds of such moduli spaces:

- those with $\mu(D) = 1$ and trivial $\vec{\lambda}$ (that is, $\vec{N} = \vec{0}$); then D must be a rectangle R on the grid, and we are looking at the moduli spaces $\mathcal{M}_0(R)$;
- those with $\mu(D) = 0$ and $\lambda_j = (N)$ for some j, where N denotes N_j, and we have $N_i = 0$ for all $i \neq j$; then D is the constant domain c_x for some $x \in \mathbb{S}$, and we write the moduli spaces as $\mathcal{M}_{N\vec{e}_j,(N)_j}(c_x)$, with the notation from Remark 9.2.

The moduli spaces of the first kind have thick dimension 0. We define them to be single points, embedded in $\mathbb{E}^d_0 = \mathbb{R}^d$ in any way, and framed so that the resulting element in $\Omega^0_{\mathrm{fr}} \cong \mathbb{Z}$ is the sign $s(R) \in \{\pm 1\}$ from (G-16). For instance, if $\vec{e}_1, \dots, \vec{e}_d$ denote the standard unit vectors in $\mathbb{R}^d$, we may frame the point $\mathcal{M}_0(R)$ by $[(-1)^{s(R)}\vec{e}_1, \vec{e}_2, \dots, \vec{e}_d]$. Since the thick dimension is zero, the point itself is its own thickening, so this completes both Steps (C-8) and (C-9).

The moduli spaces of the second kind have thick dimension $l = 2N - 1$. We define them to be single points as well, embedded arbitrarily in the interior of $\mathbb{E}^d_l$, and framed positively. For instance, if $\vec{e}^+_1, \dots, \vec{e}^+_l$ denote the standard unit vectors in $\mathbb{R}^l_+$, and $\vec{e}_1, \dots, \vec{e}_{d(l+1)}$ denote the standard unit vectors in $\mathbb{R}^{d(l+1)}$ (as before), then we may choose to frame the point $\mathcal{M}_{N\vec{e}_j,(N)_j}(c_x) \in \mathring{\mathbb{E}}^d_l \cong \mathring{\mathbb{R}}^l_+ \times \mathbb{R}^{d(l+1)}$ by $[\vec{e}^+_1, \dots, \vec{e}^+_l, \vec{e}_1, \dots, \vec{e}_{d(l+1)}]$. This completes Step (C-8).

For Step (C-9), declare the first l vectors to be the internal frame, and the rest to be the external frame. For the thickening, we choose an open embedding of the local model Z_N in the interior of $\mathbb{E}^d_l$, with the origin in Z_N mapped to $\mathcal{M}_{N\vec{e}_j,(N)_j}(c_x)$, such that the standard frame at the origin maps to the internal frame at $\mathcal{M}_{N\vec{e}_j,(N)_j}(c_x)$. For instance, we may choose to embed a neighborhood of the origin in Z_N to the interior of $\mathbb{E}^d_l$ by an affine map, with the origin mapping to $\mathcal{M}_{N\vec{e}_j,(N)_j}(c_x)$, and the standard frame mapping to the internal frame. Declare the thickening $U_{N\vec{e}_j,(N)_j}(c_x)$ to be the image of this embedding, with the induced stratification, and extend the external framing at $\mathcal{M}_{N\vec{e}_j,(N)_j}(c_x)$ to a normal framing of this entire thickening.

Note that $\mathscr{T}^\dagger$ contains exactly $(n-1)$ domains $(c_{x^{\mathrm{Id}}}, \vec{e}_j, (1)_j)$, $j = 2, \dots, n$ with $\mathrm{gr} = 1$. Their 0-dimensional moduli spaces have already been constructed (although that construction is described in the next section), along with neat embeddings and external framings. When the reader compares the construction here and the one from Section 13, it should be clear that they agree.

12.3. **Boundaries and their neighborhoods.** We now give more detailed explanations for some of the steps in the outline of the induction above. In this subsection we discuss Steps (C-1)—(C-3).

Step (C-1) just states the induction hypothesis. We assume we have constructed moduli spaces $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ for all $(D, \vec{N}, \vec{\lambda}) \in \mathscr{T}^\dagger \cup \mathscr{T}'_{\leq k-1}$, along with coherent neat embeddings (Definition 10.3) and coherent external framings (Definition 10.5).

For Step (C-2), recall that strata of $\partial\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ are of the form

$$Y = \mathcal{M}_{\vec{N}^1+\mathbb{O}(E^1)+\mathbb{O}(F^1),\vec{\lambda}^1}(D^1) \times \cdots \times \mathcal{M}_{\vec{N}^r+\mathbb{O}(E^r)+\mathbb{O}(F^r),\vec{\lambda}^r}(D^r). \tag{12.3}$$

Since $\dim Y \leq k-1$, by our induction hypothesis, each $\overline{\mathcal{M}}_{\vec{N}^i+\mathbb{O}(E^i)+\mathbb{O}(F^i),\vec{\lambda}^i}(D^i)$ already comes with a neat embedding (including a stratified thickening U_i) and internal and external framings in $\mathbb{E}^d_{l_i}$, where l_i is its thick dimension. Altogether, we obtain an embedding of the product

$$U_1 \times U_2 \times \cdots \times U_r \hookrightarrow \mathbb{E}^d_{l_1} \times \{0\} \times \mathbb{E}^d_{l_2} \times \{0\} \times \cdots \times \{0\} \times \mathbb{E}^d_{l_r} \subset \mathbb{E}^d_l,$$

where

$$l = \operatorname{tdim} \overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D) = l_1 + \cdots + l_r + (r-1)$$

and we identify

$$\mathbb{E}^d_l \cong \mathbb{E}^d_{l_1} \times \mathbb{R}_+ \times \mathbb{E}^d_{l_2} \times \mathbb{R}_+ \times \cdots \times \mathbb{R}_+ \times \mathbb{E}^d_{l_r}. \tag{12.4}$$

Let

$$U(\overline{Y}) = U_1 \times [0, \epsilon_Y) \times U_2 \times [0, \epsilon_Y) \times \cdots \times [0, \epsilon_Y) \times U_r \subset \mathbb{E}^d_l,$$

with the product stratification, and define the *thickening of the boundary* $\partial\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ to be

$$U = U(\partial\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)) = \bigcup_{Y \leq \partial\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)} U(\overline{Y}),$$

where $\epsilon_Y > 0$ are chosen so that $\epsilon_Y \ll \epsilon_Z$ for $Z < Y$. Since the stratifications of the thickenings of all moduli spaces up to dimension $k-1$ are coherent, they induce a stratification of U, with each Y as an open stratum, whose local model in this thickening is same as the local model Y in $\overline{\mathcal{M}}([\widetilde{D}])$ (where $\widetilde{D} = D + \widetilde{E} + \widetilde{F}$, with $\widetilde{E}, \widetilde{F}$ a sum of rows, columns satisfying $\mathbb{O}(\widetilde{E}) + \mathbb{O}(\widetilde{F}) = \vec{N}$), cf. Definition 10.1 Item (NE-3) and Remark 10.2. Similarly, since the external framings of all these $U(\overline{Y})$ are coherent, they induce an external framing of U in $\mathbb{E}^d_l$.

Thus, we have constructed a stratified thickening

$$\partial\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D) \hookrightarrow U \subset \mathbb{E}^d_l$$

with an external framing. Let $V = V(\partial\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D))$ be the union of the closed strata of U that correspond to $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$. Then V is a stratified open neighborhood of $\partial\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ inside $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$. This neighborhood V also has the same thickening U and the same external framing. That is, we have constructed $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ near its boundary, thus completing Step (C-2).

Next, for Step (C-3), the stratified space U is Whitney and hence Thom-Mather stratified. Therefore, each stratum $Y \leq \partial\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ has an open tubular neighborhood of the form $\rho_Y^{-1}([0,\epsilon_Y))$ inside U (similar to Lemma 6.17). Taking their union over Y (and ensuring $\epsilon_Y \ll \epsilon_Z$ for $Z < Y$) produces an open neighborhood U'' of $\partial\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ inside U. Its complement $U \setminus U''$ is an l-dimensional $\langle l\rangle$-manifold with compact boundary, and has a smoothing $\mathrm{sm}[U \setminus U'']$ (similar to Definition 6.18). The complement of the smoothing,

$$U' = U \setminus \mathrm{sm}[U \setminus U''],$$

is a new open neighborhood of $\partial\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ inside U. (By construction, we have $\overline{U}'' \subset U' \subset \overline{U}' \subset U$.)

Define $V'' = U'' \cap V$ and $V' = U' \cap V$. Then $\overline{V}''$ is exactly the closed tubular neighborhood of $\partial V = \partial\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ inside V from Lemma 6.17, and V' is the complement of its smoothing from Definition 6.18. See Figure 6 once again, with the whole square playing the role of V, ∂X playing the role of $\partial V = \partial\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$, $\mathcal{N}$ playing the role of $\overline{V}''$, and the complement of $\mathrm{sm}[X]$ playing the role of V'. By construction, $\partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D) = \overline{V}' \setminus V'$ is a smooth $(k-1)$-dimensional manifold. This completes Step (C-3).

12.4. **Obtaining a cochain.** In this subsection, we discuss Steps (C-4) and (C-5).

For Step (C-4), note that $\partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ is a smooth $(k-1)$-dimensional submanifold of $\mathrm{int}(\mathbb{E}_l^d)$, and we can identify $\mathrm{int}(\mathbb{E}_l^d)$ with a Euclidean space. We want $\partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ to give an element of $\tilde{\Omega}_{\mathrm{fr}}^{k-1}$. For this, we equip it with $\vec{v}$ (the outer normal to $\overline{V}'$), and we are left to specify a normal framing to

$$T(\partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)) \oplus \langle\vec{v}\rangle = TV|_{\partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)}.$$

Consider the thickening U of $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ near its boundary, constructed in Step (C-2) (in Section 12.3). The normal bundle of V inside U is equipped with an internal frame (cf. Definition 10.1 Item (NE-3)), and the normal bundle of U has an external frame. Concatenating the internal frame and the external frame, we get a framing of the normal bundle of V inside $\mathbb{E}_l^d$, thus completing Step (C-4).

For Step (C-5), by collecting all these elements $[\partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)] \in \tilde{\Omega}_{\mathrm{fr}}^{k-1}$ for all $(D,\vec{N},\vec{\lambda}) \in \mathscr{T}_k'$, we get an cochain $\mathfrak{o}_k \in \mathrm{Hom}(CDP_{k+1}', \tilde{\Omega}_{\mathrm{fr}}^{k-1})$. We remark that $\mathfrak{o}_k(D,\vec{N},\vec{\lambda})$ is precisely the obstruction to filling in $\partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$. (This will discussed in more detail for Steps (C-8) and (C-9), after we make the obstruction class vanish, and actually fill in $\partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$.)

12.5. **The cocycle condition.** This subsection is devoted to Step (C-6). We split the discussion into two cases, according to whether $k = 1$ or $k \geq 2$. In the case $k = 1$, we obtain a stronger conclusion:

Proposition 12.1. *We have $\mathfrak{o}_1 = 0 \in \mathrm{Hom}(CDP_2', \tilde{\Omega}_{\mathrm{fr}}^0)$.*

Proof. We seek to show that for every 1-dimensional moduli space $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$, the smoothed boundary $\partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ represents the zero element in $\tilde{\Omega}_{\mathrm{fr}}^0$. Using Formula (12.1), we find that 1-dimensional moduli spaces are of one of the following kinds (using the notation from Remark 9.3):

(1) $\overline{\mathcal{M}}_0(D)$, where D is a positive domain of index 2 on the grid, which is either a disjoint union of two rectangles or an L-shaped hexagon (cf. Item (G-13e) from Section 2.1). Then, D has two distinct representations as concatenations of two rectangles, and therefore the boundary $\partial\overline{\mathcal{M}}_0(D)$ consists of two Type I strata of the form $\mathcal{M}_0(R_1) \times \mathcal{M}_0(R_2)$; cf. Example 8.3 and Figure 8;

(2) $\overline{\mathcal{M}}_0(D)$, where D is either a horizontal annulus H_j or a vertical annulus V_j. Then, the boundary $\partial\overline{\mathcal{M}}_0(D)$ consists of a Type I stratum $\mathcal{M}_0(R_1)\times\mathcal{M}_0(R_2)$ and a Type II stratum $\mathcal{M}_{\vec{e}_j,(1)_j}(c_x)$; cf. Example 8.5 and Figure 10;

(3) $\overline{\mathcal{M}}_{N\vec{e}_j,(N)_j}(R)$, where R is a rectangle (of index 1) from x to y, $N>0$, and $j\in\{2,\dots,n\}$. Then, $\partial\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ consists of two strata of Type IV, namely $\mathcal{M}_{N\vec{e}_j,(N)_j}(c_x)\times\mathcal{M}_0(R)$ and $\mathcal{M}_0(R)\times\mathcal{M}_{N\vec{e}_j,(N)_j}(c_y)$;

(4) $\overline{\mathcal{M}}_{N\vec{e}_i+M\vec{e}_j,(N)_i+(M)_j}(c_x)$, where c_x is a constant domain and $N,M>0$, $i\neq j$. Then, $\partial\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ has two strata of Type IV, namely $\mathcal{M}_{N\vec{e}_i,(N)_i}(c_x)\times\mathcal{M}_{M\vec{e}_j,(M)_j}(c_x)$ and $\mathcal{M}_{M\vec{e}_j,(M)_j}(c_x)\times\mathcal{M}_{N\vec{e}_i,(N)_i}(c_x)$;

(5) $\overline{\mathcal{M}}_{(N+M)\vec{e}_j,(N,M)_j}(c_x)$, where c_x is a constant domain and $N,M>0$, Then, $\partial\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ consists of the stratum $\mathcal{M}_{N\vec{e}_i,(N)_i}(c_x)\times\mathcal{M}_{M\vec{e}_i,(M)_i}(c_x)$ of Type IV, and the stratum $\mathcal{M}_{(N+M)\vec{e}_i,(N+M)_i}(c_x)$ of Type III.

In all the above situations, the boundaries $\partial\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ consist of two 0-dimensional strata, which are points according to the construction in Section 12.2. Hence, the smoothings $\partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ also consist of two points.

By a case by case analysis, using the definitions in Section 12.2 and the properties (2.1), (2.2), (2.3) of the sign assignment on rectangles, we will check that the two points in $\partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ come with opposite signs in $\tilde{\Omega}^0_{\mathrm{fr}}\cong\Omega^0_{\mathrm{fr}}\cong\mathbb{Z}$, so they sum up to 0. Recall from Step (C-4) (Section 12.4) that each point is framed by concatenating the internal and external framings. Further recall from Equation (11.3) that the isomorphism $\tilde{\Omega}^0_{\mathrm{fr}}\to\Omega^0_{\mathrm{fr}}$ is given by concatenating the vector $\vec{v}$ with the framing of the point. That is, each point, as an element in Ω^0_{fr}, is framed by concatenating $\vec{v}$, its internal frame, and its external frame. For convenience, assume the 0-dimensional moduli spaces are framed explicitly as in Section 12.2.

In Case (1), that the two points in $\mathbb{R}_+\times\mathbb{R}^{2d}$ come with opposite signs is a consequence of Equation (2.1) (with $\vec{v}=\vec{e}^{\,+}_1$ in both points).

In Case (2), the Type I stratum is positively (respectively negatively) oriented if $D=H_j$ (respectively $D=V_j$), according to Equation (2.2) (respectively (2.3)), with $\vec{v}=\vec{e}^{\,+}_1$; for the Type II stratum, there is no internal frame, the external frame is the standard (positive) frame for $\mathbb{R}^{2d}$, and the vector $\vec{v}$ comes from the inward normal vector of the point $Z(0,1,0)$ in $\overline{Z}(1,0,0)$ (respectively $\overline{Z}(0,0,1)$), which is the opposite (respectively same) as the internal frame $\delta v_{1,1}$ of $Z(0,1,0)$ (cf. Section 7.3), and hence maps to the frame $-\vec{e}^{\,+}_1$ (respectively $\vec{e}^{\,+}_1$) of $\mathbb{R}_+$.

In Case (3), for either point in $\mathbb{R}^{2N}_+\times\mathbb{R}^{(2N+1)d}$, its external frame is a frame for $\mathbb{R}^{(2N+1)d}$ with sign $s(R)$; the point near the boundary $\mathcal{M}_{N\vec{e}_j,(N)_j}(c_x)\times\mathcal{M}_0(R)$ (respectively $\mathcal{M}_0(R)\times\mathcal{M}_{N\vec{e}_j,(N)_j}(c_y)$) has internal frame $[\vec{e}^{\,+}_1,\dots,\vec{e}^{\,+}_{2N-1}]$ and $\vec{v}=\vec{e}^{\,+}_{2N}$ (respectively internal frame $[\vec{e}^{\,+}_2,\dots,\vec{e}^{\,+}_{2N}]$ and $\vec{v}=\vec{e}^{\,+}_1$) and hence represents $-s(R)$ (respecively $s(R)$) in Ω^0_{fr}.

In Case (4), for either point in $\mathbb{R}^{2N+2M-1}_+\times\mathbb{R}^{(2N+2M)d}$, its external frame is the standard (positive) frame for $\mathbb{R}^{(2N+2M)d}$; assuming $i<j$, the point near the boundary $\mathcal{M}_{N\vec{e}_i,(N)_i}(c_x)\times\mathcal{M}_{M\vec{e}_j,(M)_j}(c_x)$ (respectively $\mathcal{M}_{M\vec{e}_j,(M)_j}(c_x)\times\mathcal{M}_{N\vec{e}_i,(N)_i}(c_x)$) has internal frame $[\vec{e}^{\,+}_1,\dots,\vec{e}^{\,+}_{2N-1},\vec{e}^{\,+}_{2N+1},\dots,\vec{e}^{\,+}_{2N+2M-1}]$ and $\vec{v}=\vec{e}^{\,+}_{2N}$ (respectively $[\vec{e}^{\,+}_{2M+1},\dots,\vec{e}^{\,+}_{2N+2M-1},\vec{e}^{\,+}_1,\dots,\vec{e}^{\,+}_{2M-1}]$ and $\vec{v}=\vec{e}^{\,+}_{2M}$) and hence represents -1 (respectively 1) in Ω^0_{fr}.

Finally in Case (5), for either point in $\mathbb{R}^{2N+2M-1}_+\times\mathbb{R}^{(2N+2M)d}$, the external frame is the standard (positive) frame for $\mathbb{R}^{(2N+2M)d}$ and the internal frame is $[\vec{e}^{\,+}_1,\dots,\vec{e}^{\,+}_{2N-1},\vec{e}^{\,+}_{2N+1},\dots,\vec{e}^{\,+}_{2N+2M-1}]$ (similar to the above case); for the the point near $\mathcal{M}_{N\vec{e}_i,(N)_i}(c_x)\times\mathcal{M}_{M\vec{e}_i,(M)_i}$, $\vec{v}=\vec{e}^{\,+}_{2N}$, but for the point near $\mathcal{M}_{(N+M)\vec{e}_i,(N+M)_i}(c_x)$, the vector $\vec{v}$ comes from the inward normal vector of the point $Z(0,N+M,0;(N+M))$ in $\overline{Z}(0,N+M,0;(N,M))$, which is the negative of the internal frame $-\delta\Delta_{1,N}$ of $Z(0,N+M,0;(N+M))$, and hence maps to the vector $-\vec{e}^{\,+}_{2N}$ in $\mathbb{R}_+^{2N+2M-1}$. ☐

For $k \geq 2$, we have:

Proposition 12.2. *The element* $\mathfrak{o}_k \in \mathrm{Hom}(CDP'_{k+1}, \tilde{\Omega}^{k-1}_{\mathrm{fr}})$ *is a cocycle.*

Proof. We need to show that $\delta\mathfrak{o}_k$ evaluates to zero on any generator $(E,\vec{M},\vec{\mu}) \in \mathscr{T}'_{k+1}$. This is equivalent to

$$\mathfrak{o}_k(\delta(E,\vec{M},\vec{\mu})) = 0.$$

Write

$$\delta(E,\vec{M},\vec{\mu}) = \sum s_{D,\vec{N},\vec{\lambda}}(D,\vec{N},\lambda),$$

where $s_{D,\vec{N},\vec{\lambda}} \in \{\pm 1\}$ and $(D,\vec{N},\vec{\lambda}) \in \mathscr{T}'_k$. (This is the differential in the quotient complex $CDP'_* = CDP_*/CDP^\dagger_*$, so if Formula (4.5) applied on $(E,\vec{M},\vec{\mu})$ produces any triples from $\mathscr{T}^\dagger$, we suppress them. Also, here and later, when we sum over $(D,\vec{N},\vec{\lambda})$, we consider only the triples in $\mathscr{T}'_k$ that appear in $\delta(E,\vec{M},\vec{\mu})$.) We aim to prove:

$$\sum_{(D,\vec{N},\vec{\lambda})} s_{D,\vec{N},\vec{\lambda}} \cdot \mathfrak{o}_k(D,\vec{N},\lambda) = 0,$$

that is,

$$\sum_{(D,\vec{N},\vec{\lambda})} s_{D,\vec{N},\vec{\lambda}}[\partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)] = 0. \tag{12.5}$$

(It is possible certain triples $(D,\vec{N},\vec{\lambda})$ appear twice in $\delta(E,\vec{M},\vec{\mu})$—for instance $(c_x, 2\vec{e}_j, (1,1)_j)$ appears twice in $\delta(H_j, \vec{e}_j, (1)_j)$—but we consider each appearance as a separate instance.)

Consider the $(k+1)$-dimensional moduli space $\overline{\mathcal{M}}_{\vec{M},\vec{\mu}}(E)$. Of course, this has not yet been constructed in our inductive procedure. Nevertheless, as mentioned in Section 9.2, we know that its codimension-1 strata are supposed to be of three types: products (Type I and IV) and single moduli spaces (Type II and III). Furthermore, let us distinguish between the products where one of the factors is zero-dimensional, and those were both factors are positive dimensional. When a factor is zero-dimensional, it must be a single point (see Section 12.2 and Remark 12.3 below), and therefore the product can be identified with the other factor, which is some $(k-1)$-dimensional moduli space $\mathcal{M}_{\vec{N},\vec{\lambda}}(D)$. The triples $(D,\vec{N},\vec{\lambda})$ that appear in $\delta(E,\vec{M},\vec{\mu})$ come from products (Type I and IV) where one factor is 0-dimensional and the other is $\mathcal{M}_{\vec{N},\vec{\lambda}}(D)$ with $(D,\vec{N},\vec{\lambda}) \notin \mathscr{T}^\dagger$, as well as from Type II and III strata $\mathcal{M}_{\vec{N},\vec{\lambda}}(D)$ (also with $(D,\vec{N},\vec{\lambda}) \notin \mathscr{T}^\dagger$); see Remark 9.3.

Recall that all the moduli spaces of dimension up to $(k-1)$ have already been constructed, as well as moduli spaces of $(D,\vec{N},\vec{\lambda}) \in \mathscr{T}^\dagger$. Let us define the *old boundary* of $\overline{\mathcal{M}}_{\vec{M},\vec{\mu}}(E)$, denoted $\partial^{\mathrm{old}}\overline{\mathcal{M}}_{\vec{M},\vec{\mu}}(E)$, to be the union of all strata of $\overline{\mathcal{M}}_{\vec{M},\vec{\mu}}(E)$ of codimension 2 or higher (that is, dimension $(k-1)$ or lower), together with the Types I and IV strata where neither factor of the product is 0-dimensional (and therefore both factors are of dimension $(k-1)$ or lower), together with the Types I and IV strata where one factor is 0-dimensional and the other factor is $\mathcal{M}_{\vec{N},\vec{\lambda}}(D)$ for some $(D,\vec{N},\vec{\lambda}) \in \mathscr{T}^\dagger$, together with Types II and III strata $\mathcal{M}_{\vec{N},\vec{\lambda}}(D)$ for some $(D,\vec{N},\vec{\lambda}) \in \mathscr{T}^\dagger$. By repeating what we did for Step (C-2) in Section 12.3, we see that $\partial^{\mathrm{old}}\overline{\mathcal{M}}_{\vec{M},\vec{\mu}}(E)$ has already been constructed in our inductive procedure, together with its embedding in $\mathbb{E}^d_l$ (where $l = \operatorname{tdim}\overline{\mathcal{M}}_{\vec{M},\vec{\mu}}(E)$), its stratified thickening U, and internal and external framings. Indeed, if we repeat the construction for Step (C-3), by taking the union of closed strata of U that correspond to $\overline{\mathcal{M}}_{\vec{M},\vec{\mu}}(E)$, we get a neighborhood W of $\partial^{\mathrm{old}}\overline{\mathcal{M}}_{\vec{M},\vec{\mu}}(E)$ inside $\overline{\mathcal{M}}_{\vec{M},\vec{\mu}}(E)$. As before, we take a smaller Thom-Mather neighborhood U'', smooth its complement $U \setminus U''$, thereby producing another neighborhood U' (with $\overline{U}'' \subset U' \subset \overline{U}' \subset U$). By intersecting these with W, we get similar neighborhoods W', W'' of $\partial^{\mathrm{old}}\overline{\mathcal{M}}_{\vec{M},\vec{\mu}}(E)$ inside $\overline{\mathcal{M}}_{\vec{M},\vec{\mu}}(E)$.

Consider any Type I or IV stratum of the form $\mathrm{pt} \times \overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ with $(D, \vec{N}, \vec{\lambda}) \notin \mathscr{T}^\dagger$. Then the neighborhoods $\mathrm{pt} \times V, \mathrm{pt} \times V', \mathrm{pt} \times V''$ of $\mathrm{pt} \times \partial\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ inside $\mathrm{pt} \times \overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ (as constructed in Section 12.3) are given by intersecting this stratum with W, W', W'', respectively. Denote these neighborhoods $\widetilde{V}, \widetilde{V}', \widetilde{V}''$, and let $\widetilde{\partial}'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D) = \mathrm{pt} \times \partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D) = \overline{\widetilde{V}}' \setminus \widetilde{V}'$. A similar discussion holds for any Type I or IV stratum of the form $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D) \times \mathrm{pt}$. Similarly, for any Type II or III stratum $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$, the neighborhoods V, V', V'' of $\partial\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ inside $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ are given by intersecting this stratum with W, W', W'', respectively. For consistency, denote these neighborhoods $\widetilde{V}, \widetilde{V}', \widetilde{V}''$ as well, and let $\widetilde{\partial}'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D) = \partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$.

Define $\partial''\overline{\mathcal{M}}_{\vec{M},\vec{\mu}}(E) = \overline{W}' \setminus W'$. Then $\partial''\overline{\mathcal{M}}_{\vec{M},\vec{\mu}}(E)$ is a compact k-dimensional smooth manifold in $\operatorname{int}(\mathbb{E}^d_l)$ with boundary

$$\partial\big(\partial''\overline{\mathcal{M}}_{\vec{M},\vec{\mu}}(E)\big) = \coprod_{(D,\vec{N},\vec{\lambda})} \widetilde{\partial}'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D), \tag{12.6}$$

see Figure 24.

Similarly to Step (C-4), we frame $\partial''\overline{\mathcal{M}}_{\vec{M},\vec{\mu}}(E)$ by concatenating the outward normal vector (of $\partial''\overline{\mathcal{M}}_{\vec{M},\vec{\mu}}(E)$ in $\overline{W}'$), the internal frame (of W inside U) and the external frame (of U inside $\mathbb{E}^d_l$). Define $[\widetilde{\partial}'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)] \in \tilde{\Omega}^{k-1}_{\mathrm{fr}}$ be to be manifold $\widetilde{\partial}'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$, embedded in $\operatorname{int}(\mathbb{E}^d_l)$, with frame induced from that of $\partial''\overline{\mathcal{M}}_{\vec{M},\vec{\mu}}(E)$, and the vector field $\vec{v}$ the inward normal vector of $\widetilde{\partial}'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ in $\partial''\overline{\mathcal{M}}_{\vec{M},\vec{\mu}}(E)$. Equation (12.6) implies

$$\sum_{(D,\vec{N},\vec{\lambda})} [\widetilde{\partial}'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)] = 0.$$

Equation (12.5) now follows from a sign check

$$[\widetilde{\partial}'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)] = (-1)^{k+1} s_{D,\vec{N},\vec{\lambda}}[\partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)], \tag{12.7}$$

which occupies the rest of the proof.

Let $\vec{w}$ denote the inward normal vector field of $\widetilde{\partial}'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ in $\partial''\overline{\mathcal{M}}_{\vec{M},\vec{\mu}}(E)$, $\vec{v}$ denote the outward normal vector field of $\widetilde{\partial}'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ in $\overline{\widetilde{V}}'$. Let $l_D = \operatorname{tdim}\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ and U_D be the l_D-dimensional thickening of $\partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$. Apply the isomorphism $\tilde{\Omega}^{k-1}_{\mathrm{fr}} \to \Omega^{k-1}_{\mathrm{fr}}$ from Equation (11.3), and we treat $(-1)^{k-1}[\widetilde{\partial}'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)] \in \Omega^{k-1}_{\mathrm{fr},l+d(l+1)}$ as the manifold $\widetilde{\partial}'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D) \subset \mathbb{E}^d_l$ framed as

$$[\vec{w}, \vec{v},\ \text{internal frame of } W \subset U,\ \text{external frame of } U \subset \mathbb{E}^d_l],$$

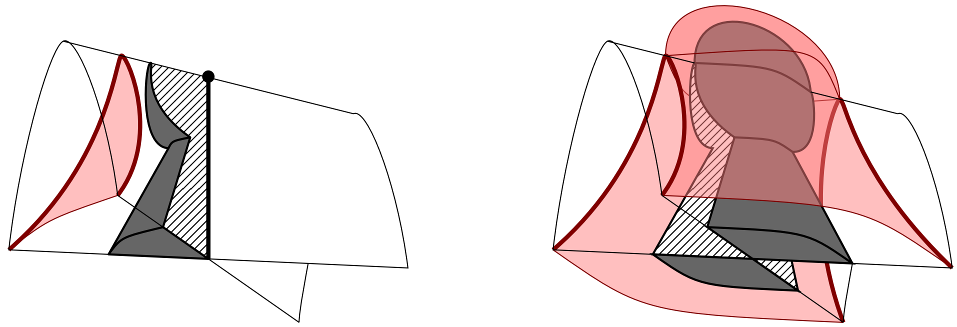


FIGURE 24. This is the Whitney umbrella from Figure 7. On the left, we assume locally $\overline{\mathcal{M}}_{\vec{M},\vec{\mu}}(E) = Z(2,0,0)$, $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D) = Z(1,1,0)$, and $\partial\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D), \partial^{\mathrm{old}}\overline{\mathcal{M}}_{\vec{M},\vec{\mu}}(E) = Z(0,2,0;(1,1))$ (drawn in thick). The neighborhood W'' is shown in gray, and its intersection with $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ is the neighborbood $\widetilde{V}''$ (shown striped); $\widetilde{\partial}'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ is the thick red curve, and $\partial''\overline{\mathcal{M}}_{\vec{M},\vec{\mu}}(E)$ is the transparent pink surface. On the right, we assume locally $\overline{\mathcal{M}}_{\vec{M},\vec{\mu}}(E) = Z(1,0,1)$, $\overline{\mathcal{M}}_{\vec{N}_1,\vec{\lambda}_1}(D_1) = Z(1,1,0)$, $\overline{\mathcal{M}}_{\vec{N}_2,\vec{\lambda}_2}(D_2) = Z(0,1,1)$, and $\partial\overline{\mathcal{M}}_{\vec{N}_1,\vec{\lambda}_1}(D_1), \partial\overline{\mathcal{M}}_{\vec{N}_2,\vec{\lambda}_2}(D_2), \partial^{\mathrm{old}}\overline{\mathcal{M}}_{\vec{M},\vec{\mu}}(E) = Z(0,2,0;(1,1))$ (not visible); the colors are same as before.

equivalently as

$$[\vec{v}, -\vec{w},\ \text{internal frame of } W \subset U,\ \text{external frame of } U \subset \mathbb{E}^d_l].$$

Similarly, treat $(-1)^{k-1}[\partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)] \in \Omega^{k-1}_{\mathrm{fr},l_D+d(l_D+1)}$ as the manifold $\partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D) \subset \mathbb{E}^d_{l_D}$ framed as

$$[\vec{v},\ \text{internal frame of } V \subset U_D,\ \text{external frame of } U_D \subset \mathbb{E}^d_{l_D}],$$

or by stabilizing, as the manifold $\partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D) \times 0$ in $\mathbb{E}^d_{l_D} \times (\mathbb{R}_+ \times \mathbb{E}^d_{l-l_D}) \cong \mathbb{E}^d_l$ framed as

$$[\vec{v},\ \text{internal frame of } V \subset U_D,\ \text{external frame of } U_D \subset \mathbb{E}^d_{l_D},\ \text{standard frame of } \mathbb{E}^d_{l_D} \subset \mathbb{E}^d_l].$$

Also recall that we had assumed that d is even, so there is an even number of vectors in each external frame, and we will freely reshuffle their orders without affecting any signs.

The sign check is fairly easy for Type II and Type III strata. In those cases, $l = l_D$, $U = U_D$ and $\pm\vec{w}$ is one of the vectors of the internal frame of $V \subset U$, and rest of the vectors constitute the frame of $W \subset U$ (cf. Remark 7.6), so we just need to calculate the position $P+1$ where $-\vec{w}$ appears in the the internal frame of $V \subset U$ and with what sign $S \in \{\pm 1\}$. As in the proof of Proposition 12.1, the outward vector $-\vec{w}$ appears with sign $S = -1$ if it is Type II stratum coming some vertical annulus V_j, and otherwise appears with sign $S = 1$. By inspecting Formulas (4.7) and (4.8), we see $(-1)^P S = s_{D,\vec{N},\vec{\lambda}}(-1)^{|\vec{\mu}|+\mu(E)+1}$ in both cases; and we have $|\vec{\mu}| + \mu(E) - 1 = k+1$ from Formula (8.1).

For Type I and Type IV strata of the form $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D) \times \mathrm{pt}$, there are embeddings

$$\widetilde{\partial}'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D) = \partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D) \times 0 \times \mathrm{pt} \hookrightarrow \mathbb{E}^d_{l_D} \times \mathbb{R}_+ \times \mathbb{E}^d_{l-l_D-1}.$$

In Type I, $l - l_D = 1$, locally $U \cong U_D \times \mathbb{R}_+ \times \mathrm{pt}$, $\vec{\lambda} = \vec{\mu}$, and the point is framed with sign $s(R)$ in $\mathbb{R}^d$; in Type IV, $l - l_D = 2|\vec{M} - \vec{N}|$, locally $U \cong U_D \times \mathbb{R}_+ \times (\mathbb{R}_+^{l-l_D-1})$ (assuming the thickening of pt inside $\mathbb{E}^d_{l-l_D-1} \cong \mathbb{R}_+^{l-l_D-1} \times \mathbb{R}^{(l-l_D)d}$ is $\mathbb{R}_+^{l-l_D-1} \times 0$), $|\vec{\lambda}| = |\vec{\mu}| - 1$, and the point is framed positively. The outward vector $-\vec{w}$ is the negative of the unit vector of the $\mathbb{R}_+$ factor, and to bring it to the second position requires a sign of $(-1)^{|\vec{\lambda}|}$; for Type IV, there is an additional sign of $(-1)^{|(\lambda_{j+1},\dots,\lambda_n)|}$ to bring the $l - l_D - 1$ odd number of vectors of the internal frame of $\mathrm{pt} \subset \mathbb{E}^d_{l-l_D-1}$ to the correct position. In either case, after inspecting Formulas (4.6) and (4.9), one gets the sign $s_{D,\vec{N},\vec{\lambda}}(-1)^{|\vec{\mu}|+\mu(E)+1}$.

For Type I and Type IV strata of the form $\mathrm{pt} \times \overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$, we need to recall the following fact: If $M^m \subset \mathbb{R}^a, N^n \subset \mathbb{R}^b$ are framed manifolds, then the framed manifolds $M \times N, N \times M \subset \mathbb{R}^{a+b}$—each with product framing—are related by the sign $(-1)^{mn+mb+na}$; therefore, for framed manifolds $M^m \subset \mathbb{R}^a, N^n \subset \mathbb{R}^b, P^p \subset \mathbb{R}^c$, $M \times N \times P$ and $P \times N \times M$ are related by the sign $(-1)^{mn+np+pm+m(b+c)+n(c+a)+p(a+b)}$. Now for a Type I stratum, we have

$$\widetilde{\partial}'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D) = \mathcal{M}_0(R) \times 0 \times \partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D) \hookrightarrow \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{E}^d_{l-1},$$

and we may assume, up to a sign $s(R)$, that it is framed as

$$[\text{standard frame in } \mathbb{R}^d, \vec{w} = e_1^+, \vec{v},\ \text{internal frame of } V \subset U_D,\ \text{external frame of } U_D \subset \mathbb{E}^d_{l-1}].$$

By the above discussion, the latter is same as

$$\partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D) \times 0 \times \mathcal{M}_0(R) \hookrightarrow \mathbb{E}^d_{l-1} \times \mathbb{R}_+ \times \mathbb{R}^d,$$

framed as

$$[\vec{v},\ \text{internal frame of } V \subset U_D,\ \text{external frame of } U_D \subset \mathbb{E}^d_{l-1}, \vec{e}^{\,+}_{l-1},\ \text{standard frame in } \mathbb{R}^d],$$

up to a further sign of $(-1)^{\dim \partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)} = (-1)^{k-1}$. Instead for a Type IV stratum,

$$\widetilde{\partial}'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D) = \mathcal{M}_{N\vec{e}_j,(N)_j}(c_x) \times 0 \times \partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D) \hookrightarrow \mathbb{E}^d_{2N-1} \times \mathbb{R}_+ \times \mathbb{E}^d_{l_D},$$

framed as

$$[\text{standard frame in } \mathbb{E}^d_{2N-1}, \vec{w} = e^+_{2N}, \vec{v},\ \text{internal frame of } V \subset U_D,\ \text{external frame of } U_D \subset \mathbb{E}^d_{l_D}],$$

up to a sign of $(-1)^{|(\mu_1,\dots,\mu_{j-1})|} = s_{D,\vec{N},\vec{\lambda}}(-1)^{|\vec{\mu}|+\mu(E)+1}$ which comes from bringing the odd-dimensional internal frame of $\mathrm{pt} \in \mathbb{E}^d_{2N-1}$ to the front. We switch the order of the three factors again, but this time, do not incur any additional sign. □

12.6. **Concluding the induction.** This section carries out the remaining steps of the induction. In Step (C-7), we get a cochain $\mathfrak{b} \in \mathrm{Hom}(CDP'_k, \tilde{\Omega}^{k-1}_{\mathrm{fr}})$ satisfying $\delta\mathfrak{b} = \mathfrak{o}_k$. For each $(F, \vec{P}, \vec{\nu}) \in \mathscr{T}'_{k-1}$, we pick a framed manifold $M(F, \vec{P}, \vec{\nu}) \subset \mathrm{int}(\mathbb{E}^d_l)$ (where $l = \mathrm{tdim}\,\overline{\mathcal{M}}_{\vec{P},\vec{\nu}}(F)$), such that $[M(F, \vec{P}, \vec{\nu})] \in \Omega^{k-1}_{\mathrm{fr}}$ is the image of $-\mathfrak{b}(F, \vec{P}, \vec{\nu}) \in \tilde{\Omega}^{k-1}_{\mathrm{fr}}$ under the isomorphism $\tilde{\Omega}^{k-1}_{\mathrm{fr}} \to \Omega^{k-1}_{\mathrm{fr}}$. Change all these $(k-1)$-dimensional moduli spaces by taking disjoint union with $M(F, \vec{P}, \vec{\nu})$, that is, define

$$\overline{\mathcal{M}}^{\mathrm{new}}_{\vec{P},\vec{\nu}}(F) = \overline{\mathcal{M}}_{\vec{P},\vec{\nu}}(F) \amalg M(F, \vec{P}, \vec{\nu}).$$

This new piece $M(F, \vec{P}, \vec{\nu})$ is equipped with a framing of its normal bundle, but does not yet have a stratified thickening with internal framings, nor an external framing. We declare the first $l - k + 1$ vectors of its frame to be the internal frame, and the remaining $d(l + 1)$ vectors to be the external frame. To construct the stratified thickening, we reuse our strategy from Section 12.2; fix a point p

in the open stratum $Z(\vec{0},\vec{P},\vec{0};\vec{\nu}) \subset Z_{\vec{P}}$ (from Equation (7.9)), and a small open disk D_p around p in the normal direction (that is, in the affine subspace spannned by the standard frame at p); for each point $q \in M(F,\vec{P},\vec{\nu})$, embed D_p in $\mathrm{int}(\mathbb{E}^d_l)$ by an affine map sending p to q and the standard frame at p to the internal frame at q; assuming D_p is small enough, the union of these affine embeddings of D_p forms the stratified thickening of $M(F,\vec{P},\vec{\nu})$. (This construction of the thickening is similar to—but simpler than—what we will do in Step (C-9) below.)

This modification changes $\partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ for $(D,\vec{N},\vec{\lambda}) \in \mathscr{T}^{\dagger}_{k+1}$. If we write the differential in CDP'_* as

$$\delta(D,\vec{N},\vec{\lambda}) = \sum s_{F,\vec{P},\vec{\nu}}(F,\vec{P},\vec{\nu}),$$

with $s_{F,\vec{P},\vec{\nu}} \in \{\pm 1\}$, then the terms that appear in the sum correspond to certain strata of $\partial\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ (as in proof of Proposition 12.2). Changing such a stratum by taking disjoint union with $M(F,\vec{P},\vec{\nu})$ has the effect of also changing $\partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ by taking disjoint union with $M(F,\vec{P},\vec{\nu})$ (if Type II, III) or by taking disjoint union with $\mathrm{pt}\times M(F,\vec{P},\vec{\nu})$ or $M(F,\vec{P},\vec{\nu})\times\mathrm{pt}$ (if Type I, IV); let us denote it $\widetilde{M}(F,\vec{P},\vec{\nu})$ for uniformity. Then

$$\partial'^{,\mathrm{new}}\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D) = \partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D) \amalg \coprod_{(F,\vec{P},\vec{\nu})} \widetilde{M}(F,\vec{P},\vec{\nu}).$$

A sign check (left to the reader) similar to that in Proposition 12.2 shows that $[\widetilde{M}(F,\vec{P},\vec{\nu})]$, viewed as an element in $\tilde{\Omega}^{k-1}_{\mathrm{fr}}$, differs from $[M(F,\vec{P},\vec{\nu})]$, viewed as an element of $\Omega^{k-1}_{\mathrm{fr}}$, by precisely the sign $s_{F,\vec{P},\vec{\nu}}$. Therefore in $\tilde{\Omega}^{k-1}_{\mathrm{fr}}$ we get

$$\mathfrak{o}^{\mathrm{new}}_k(D,\vec{N},\vec{\lambda}) = \mathfrak{o}_k(D,\vec{N},\vec{\lambda}) - \sum_{(F,\vec{P},\vec{\nu})} s_{F,\vec{P},\vec{\nu}}\mathfrak{b}(F,\vec{P},\vec{\nu}) = 0.$$

Remark 12.3. In view of Proposition 12.1, we see that Step (C-7) is unnecessary when $k=1$. Thus, the 0-dimensional moduli spaces are not changed in the process, and they will always remain single points, as they were defined in Section 12.2.

In Step (C-8), once we have that $\partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ is framed null-cobordant, we choose a filling $\mathcal{M}'_{\vec{N},\vec{\lambda}}(D) \subset \mathrm{int}(\mathbb{E}^d_l)$ and define

$$\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D) \coloneqq \overline{V}' \cup \mathcal{M}'_{\vec{N},\vec{\lambda}}(D).$$

Finally, for Step (C-9), note that, by construction, $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ comes equipped with a framing of its normal bundle in $\mathbb{E}^d_l$. As before, declare the first $l-k$ vectors to be the internal frame and the remaining $d(l+1)$ vectors to be the external frame. For the stratified thickening, observe that the thickening U is already defined around $\overline{V}' \subset \overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$. Fix a point p in the open stratum $Z(\vec{0},\vec{N},\vec{0};\vec{\lambda}) \subset Z_{\vec{N}}$, and let D_p be a small open disk around p in the normal direction (spanned by the standard frame). By construction, for any point in $\overline{V}'\setminus\partial\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$, the local model in the normal direction inside U agrees with D_p. For every point $q \in \partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$, fix an embedding $D_p \hookrightarrow U$ sending p to q that identifies D_p with the local model at q. Since the stratified thickening U is already defined around q by induction, we may not be able to ensure that this is an affine embedding; but we may choose these embeddings smoothly over $\partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ to get a codimension-one embedding

$\iota\colon \partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D) \times D_p \hookrightarrow U$, respecting the stratifications. We may assume $\mathrm{im}(\iota)$ divides U into two pieces, and that U' is one the pieces.

Push forward the $(l-k)$-dimensional standard frame (in the affine spaces D_p) to get $(l-k)$ pairwise commuting linearly independent vector fields on $\mathrm{im}(\iota)$. In some neighborhood $N \subset \mathrm{int}(\mathbb{E}^d_l)$ of $\mathcal{M}'_{\vec{N},\vec{\lambda}}(D) \cup_{\partial'\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)} \mathrm{im}(\iota)$, extend the internal frame of $\mathcal{M}'_{\vec{N},\vec{\lambda}}(D)$ and these vector fields on $\mathrm{im}(\iota)$ to $(l-k)$ pairwise commuting vector fields; if N is small enough, the extended vector fields are also linearly independent. By integrating along these vector fields, we get an embedding $\mathcal{M}'_{\vec{N},\vec{\lambda}}(D) \times D_p \hookrightarrow N$, and define the stratified thickening $U_{\vec{N},\vec{\lambda}}(D)$ to be union of $\overline{U}'$ and the image of this embedding (along $\mathrm{im}(\iota)$).

Step (C-10) just states the conclusion of the induction step and needs no additional explanation.

12.7. **Gluing moduli spaces.** Recall the equivalence relation on domains given by Equation (8.2):

$$D \sim D' \iff (D - D' \in \mathscr{P} \text{ and } \mathbb{O}(D) = \mathbb{O}(D')).$$

Now that we have constructed all the framed moduli spaces $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$, we glue $\overline{\mathcal{M}}_0(D)$ for all D in the same equivalence class. The results are the required moduli spaces $\overline{\mathcal{M}}([D])$ as in Equation (8.3), which are $\langle l\rangle$-manifolds neatly embedded in $\mathbb{E}^d_l$, and equipped with normal (external) framings there. The fact that we can glue the different $\overline{\mathcal{M}}_0(D)$ is automatic since we constructed them along with their thickenings $U_0(D) \subset \mathbb{E}^d_l$, and the local models of $\overline{\mathcal{M}}_0(D) \subset U_0(D)$ were defined to be the same as our required local models of $\overline{\mathcal{M}}_0(D) \subset \overline{\mathcal{M}}([D])$, cf. Remark 10.2.

13. Embedding and framing the permutohedra

In this section we construct the moduli spaces $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ (along with coherent neat embeddings and external framings), where $D = c_{x^{\mathrm{Id}}}$, the constant domain from the fixed generator x^{Id} to itself, and each entry in $\vec{N}$ is 0 or 1 (so $\vec{\lambda}$ is a trivial partition). These correspond to the triples $(D,\vec{N},\vec{\lambda}) \in \mathscr{T}^\dagger$ generating the subcomplex $CDP^\dagger_* \subset CDP$ that carries all the homology of CDP; cf. Proposition 4.8.

Suppose $\vec{N}$ is as above and let $|\vec{N}| = n$. (In this section, n no longer denotes the grid index.) We will use the notation

$$I = \{i \mid N_i = 1\} = \{p_1 < \cdots < p_n\} \tag{13.1}$$

and

$$X_I \coloneqq \overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(c_{x^{\mathrm{Id}}}). \tag{13.2}$$

Note that X_I is supposed to be an $(n-1)$-dimensional $\langle n-1\rangle$-manifold. Its thick dimension is $2n-1$.

We will define X_I to be the permutohedron Π_n, which is the convex hull of the $n!$ points in $\mathbb{R}^n$ obtained by permuting the coordinates of $(1,2,\dots,n)$; cf. Example 5.2. We will then neatly embed and frame Π_n inside

$$\mathbb{R}^d \times \mathring{\mathbb{R}}_+ \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^d \times \cdots \times \mathbb{R}_+ \times \mathbb{R}^d \times \mathring{\mathbb{R}}_+ \times \mathbb{R}^d \cong \mathring{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1}_+ \times \mathbb{R}^{2dn} \subset \mathbb{E}^d_{2n-1},$$

coherently with respect to the neat embeddings and framings of the lower dimensional strata. We will do this for the case $d=0$, so we do not actually need to construct the external frames. For larger d, we can then simply compose with the standard inclusion

$$\mathring{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1}_+ \hookrightarrow \mathring{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1}_+ \times \mathbb{R}^{2dn}, \quad x \mapsto (x,0),$$

and define the external frame to be the standard frame of the new $\mathbb{R}^{2dn}$.

We will also only construct the internal frames (coherently), and will not construct the thickenings. This is because the local model (in the normal direction) of X_I inside its thickening is the product $\prod_I Z_1$, where Z_1 is a line, cf. Equation (7.8) and Example 7.2. Therefore, if we are given the internal frames at each point $x \in X_I$, we can define the thickening by simply taking the union of small open disks in the affine planes spanned by the internal frames.

All that remains to do is to embed $X_I = \Pi_n$ inside $\mathring{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1}_+$, along with its internal frame, which consists of n vectors fields $(v_{p_1}, \dots, v_{p_n})$ indexed by elements of I, and everything should be done coherently with respect to the data for the lower-dimensional strata. For this we will use the fact that the quotient of $\mathbb{R}^n \times \Pi_n$ by the symmetric group S_n is diffeomorphic to $\mathring{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1}_+$. We will then embed X_I inside $\mathbb{R}^n \times \Pi_n$ by the map $x \mapsto ((p_1, \dots, p_n), x)$ and then quotient by the symmetric group action. The construction of X_I is described in Section 13.2, and its embedding and framing, as outlined above, is described in detail in Section 13.4. The main work is in Section 13.3, which is devoted to proving $(\mathbb{R}^n \times \Pi_n)/S_n \cong \mathring{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1}_+$.

13.1. **The permutohedron.** In this section, we will collect well-known facts about the permutohedron Π_n. See for instance [55, Example 0.10] and [24, Section 3.3].

(Π-1) Letting S_n denote the group of permutations of $\{1, 2, \dots, n\}$, the permutohedron Π_n is the convex hull of the $n!$ points $v_\sigma = (\sigma^{-1}(1), \dots, \sigma^{-1}(n))$ in $\mathbb{R}^n$, for $\sigma \in S_n$.

(Π-2) The permutohedron Π_n is $(n-1)$-dimensional and lies in the hyperplane

$$\mathbb{A}^{n-1} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_j x_j = n(n+1)/2\}$$

and the points v_σ are its vertices.

(Π-3) For any non-empty proper subset $S \subset \{1, 2, \dots, n\}$ of cardinality say k, let $\mathbb{H}_S \subset \mathbb{A}^{n-1}$ denote the half-space $\{(x_1, \dots, x_n) \in \mathbb{A}^{n-1} \mid \sum_{j \in S} x_j \geq k(k+1)/2\}$. Then Π_n is also the intersection of the $2^n - 2$ half-spaces $\mathbb{H}_S$, and the facets of Π_n are $F_S = \Pi_n \cap \partial \mathbb{H}_S$.

(Π-4) The vertices in the facet F_S are precisely the v_σ so that $\{\sigma(1), \sigma(2), \dots, \sigma(k)\} = S$.

(Π-5) The permutohedron carries the structure of $\langle n-1 \rangle$-manifold by declaring

$$\partial_k \Pi_n = \bigcup_{\{S,\ |S|=k\}} F_S.$$

(Π-6) Each of the facets $F_S \subset \partial_k \Pi_n$ can be identified with products of lower dimensional permutohedra $\Pi_k \times \Pi_{n-k}$. Identify $\mathbb{R}^n$ with $\prod_{j \in \{1,\dots,n\}} \mathbb{R}$, and using the linear ordering of the elements of S and $S^c = \{1, 2, \dots, n\} \setminus S$, identify $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$ with $\prod_{j \in S} \mathbb{R}$ and $\prod_{j \in S^c} \mathbb{R}$, respectively. Then the map

$$\prod_{j \in S} \mathbb{R} \times \prod_{j \in S^c} \mathbb{R} \xrightarrow{+(0,\dots,0,k,\dots,k)} \prod_{j \in S} \mathbb{R} \times \prod_{j \in S^c} \mathbb{R} \cong \prod_{j \in \{1,\dots,n\}} \mathbb{R}$$

identifies $\Pi_k \times \Pi_{n-k} \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$ with the facet $F_S \subset \partial_k \Pi_n \subset \mathbb{R}^n$.

(Π-7) We will also need the action of S_n on Π_n. Consider the left action of S_n on $\mathbb{R}^n$ given by:

$$\sigma \cdot (x_1, \dots, x_n) = (x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)}).$$

This restricts to an action on $\mathbb{A}^{n-1}$ and Π_n. On the vertices of the permutohedron, S_n acts by $\sigma \cdot v_\tau = v_{\sigma\tau}$.

See Figure 25 for an illustration of some of these concepts.

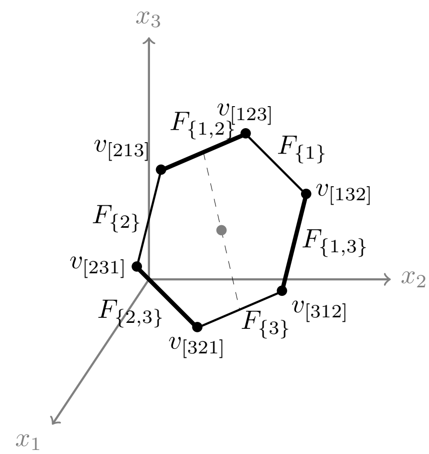


FIGURE 25. The permutohedron $\Pi_3 \subset \mathbb{R}^3$. Here the permutations $\sigma \in S_3$ are denoted as $[\sigma(1)\sigma(2)\sigma(3)]$. The facets in $\partial_1\Pi_3$ are drawn thin and the facets in $\partial_2\Pi_3$ are drawn thick. The action of S_3 is also shown; S_3 is generated by the transposition [213] and the 3-cycle [231], and they act on Π_3 by reflection across the dashed line, and by positive rotation by $120°$ around the gray center, respectively.

13.2. **Construction of the moduli spaces.** We define the moduli spaces X_I from Equation (13.2) as permutohedra:

$$X_I = \Pi_n \subset \mathbb{R}^n \cong \prod_I \mathbb{R}, \tag{13.3}$$

where for convenience, we have identified the ambient space $\mathbb{R}^n$ with $\prod_I \mathbb{R}$ using the linear ordering of the elements of I (cf. Equation (13.1)), that is, using the bijection $\{1,\dots,n\} \to I$, $i \mapsto p_i$. (It is understood that for distinct subsets I, I' of cardinality n, the spaces X_I and $X_{I'}$ are *different* copies of Π_n; for this it might be useful to regard them as living in different ambient spaces $\prod_I \mathbb{R}$ and $\prod_{I'} \mathbb{R}$.)

We need to check that the stratification on X_I is as described in Section 9.1. Indeed, it follows from Items (Π-5) and (Π-6) that X_I is a $\langle n-1\rangle$-manifold with $\partial_k X_I$ identified with

$$\coprod_{\substack{J\subset I\\ |J|=k}} X_J \times X_{I\setminus J}.$$

Moreover, these identifications are coherent, that is, for any $k<\ell$, the two identifications of $\partial_{\{k,\ell\}} X_I$ with

$$\coprod_{\substack{K\subset J\subset I\\ |K|=k,|J|=\ell}} X_K \times X_{J\setminus K} \times X_{I\setminus J}$$

are the same. See for instance [24, Lemma 3.17].

13.3. **Quotienting by the symmetric group.** In this section, we will consider the diagonal actions by the symmetric group on $\mathbb{R}^n \times \mathbb{R}^n$, $\mathbb{R}^n \times \mathbb{A}^{n-1}$, and $\mathbb{R}^n \times \Pi^{n-1}$. We will prove that the quotient $(\mathbb{R}^n \times \Pi_n)/S_n$ is diffeomorphic to $\mathbb{R}^n \times \mathbb{R}^{n-1}_+$; this is stated more precisely as Proposition 13.1 below.

Let $\pi\colon \mathbb{R}^n\times\mathbb{R}^n \to \big(\mathbb{R}^n\times\mathbb{R}^n\big)/S_n$ denote the projection to the quotient. Consider the well-known homeomorphism

$$\psi_n\colon \big(\mathbb{R}^n\times\mathbb{R}^n\big)/S_n \to \mathbb{R}^{2n}$$

given by the Viète relations. In more detail, let $\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n$ be the coordinates on $\mathbb{R}^n\times\mathbb{R}^n$, which we group into complex variables $\alpha_j+i\beta_j$, and let $a_1,\dots,a_n,b_1,\dots,b_n$ be the coordinates of the target $\mathbb{R}^{2n}$, which we also group into complex variables a_j+ib_j. Then the function ψ_n is induced by the smooth function $\psi_n\circ\pi\colon \mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^{2n}$, which is given by equating the coefficients of the polynomial

$$\prod_{j=1}^{n}(z-\alpha_j-i\beta_j) = z^n-(a_1+ib_1)z^{n-1}+(a_2+ib_2)z^{n-2}-\dots+(-1)^n(a_n+ib_n). \tag{13.4}$$

Note that ψ_n restricts to a homeomorphism $\big(\mathbb{R}^n\times\mathbb{A}^{n-1}\big)/S_n\to\mathbb{R}^{2n-1}$, which we also denote by ψ_n. To wit, $\mathbb{R}^n\times\mathbb{A}^{n-1}$ is the subspace given by $\beta_1+\dots+\beta_n=n(n+1)/2$, and so its image is given by the subspace $b_1=n(n+1)/2$.

Proposition 13.1. *The image of $\big(\mathbb{R}^n\times\Pi_n\big)/S_n$ under ψ_n is a smooth submanifold with corners of $\mathbb{R}^{2n-1}$, and there is a diffeomorphism*

$$\Psi_n\colon \mathbb{R}^n\times\mathbb{R}_+^{n-1}\to\psi_n(\big(\mathbb{R}^n\times\Pi_n\big)/S_n). \tag{13.5}$$

Moreover, the composition $\iota\circ\Psi_n\colon\mathbb{R}^n\times\mathbb{R}_+^{n-1}\to\mathbb{R}^{2n-1}$ (where ι is the inclusion) is a proper smooth embedding and the composition $\Psi_n^{-1}\circ\psi_n\circ\pi\colon\mathbb{R}^n\times\Pi_n\to\mathbb{R}^n\times\mathbb{R}_+^{n-1}$ is a smooth map of $\langle n-1\rangle$ manifolds which is a smooth covering map away from the big diagonal (i.e., the subset on which the S_n-action is not free). That is, we have the following diagram

$$\begin{array}{ccccccc}
 & & \big(\mathbb{R}^n\times\Pi_n\big)/S_n & & & & \\
\mathbb{R}^n\times\Pi_n & \xrightarrow{\pi} & \hookrightarrow & \xrightarrow{\psi_n} & \psi_n(\big(\mathbb{R}^n\times\Pi_n\big)/S_n) & \xleftarrow{\Psi_n} & \mathbb{R}^n\times\mathbb{R}_+^{n-1} \\
\cup & & \big(\mathbb{R}^n\times\mathbb{A}^{n-1}\big)/S_n & & \cup & & \\
\mathbb{R}^n\times\mathbb{A}^{n-1} & \xrightarrow{\pi} & \hookrightarrow & \xrightarrow{\psi_n} & \mathbb{R}^{2n-1} & & \\
\cup & & \big(\mathbb{R}^n\times\mathbb{R}^n\big)/S_n & & \cup & & \\
\mathbb{R}^n\times\mathbb{R}^n & \xrightarrow{\pi} & & \xrightarrow{\psi_n} & \mathbb{R}^{2n} & &
\end{array} \tag{13.6}$$

Here, the thick arrows are smooth and the solid-head arrows are homeomorphisms (and the solid-head thick arrow is the diffeomorphism Ψ_n) and all the embeddings are proper.

Moreover, these maps will be compatible with the identifications of facets $F_S\subset\partial_k\Pi_n$ with $\Pi_k\times\Pi_{n-k}$ from Item (Π-6). Specifically, the following diagram will commute:

$$\begin{array}{ccccc}
(\mathbb{R}^k\times\Pi_k)\times(\mathbb{R}^{n-k}\times\Pi_{n-k}) & \xleftarrow{\cong} & \mathbb{R}^n\times F_S & \hookrightarrow & \mathbb{R}^n\times\Pi_n \\
\downarrow{\scriptstyle (\Psi_k^{-1}\psi_k\pi,\,\Psi_{n-k}^{-1}\psi_{n-k}\pi)} & & & & \downarrow{\scriptstyle\pi} \\
(\mathbb{R}^k\times\mathbb{R}_+^{k-1})\times(\mathbb{R}^{n-k}\times\mathbb{R}_+^{n-k-1}) & & & & \big(\mathbb{R}^n\times\Pi_n\big)/S_n \\
\downarrow{\scriptstyle\cong} & & & & \downarrow{\scriptstyle\psi_n} \\
\mathbb{R}^n\times(\mathbb{R}_+^{k-1}\times\{0\}\times\mathbb{R}_+^{n-k-1}) & \hookrightarrow & \mathbb{R}^n\times\mathbb{R}_+^{n-1} & \xhookrightarrow{\Psi_n} & \mathbb{R}^{2n-1}
\end{array} \tag{13.7}$$

Here, the identification $\mathbb{R}^k\times\mathbb{R}^{n-k}\xrightarrow{\cong}\mathbb{R}^n$ in the bottom-left vertical arrow is the usual one. However, the identification $\mathbb{R}^n\xrightarrow{\cong}\mathbb{R}^k\times\mathbb{R}^{n-k}$ on the top-left horizontal arrow is similar to the one from

Item (Π-6); that is, we identify $\mathbb{R}^n$, $\mathbb{R}^k$, $\mathbb{R}^{n-k}$ with $\prod_{j\in\{1,\dots,n\}}\mathbb{R}$, $\prod_{j\in S}\mathbb{R}$ and $\prod_{j\in S^c}\mathbb{R}$, respectively, and then identify $\prod_{j\in\{1,\dots,n\}}\mathbb{R}$ with $\prod_{j\in S}\mathbb{R}\times\prod_{j\in S^c}\mathbb{R}$.

Proof. The construction of the diffeomorphism from Equation (13.5) is inductive. For the base case $n=1$, the maps π and ψ_1 are identity (after naturally identifying the relevant spaces with $\mathbb{R}$), and we may define Ψ_1 to be the identity as well.

We explicitly also do the next case $n=2$ to help build intuition. (Also, a portion of the proof for $n>2$ does not generalize to $n=2$). Recall that $\mathbb{R}^2\times\Pi_2$ is the subset of $\mathbb{R}^2\times\mathbb{R}^2$ given by

$$\beta_1\geq 1,\qquad \beta_2\geq 1,\qquad \beta_1+\beta_2=3,$$

and $\mathbb{R}^2\times\partial\Pi_2$ is given by $\beta_1=1$ (and hence $\beta_2=2$) or $\beta_2=1$ (and hence $\beta_1=2$). We want to satisfy the compatibility from Equation (13.7), so we need to analyse the image of $\big(\mathbb{R}^2\times\partial\Pi_2\big)/S_2$ under ψ_2. Since we are quotienting by the action of S_2, we may assume $\beta_1=1$ and $\beta_2=2$.

From Equation (13.4), the image of the boundary is given by the coefficients of the polynomial

$$z^2-(a_1+ib_1)z+(a_2+ib_2)=(z-\alpha_1-i)(z-\alpha_2-2i)=z^2-(\alpha_1+\alpha_2+3i)z+(\alpha_1\alpha_2-2)+i(2\alpha_1+\alpha_2).$$

As we already observed, this lies in the subspace $\mathbb{R}^3\subset\mathbb{R}^4$ given by $b_1=3$. The image is a parametrized surface in this $\mathbb{R}^3$, given by

$$a_1=\alpha_1+\alpha_2,\qquad a_2=\alpha_1\alpha_2-2,\qquad b_2=2\alpha_1+\alpha_2. \tag{13.8}$$

Here, a_1,a_2,b_2 are the coordinates of $\mathbb{R}^3$ and α_1,α_2 are the parameters. We may solve for α_1,α_2 in terms of a_1,b_2, and so this surface can also be written as $\{a_2=3a_1b_2-2a_1^2-b_2^2-2\}$. This is a graph of 2-variable function in a_1,b_2, and therefore represents a properly embedded $\mathbb{R}^2$ via the map $(a_1,b_2)\mapsto(a_1,3a_1b_2-2a_1^2-b_2^2-2,b_2)$. Furthermore, its complement has two components, say $A=\{a_2<3a_1b_2-2a_1^2-b_2^2-2\}$ and $B=\{a_2>3a_1b_2-2a_1^2-b_2^2-2\}$, and the closure of each is a properly embedded $\mathbb{R}^2\times\mathbb{R}_+$ via the maps $(a_1,b_2,t)\mapsto(a_1,3a_1b_2-2a_1^2-b_2^2-2-t,b_2)$ and $(a_1,b_2,t)\mapsto(a_1,3a_1b_2-2a_1^2-b_2^2-2+t,b_2)$, respectively.

On the ambient $\mathbb{R}^3$, define the continuous function

$$\Sigma_1\colon\mathbb{R}^3\to\mathbb{R}$$

as the minimum of the two imaginary parts of the two roots of $z^2-(a_1+3i)z+(a_2+ib_2)$. Then the given surface $\psi_2(\big(\mathbb{R}^2\times\partial\Pi_2\big)/S_2)$ is precisely the subspace $\{\Sigma_1=1\}$, while $\psi_2(\big(\mathbb{R}^2\times\Pi_2\big)/S_2)$ is the subspace $\{\Sigma_1\geq 1\}$. By the intermediate value theorem, the latter is one of the closures $\bar{A}$ or $\bar{B}$, which is indeed diffeomorphic to $\mathbb{R}^2\times\mathbb{R}_+$. Indeed, we can determine that it is $\bar{A}$ since the value of ψ_2 at any specific point of $\big(\mathbb{R}^2\times\mathring{\Pi}_1\big)/S_2$ (say $\alpha_1=\alpha_2=0,\beta_1=\beta_2=3/2$) lies in A (at the point $a_1=b_2=0,a_2=-9/4$).

Although in this case we were explicitly able to see the parametrized surface as a properly embedded $\mathbb{R}^2\subset\mathbb{R}^3$, there is a more abstract argument which also works. The abstract argument is also needed to check compatibility. Let us take $S=\{1\}$ (respectively, $S=\{2\}$), and start with the point $((\alpha_1,\alpha_2),(1,2))\in\mathbb{R}^2\times F_S$ (respectively, $((\alpha_1,\alpha_2),(2,1))\in\mathbb{R}^2\times F_S$) in the top-middle vertex of Equation (13.7). Starting left and following four arrows, we get

$$((\alpha_1,\alpha_2),(1,2))\mapsto((\alpha_1,1),(\alpha_2,1))\mapsto((\alpha_1),(\alpha_2))\mapsto((\alpha_1,\alpha_2),(0))\mapsto((\alpha_1,\alpha_2),(0))\in\mathbb{R}^2\times\mathbb{R}^1_+$$

(respectively,

$$((\alpha_1,\alpha_2),(2,1))\mapsto((\alpha_2,1),(\alpha_1,1))\mapsto((\alpha_2),(\alpha_1))\mapsto((\alpha_2,\alpha_1),(0))\mapsto((\alpha_2,\alpha_1),(0))\in\mathbb{R}^2\times\mathbb{R}^1_+),$$

while starting right, and following three arrows, we get

$$((\alpha_1,\alpha_2),(1,2))\mapsto((\alpha_1,\alpha_2),(1,2))\mapsto[((\alpha_1,\alpha_2),(1,2))]\mapsto(\alpha_1+\alpha_2,\alpha_1\alpha_2-2,2\alpha_1+\alpha_2)$$

(respectively,

$$((\alpha_1,\alpha_2),(2,1)) \mapsto ((\alpha_1,\alpha_2),(2,1)) \mapsto [((\alpha_2,\alpha_1),(1,2))] \mapsto (\alpha_2+\alpha_1, \alpha_2\alpha_1 - 2, 2\alpha_2+\alpha_1)).$$

Therefore, in either case, the compatibility condition tells us that the map $\Psi_2 \colon \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}^3$ must restrict on the boundary to the given parametrization from Equation (13.8). Letting $\Psi_|$ denote the map $\mathbb{R}^2 \to \mathbb{R}^3$ given by Equation (13.8), we then have to prove the following:

(1) **Properly embedded:** We have to prove $\Psi_| \colon \mathbb{R}^2 \to \mathbb{R}^3$ is a proper smooth embedding, that is, it is a proper map which is an immersion and a diffeomorphism onto its image.

For injectivity, note that image of $\Psi_|$ is the subspace $\{\Sigma_1 = 1\}$ of $\mathbb{R}^3$, which consists of monic quadratic polynomials whose one root has imaginary part 1 and the other root has imaginary part 2; and such polynomials *uniquely* factorize as $(z-\alpha_1-i)(z-\alpha_2-2i)$.

To show it is an immersion, consider the map $\psi_2 \circ \pi \colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^4$ given by the Viète relations, as shown in the bottom row of Equation (13.6). It is well-known that away from the diagonal this is a local diffeomorphism (that is, roots of a polynomial are smooth functions of its coefficients), so $d(\psi_2 \circ \pi)$ has rank 4 at each point. Our map $\Psi_|$ is the restriction to the 2-dimensional affine subspace $\beta_1 = 1, \beta_2 = 2$ (which lies in the complement of the diagonal), and so $d\Psi_|$ has rank 2 at each point.

The statement that the inverse map is smooth is again just the statement that the roots of a polynomial are smooth functions of its coefficients.

Finally, $\Psi_|$ is automatically proper since it is an embedding with closed image $\{\Sigma_1 = 1\} \subset \mathbb{R}^3$.

(2) **Extendable:** We have to prove the proper smooth embedding $\Psi_| \colon \mathbb{R}^2 \to \mathbb{R}^3$ extends to a proper smooth embedding $\Psi_2 \colon \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}^3$ with image $\bar{A} = \{\Sigma_1 \geq 1\} = \psi_2\big((\mathbb{R}^2 \times \Pi_2)/S_2\big)$. This is the part where the general proof for $n > 2$ does not work, so we have to fall back to our explicit computation from before and simply set

$$\Psi_2(\alpha_1,\alpha_2,t) = (\alpha_1+\alpha_2, \alpha_1\alpha_2 - 2 - t, 2\alpha_1+\alpha_2). \tag{13.9}$$

Let us now do the general case for $n \geq 3$. We will use the following.

Proposition 13.2. *Assume $m \geq 4$. Let $i \colon \mathbb{R}^m \hookrightarrow \mathbb{R}^{m+1}$ be a proper smooth embedding. Then the embedding splits $\mathbb{R}^{m+1}$ into two pieces, each the image of a proper smooth embedding of $\mathbb{R}^m \times \mathbb{R}_+$; moreover, the embeddings of $\mathbb{R}^m \times \mathbb{R}_+$ may be chosen to agree with i on the boundary.*

Proof. The Jordan-Brouwer theorem says that the complement of $i(\mathbb{R}^m)$ has two connected components. Let A be the closure of one of these components. Then A is a smooth manifold with boundary, and we need to show it is a properly embedded $\mathbb{R}^m \times \mathbb{R}_+$.

By taking one-point compactifications, we can extend i to an embedding of S^m into S^{m+1}, smooth away from a point. By a result of Kirby [19], since $m \geq 4$, the embedding cannot fail to be locally flat at exactly one point. Therefore, it is locally flat. By the topological Schönflies theorem [9, 35, 36], S^m splits S^{m+1} into two pieces, each homeomorphic to B^m. After removing the point at infinity, we get that A is homeomorphic to $\mathbb{R}^m \times \mathbb{R}_+$. We will show that A is diffeomorphic to $\mathbb{R}^m \times \mathbb{R}_+$.

We also know that A is smooth, and ∂A is a properly embedded $\mathbb{R}^m$. Using the tubular neighborhood theorem for proper smooth embeddings, choose a tubular neighborhood V of ∂A in A, with closure $\bar{V}$, such that we have a diffeomorphism

$$\phi_1 \colon \bar{V} \to \mathbb{R}^m \times [0,\epsilon].$$

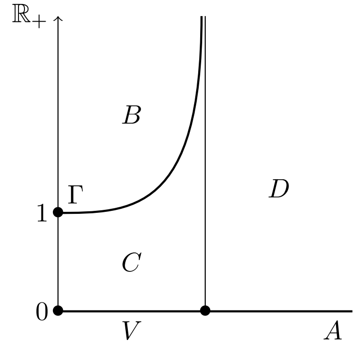


FIGURE 26. The smooth manifold (with corners) $A \times \mathbb{R}_+$. The region $C \cup D$ is a proper h-cobordism with boundary, from A to Γ.

By the same argument as for A (taking one-point compactifications), we find that $A \setminus V$ is homeomorphic to $\mathbb{R}^m \times \mathbb{R}_+$. Let

$$\phi_2 \colon (A \setminus V) \to \mathbb{R}^m \times [\epsilon, \infty)$$

be a homeomorphism.

Let us identify $\mathbb{R}^m \times \{\epsilon\}$ with $\mathbb{R}^m$, and view the restriction of $\phi_1 \circ \phi_2^{-1}$ as a self-homeomorphism $h \colon \mathbb{R}^m \to \mathbb{R}^m$. This can be extended to a self-homeomorphism

$$\tilde{h} \colon \mathbb{R}^m \times [\epsilon, \infty) \to \mathbb{R}^m \times [\epsilon, \infty), \qquad \tilde{h}(x,t) = (h(x), t).$$

By replacing ϕ_2 with $\tilde{h} \circ \phi_2$, we can assume without loss of generality that ϕ_1 and ϕ_2 coincide on the boundary $\partial(A \setminus V)$. Let $\bar{\phi} \colon A \to \mathbb{R}^m \times [0, \infty)$ be the homeomorphism obtained by gluing ϕ_1 and ϕ_2.

Consider the product $A \times \mathbb{R}_+$, which is homeomorphic to $\mathbb{R}^m \times \mathbb{R}_+ \times \mathbb{R}_+$ by $\bar{\phi} \times \mathrm{id}$. This is an $(m+1)$-dimensional smooth manifold with corners. Fix a strictly increasing smooth function $f \colon [0, \epsilon) \to \mathbb{R}_+$ with $f(0) = 1$, $f'(0) = 0$, and $\lim_{t \to \epsilon} f(t) = \infty$, for instance $f(t) = \sec\left(\frac{\pi t}{2\epsilon}\right)$. Inside the half-strip $V \times \mathbb{R}_+$, consider the graph Γ of the smooth composition function

$$V \xrightarrow{\phi_1} \mathbb{R}^m \times [0, \epsilon) \xrightarrow{\pi_2} [0, \epsilon) \xrightarrow{f} \mathbb{R}_+.$$

Note that Γ is diffeomorphic to $\mathbb{R}^m \times \mathbb{R}_+$ and splits the collar $V \times \mathbb{R}_+$ into two pieces B (above Γ) and C (below Γ), as in Figure 26.

Let $D = (A \setminus V) \times \mathbb{R}_+$. Then $C \cup D$ is a proper cobordism with boundary, from A to Γ. In fact, the homeomorphism $\bar{\phi} \times \mathrm{id}$ sends $C \cup D$ to $\mathbb{R}^m \times E$, where $E \subset \mathbb{R}^2_+$ is the region in the first quadrant that lies below and to the right of the graph of the function $f(t), 0 \leq t < \epsilon$. Since E is diffeomorphic to $\mathbb{R}_+ \times [0,1]$, we find that $C \cup D$ is homeomorphic to $\mathbb{R}^m \times \mathbb{R}_+ \times [0,1]$. Thus, $C \cup D$ is a proper h-cobordism with boundary, with inclusion of either end a simple equivalence, and has dimension $m+2 \geq 6$. Siebenmann's proper h-cobordism theorem [49] (applied to cobordisms of manifolds with boundary, where the boundary cobordism is a cylinder, cf. [49, Footnote 1 on pp. 484]) implies that $C \cup D$ is diffeomorphic to a cylinder. Therefore, A is diffeomorphic to $\Gamma \cong \mathbb{R}^m \times \mathbb{R}_+$.

If $j \colon \mathbb{R}^m \times \mathbb{R}_+ \xrightarrow{\cong} A$ is any diffeomorphism, then let $g \colon \mathbb{R}^m \to \mathbb{R}^m$ denote the self-diffeormorphism $j^{-1} \circ i$; extend this to a self-diffeomorphism $\tilde{g} \colon \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}^m \times \mathbb{R}_+$ by $\tilde{g}(x,t) = (g(x), t)$. Then $j \circ \tilde{g} \colon \mathbb{R}^m \times \mathbb{R}_+ \to A$ is a diffeomorphism that agrees with given embedding i on the boundary.

Moreover, the composition $\mathbb{R}^m \times \mathbb{R}_+ \xrightarrow{j\circ\tilde{g}} A \hookrightarrow \mathbb{R}^{m+1}$ is a smooth embedding with closed image, so it is a proper embedding. □

We will also need the following in order to glue diffeomorphisms on the boundary.

Proposition 13.3. *Let $i\colon \mathbb{R}^m \times \mathbb{R}_+ \hookrightarrow \mathbb{R}^{m+1}$ be a proper smooth embedding. Let $j\colon \mathbb{R}^m \times [0,1] \hookrightarrow \mathbb{R}^{m+1}$ be a collar neighborhood of $i(\mathbb{R}^m \times \{0\})$ inside $i(\mathbb{R}^m \times \mathbb{R}_+)$ which agrees with i on $\mathbb{R}^m \times \{0\}$. Then there exists a proper smooth embedding $i'\colon \mathbb{R}^m \times \mathbb{R}_+ \hookrightarrow \mathbb{R}^{m+1}$ with the same image as i which agrees with j on some open neighborhood of $\mathbb{R}^m \times \{0\}$ inside $\mathbb{R}^m \times [0,1]$.*

Proof. By pulling back by the diffeomorphism i, we may assume i is the identity map. That is, we have a collar neighborhood $j\colon \mathbb{R}^m \times [0,1] \to \mathbb{R}^m \times \mathbb{R}_+$ with $j(x,0) = (x,0)$ for all $x \in \mathbb{R}^m$, and we want to construct a diffeomorphism $i'\colon \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}^m \times \mathbb{R}_+$ which agrees with j on some open set around $\mathbb{R}^m \times \{0\}$.

Apply the local collaring uniqueness theorem [20, Theorem A.1] with $M = \mathbb{R}^m \times \{0\}$, $W = \mathbb{R}^m \times \mathbb{R}_+$, $C = \varnothing$, $D = M$, $f\colon \mathbb{R}^m \times [0,1] \to \mathbb{R}^m \times \mathbb{R}_+$ the given collar neighborhood j, and $g\colon \mathbb{R}^m \times [0,1] \to \mathbb{R}^m \times \mathbb{R}_+$ the standard inclusion. The required diffeomorphism i' is then the map $h_1\colon \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}^m \times \mathbb{R}_+$. □

We will now construct a proper smooth embedding $\Psi_n\colon \mathbb{R}^n \times \mathbb{R}_+^{n-1} \to \mathbb{R}^{2n-1}$ with image $\psi_n\big((\mathbb{R}^n \times \Pi_n)/S_n\big)$, as in Equation (13.6). Moreover, it will respect the $\langle n-1\rangle$-manifold structures, so it will map $\mathbb{R}^n \times \partial_J\mathbb{R}_+^{n-1}$ to $\psi_n\big((\mathbb{R}^n \times \partial_J\Pi_n)/S_n\big)$ for any $J \subset \{1,2,\dots,n-1\}$, where ∂_J denotes $\bigcap_{j\in J} \partial_j$.

The map Ψ_n is required to satisfy the compatibility condition from Equation (13.7); therefore, it is already defined on $\mathbb{R}^n \times \partial\mathbb{R}_+^{n-1}$ respecting the stratification given by the closed strata $\mathbb{R}^n \times \partial_J\mathbb{R}_+^{n-1}$. This statement deserves further details.

Define continuous functions $\Sigma_k\colon \mathbb{R}^{2n-1} \to \mathbb{R}$, $k = 1,\dots,n-1$, as follows: $\Sigma_k(a_1,\dots,a_n,b_2,\dots,b_n)$ is the sum of the k smallest imaginary parts among the n roots of the polynomial

$$z^n - (a_1 + in(n+1)/2)z^{n-1} + (a_2 + ib_2)z^{n-2} - \cdots + (-1)^n(a_n + ib_n).$$

Recall from Item (Π-3) that the permutohedron $\Pi_n \subset \mathbb{A}^{n-1}$ is the intersection of the half-spaces $\mathbb{H}_S = \{(x_1,\dots,x_n) \in \mathbb{A}^{n-1} \mid \sum_{j\in S} x_j \geq k(k+1)/2\}$; therefore, $\psi_n\big((\mathbb{R}^n \times \Pi_n)/S_n\big)$ is precisely the subspace

$$\{\Sigma_1 \geq 1, \Sigma_2 \geq 3, \dots, \Sigma_k \geq k(k+1)/2, \dots, \Sigma_{n-1} \geq (n-1)n/2\} \subset \mathbb{R}^{2n-1}.$$

Moreover, if $|S| = k$, then $F_S = \Pi_n \cap \partial\mathbb{H}_S$, and therefore $\psi_n\big((\mathbb{R}^n \times \partial_k\Pi_n)/S_n\big)$ is precisely the subspace

$$\{\Sigma_1 \geq 1, \Sigma_2 \geq 3, \dots, \Sigma_k = k(k+1)/2, \dots, \Sigma_{n-1} \geq (n-1)n/2\} \subset \mathbb{R}^{2n-1}.$$

More tersely, define the continuous function $\bar{\Sigma}\colon \mathbb{R}^{2n-1} \to \mathbb{R}$ by

$$\bar{\Sigma} = \min\big\{\Sigma_1 - 1, \Sigma_2 - 3, \dots, \Sigma_k - k(k+1)/2, \dots, \Sigma_{n-1} - (n-1)n/2\big\}. \tag{13.10}$$

Then $\psi_n\big((\mathbb{R}^n \times \partial\Pi_n)/S_n\big) = \{\bar{\Sigma} = 0\}$ and $\psi_n\big((\mathbb{R}^n \times \Pi_n)/S_n\big) = \{\bar{\Sigma} \geq 0\}$.

Consider the subset $S = \{1,2,\dots,k\}$, and the corresponding facet $F_S \subset \partial_k\Pi_n$. Fix any point $(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n) \in \mathbb{R}^n \times F_S$; so we have

$$\beta_1 + \cdots + \beta_k = k(k+1)/2, \qquad \beta_{k+1} + \cdots + \beta_n = \big(n(n+1) - k(k+1)\big)/2.$$

Starting from this point on the top-middle vertex of Equation (13.7), and following three arrows on the right, we get the point $(a_1,\dots,a_n,b_2,\dots,b_n) \in \mathbb{R}^{2n-1}$ by equating the coefficients of the polynomial—call it $P(z)$—from Equation (13.4). However, starting left, the first arrow takes it to the point

$$((\alpha_1,\dots,\alpha_k,\beta_1,\dots,\beta_k),(\alpha_{k+1},\dots,\alpha_n,\beta_{k+1}-k,\dots,\beta_n-k))\in(\mathbb{R}^k\times\Pi_k)\times(\mathbb{R}^{n-k}\times\Pi_{n-k}).$$

Under the map $\psi_k\circ\pi$, the point $(\alpha_1,\dots,\alpha_k,\beta_1,\dots,\beta_k)$ maps to $(c_1,\dots,c_k,d_2,\dots,d_k)\in\mathbb{R}^{2k-1}$ given by equating the coefficients of the polynomial $Q_0(z)$, again from Equation (13.4) (but with k instead of n, c_j instead of a_j, and d_j instead of b_j). Under the map $\psi_{n-k}\circ\pi$, the point $(\alpha_{k+1},\dots,\alpha_n,\beta_{k+1}-k,\dots,\beta_n-k)$ maps to the point $(\tilde{e}_1,\dots,\tilde{e}_{n-k},\tilde{f}_2,\dots,\tilde{f}_{n-k})\in\mathbb{R}^{2n-2k-1}$ given by equating the coefficients of the polynomial

$$\tilde{Q}_1(z)=\prod_{j=k+1}^{n}(z-\alpha_j-i(\beta_j-k))=z^{n-k}-(\tilde{e}_1+i\tilde{f}_1)z^{n-k-1}+\cdots+(-1)^{n-k}(\tilde{e}_{n-k}+i\tilde{f}_{n-k}),$$

with $\tilde{f}_1=\frac{(n-k)(n-k+1)}{2}$. Let $(e_1,\dots,e_{n-k},f_2,\dots,f_{n-k})\in\mathbb{R}^{2n-2k-1}$ be given by the polynomial $Q_1(z)$ from Equation (13.4) (but with $n-k$ instead of n, e_j instead of a_j, f_j instead of b_j, and roots $\alpha_j+i\beta_j$ for $k<j\le n$):

$$Q_1(z)=\prod_{j=k+1}^{n}(z-\alpha_j-i\beta_j)=z^{n-k}-(e_1+if_1)z^{n-k-1}+\cdots+(-1)^{n-k}(e_{n-k}+if_{n-k}),$$

with $f_1=\frac{n(n+1)-k(k+1)}{2}$. Then $(e_1,\dots,e_{n-k},f_2,\dots,f_{n-k})$ can be obtained from $(\tilde{e}_1,\dots,\tilde{e}_{n-k},\tilde{f}_2,\dots,\tilde{f}_{n-k})$ by applying a diffeomorphism

$$\xi_{n,k}\colon\mathbb{R}^{2n-2k-1}\xrightarrow{\cong}\mathbb{R}^{2n-2k-1},\tag{13.11}$$

namely, by equating the coefficients of the polynomial $Q_1(z)=\tilde{Q}_1(z-ik)$:

$$\begin{aligned}z^{n-k}&-(e_1+i\frac{n(n+1)-k(k+1)}{2})z^{n-k-1}+\cdots+(-1)^{n-k}(e_{n-k}+if_{n-k})\\&=(z-ik)^{n-k}-(\tilde{e}_1+i\frac{(n-k)(n-k+1)}{2})(z-ik)^{n-k-1}+\cdots+(-1)^{n-k}(\tilde{e}_{n-k}+i\tilde{f}_{n-k}).\end{aligned}$$

Assume

$$\begin{aligned}\Psi_k^{-1}(c_1,\dots,c_k,d_2,\dots,d_k)&=(s_1,\dots,s_k,t_1,\dots,t_{k-1})\in\mathbb{R}^k\times\mathbb{R}_+^{k-1}\\\Psi_{n-k}^{-1}(\tilde{e}_1,\dots,\tilde{e}_{n-k},\tilde{f}_2,\dots,\tilde{f}_{n-k})&=(s_{k+1},\dots,s_n,t_{k+1},\dots,t_{n-1})\in\mathbb{R}^{n-k}\times\mathbb{R}_+^{n-k-1}.\end{aligned}$$

Then the compatibility condition from Equation (13.7) states that Ψ_n must be defined on $\mathbb{R}^n\times\partial_k\mathbb{R}_+^{n-1}$ as follows:

$$\Psi_n((s_1,\dots,s_n),(t_1,\dots,t_{k-1},0,t_{k+1},\dots,t_{n-1}))=(a_1,\dots,a_n,b_2,\dots,b_n).$$

This map has an explicit description in terms of Ψ_k and Ψ_{n-k}. We have

$$\begin{aligned}\Psi_k(s_1,\dots,s_k,t_1,\dots,t_{k-1})&=(c_1,\dots,c_k,d_2,\dots,d_k)\in\mathbb{R}^{2k-1}\\\xi_{n,k}\circ\Psi_{n-k}(s_{k+1},\dots,s_n,t_{k+1},\dots,t_{n-1})&=(e_1,\dots,e_{n-k},f_2,\dots,f_{n-k})\in\mathbb{R}^{2n-2k-1},\end{aligned}$$

and since $P(z)=Q_0(z)Q_1(z)$, the point $(a_1,\dots,a_n,b_2,\dots,b_n)$ can be obtained from $(c_1,\dots,c_k,d_2,\dots,d_k)$ and $(e_1,\dots,e_{n-k},f_2,\dots,f_{n-k})$ by equating the coefficients of

$$z^n-(a_1+i\frac{n(n+1)}{2})z^{n-1}+\cdots+(-1)^n(a_n+ib_n)$$

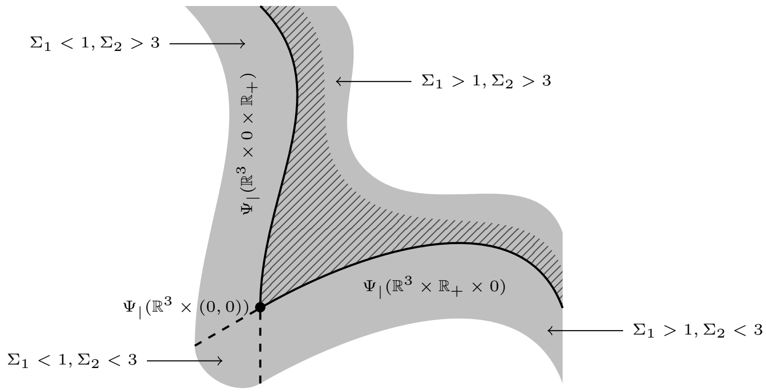


FIGURE 27. The embedding $\Psi_|\colon \mathbb{R}^3 \times \partial\mathbb{R}^2_+ \to \mathbb{R}^5$ (for the case $n = 3$). We have indicated the signs of the functions $\Sigma_1 - 1$ and $\Sigma_2 - 3$ in a neighborhood (shown in gray) of $\Psi_|(\mathbb{R}^3 \times \partial\mathbb{R}^2_+)$. A collar extension (drawn striped) is also shown, which will be used to extend $\Psi_|$ to $\Psi_3\colon \mathbb{R}^3 \times \mathbb{R}^2_+ \to \mathbb{R}^5$.

$$\begin{aligned} &= \big(z^k - (c_1 + i\frac{k(k+1)}{2})z^{k-1} + \cdots + (-1)^k(c_n + id_n)\big) \\ &\quad \times \big(z^{n-k} - (e_1 + i\frac{n(n+1) - k(k+1)}{2})z^{n-k-1} + \cdots + (-1)^{n-k}(e_{n-k} + if_{n-k})\big). \end{aligned}$$

As before, let $\Psi_|$ denote this map $\mathbb{R}^n \times \partial\mathbb{R}^{n-1}_+ \to \mathbb{R}^{2n-1}$ with image $\psi_n\big((\mathbb{R}^n \times \partial\Pi_n)/S_n\big)$. We have to extend it to a map $\Psi_n\colon \mathbb{R}^n \times \mathbb{R}^{n-1}_+ \to \mathbb{R}^{2n-1}$ which is a diffeomorphism onto its image $\psi_n\big((\mathbb{R}^n \times \Pi_n)/S_n\big)$. Therefore, we need to check the following.

(1) **Well-defined:** For well-definedness we have to prove the following couple of things.

The map $\Psi_|$ on $\mathbb{R}^n \times \partial_k\mathbb{R}^{n-1}_+$ was defined from the subset $S = \{1, 2, \ldots, k\}$ via Equation (13.7). We need to show that for any other subset S' with $|S'| = k$, Equation (13.7) would have produced the same map. Fix any point $p' = (\alpha'_1, \ldots, \alpha'_n, \beta'_1, \ldots, \beta'_n) \in \mathbb{R}^n \times F_{S'}$. There exists some permutation $\sigma \in S_n$ which maps this point to some point $p = (\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n) \in \mathbb{R}^n \times F_S$. Since we quotient by S_n, if we start at either p or p' at the top-middle vertex of Equation (13.7) and proceed rightwards by three arrows, we will end up at the same point $(a_1, \ldots, a_n, b_2, \ldots, b_n) \in \mathbb{R}^{2n-1}$. However, if we proceed leftwards by one arrow, we will end up at the point

$$((\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k), (\alpha_{k+1}, \ldots, \alpha_n, \beta_{k+1} - k, \ldots, \beta_n - k)) \in (\mathbb{R}^k \times \Pi_k) \times (\mathbb{R}^{n-k} \times \Pi_{n-k});$$

this is due to way the identification $\mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^k \times \mathbb{R}^{n-k}$ is set up in Equation (13.7). Therefore, the map $\Psi_|$ defined using the subset S' will be same as the map $\Psi_|$ defined using the subset $S = \{1, 2, \ldots, k\}$.

We also need to check that the maps $\Psi_|$, as defined on $\mathbb{R}^n \times \partial_k\mathbb{R}^{n-1}_+$ and $\mathbb{R}^n \times \partial_\ell\mathbb{R}^{n-1}_+$, agree on their common boundary $\mathbb{R}^n \times \partial_{\{k,\ell\}}\mathbb{R}^{n-1}_+$. Let $k < \ell$, and let $S = \{1, 2, \ldots, k\}$ and $T = \{1, 2, \ldots, \ell\}$. Using repeated applications of Equation (13.7), it is not hard to show that in either case, the map $\Psi_|\colon \mathbb{R}^n \times \partial_{\{k,\ell\}}\mathbb{R}^{n-1}_+ \to \mathbb{R}^{2n-1}$ is given as follows. For any point

$p = ((s_1, \dots, s_n), (t_1, \dots, t_{k-1}, 0, t_{k+1}, \dots, t_{\ell-1}, 0, t_{\ell+1}, \dots, t_{n-1})) \in \mathbb{R}^n \times \partial_{\{k,\ell\}}\mathbb{R}^{n-1}_+$, let

$$\begin{aligned}(c_1, \dots, c_k, d_2, \dots, d_k) &= \Psi_k(s_1, \dots, s_k, t_1, \dots, t_{k-1})\\ (e_1, \dots, e_{\ell-k}, f_2, \dots, f_{\ell-k}) &= \xi_{\ell,k} \circ \Psi_{\ell-k}(s_{k+1}, \dots, s_\ell, t_{k+1}, \dots, t_{\ell-1})\\ (g_1, \dots, g_{n-\ell}, h_2, \dots, h_{n-\ell}) &= \xi_{n,\ell} \circ \Psi_{n-\ell}(s_{\ell+1}, \dots, s_n, t_{\ell+1}, \dots, t_{n-1}).\end{aligned}$$

where $\xi_{\ell,k} \colon \mathbb{R}^{2\ell-2k-1} \to \mathbb{R}^{2\ell-2k-1}$ and $\xi_{n,\ell} \colon \mathbb{R}^{2n-2\ell-1} \to \mathbb{R}^{2n-2\ell-1}$ are as defined in Equation (13.11). Then $\Psi_|(p) = (a_1, \dots, a_n, b_2, \dots, b_n)$ is obtained by equating the coefficients of

$$\begin{aligned}&z^n - (a_1 + i\frac{n(n+1)}{2})z^{n-1} + \dots + (-1)^n(a_n + ib_n)\\ &\quad= \big(z^k - (c_1 + i\frac{k(k+1)}{2})z^{k-1} + \dots + (-1)^k(c_n + id_n)\big)\\ &\qquad\times \big(z^{\ell-k} - (e_1 + i\frac{\ell(\ell+1) - k(k+1)}{2})z^{\ell-k-1} + \dots + (-1)^{\ell-k}(e_{\ell-k} + if_{\ell-k})\big)\\ &\qquad\times \big(z^{n-\ell} - (g_1 + i\frac{n(n+1) - \ell(\ell+1)}{2})z^{n-\ell-1} + \dots + (-1)^{n-\ell}(g_{n-\ell} + ih_{n-\ell})\big).\end{aligned}$$

(2) **Properly embedded:** We next have to prove $\Psi_| \colon \mathbb{R}^n \times \partial\mathbb{R}^{n-1}_+ \to \mathbb{R}^{2n-1}$ is a proper smooth embedding, that is, it is a proper map which is an immersion and a diffeomorphism onto its image.

For injectivity, we first check that the image of different open strata of $\mathbb{R}^n \times \partial\mathbb{R}^{n-1}_+$ are disjoint. This follows from the observation that for any non-empty subset $J \subset \{1, 2, \dots, n-1\}$, the image of the interior of $\mathbb{R}^n \times \partial_J\mathbb{R}^{n-1}_+$ is given by

$$\{p \in \mathbb{R}^{2n-1} \mid \Sigma_k(p) = \frac{k(k+1)}{2}, \forall k \in J \text{ and } \Sigma_k(p) > \frac{k(k+1)}{2}, \forall k \notin J\}. \tag{13.12}$$

So it is now enough to show that $\Psi_|$, restricted to any open stratum, is injective. Fix any stratum $\mathbb{R}^n \times \partial_J(\mathbb{R}^n)$ with $J = \{k_1 < k_2 < \dots < k_m\}$, and fix any point

$$p = ((s_1, \dots, s_n), (t_1, \dots, t_{k_1-1}, 0, t_{k_1+1}, \dots, t_{k_2-1}, 0, t_{k_2+1}, \dots, t_{k_m-1}, 0, t_{k_m+1}, \dots, t_{n-1}))$$

in its interior. Let $\Psi_|(p) = (a_1, \dots, a_n, b_2, \dots, b_n) \in \mathbb{R}^{2n-1}$ and consider the polynomial

$$P(z) = z^n - (a_1 + ib_1)z^{n-1} + \dots + (-1)^n(a_n + ib_n) \tag{13.13}$$

with $b_1 = \frac{n(n+1)}{2}$. In the same vein as the definition of $\Psi_|$ (and also the second part of the above argument for well-definedness), for $\ell = 0, 1, \dots, m$, define

$$(c^\ell_1, \dots, c^\ell_{k_{\ell+1}-k_\ell}, d^\ell_2, \dots, d^\ell_{k_{\ell+1}-k_\ell}) = \xi_{k_{\ell+1},k_\ell} \circ \Psi_{k_{\ell+1}-k_\ell}(s_{k_\ell+1}, \dots, s_{k_{\ell+1}}, t_{k_\ell+1}, \dots, t_{k_{\ell+1}-1})$$

(with the understanding that $k_0 = 0$ and $k_{m+1} = n$), and consider the polynomial

$$Q_\ell(z) = z^{k_{\ell+1}-k_\ell} - (c^\ell_1 + id^\ell_1)z^{k_{\ell+1}-k_\ell-1} + \dots + (-1)^{k_{\ell+1}-k_\ell}(c^\ell_{k_{\ell+1}-k_\ell} + id^\ell_{k_{\ell+1}-k_\ell}) \tag{13.14}$$

with $d^\ell_1 = \frac{k_{\ell+1}(k_{\ell+1}+1) - k_\ell(k_\ell+1)}{2}$. Then we have

$$P(z) = Q_0(z)Q_1(z)\cdots Q_m(z) \tag{13.15}$$

and expanding the coefficients, we get $(a_1, \dots, a_n, b_2, \dots, b_n)$ as a function of $(c^1_1, \dots, c^m_{n-k_m}, d^1_2, \dots, d^m_{n-k_m})$. The maps $\Psi_{k_{\ell+1}-k_\ell}$ are embeddings and the maps $\xi_{k_{\ell+1},k_\ell}$ are diffeomorphisms, so the only place non-injectivity might arise is in this final map. Let $\alpha_1 + i\beta_1, \dots, \alpha_n + i\beta_n$ be the roots of $P(z)$ arranged in increasing order of their imaginary parts, that is, $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_n$. From Equation (13.12), we know:

$$\sum_{j=1}^{k} \beta_j = \frac{k(k+1)}{2}, \forall k \in J \cup \{n\}, \qquad \sum_{j=1}^{k} \beta_j > \frac{k(k+1)}{2}, \forall k \in \{1, 2, \dots, n-1\} \setminus J$$

Therefore, for every $k \in J$,

$$\begin{aligned} \beta_k &= \sum_{j=1}^{k} \beta_j - \sum_{j=1}^{k-1} \beta_j \leq \frac{k(k+1)}{2} - \frac{(k-1)k}{2} = k \\ \beta_{k+1} &= \sum_{j=1}^{k+1} \beta_j - \sum_{j=1}^{k} \beta_j \geq \frac{(k+1)(k+2)}{2} - \frac{k(k+1)}{2} = k+1. \end{aligned} \tag{13.16}$$

Therefore, the set of k smallest β_j's is well-defined, for every $k \in J$. Note that for every $1 \leq \ell \leq m$, the polynomial $Q_0(z)Q_1(z) \cdots Q_{\ell-1}(z)$ has roots $\alpha_j + i\beta_j$, for the k_ℓ smallest β_j's. Therefore, the factorization from Equation (13.15) is the unique one of that form, thus completing the proof of injectivity.

Being an immersion is a local condition. Since $\mathbb{R}^n \times \partial\mathbb{R}^{n-1}_+$ is not a smooth manifold, rather the boundary of a manifold with corners, let us clarify what we mean by an immersion. For any point $p \in \mathbb{R}^n \times \partial\mathbb{R}^{n-1}_+$, we will construct a neighborhood U of p inside $\mathbb{R}^n \times \mathbb{R}^{n-1}$ and an extension of $\Psi_|$ to U which is a smooth embedding. Continuing from the proof of injectivity, fix a point p in some open stratum $\mathbb{R}^n \times \mathring{\partial}_J\mathbb{R}^{n-1}_+$, and let us reuse the same notation. Instead of parametrizing $\Psi_|(\mathbb{R}^n \times \mathring{\partial}_J(\mathbb{R}^n)$ by the parameters s_j $(1 \leq j \leq n)$ and t_j $(j \in \{1, \dots, n-1\} \setminus J)$, let us parametrize it by the variables c^ℓ_j $(0 \leq \ell \leq m,\ 1 \leq j \leq k_{\ell+1} - k_\ell)$ and d^ℓ_j $(0 \leq \ell \leq m,\ 2 \leq j \leq k_{\ell+1} - k_\ell)$, which is a valid reparametrization since the maps $\Psi_{k_{\ell+1}-k_\ell}$ are smooth embeddings and the maps $\xi_{k_{\ell+1},k_\ell}$ are diffeomorphisms. As before, create and set additional variables $d^\ell_1 = \frac{k_{\ell+1}(k_{\ell+1}+1)-k_\ell(k_\ell+1)}{2}$, for $0 \leq \ell \leq m$. Define the polynomials $Q_\ell(z)$ as in Equation (13.14), and if we define the polynomial $P(z)$ using Equations (13.13) and (13.15), this determines the variables $a_1, \dots, a_n, b_1, \dots, b_n$ as smooth functions of the variables (c^ℓ_j, d^ℓ_j) (with $b_1 = \frac{n(n+1)}{2}$). We now let the variables d^ℓ_1 vary in small neighborhoods, and apply the same function to get the variables a_j, b_j (of course, now b_1 also varies). Since the inequalities from Equation (13.16) still hold up to some small $\epsilon > 0$, we still have $\beta_k < \beta_{k+1}$ for all $k \in J$, and therefore, the factorization from Equation (13.15) is still well-defined, and so the function $(c^\ell_j, d^\ell_j) \mapsto (a_j, b_j)$ is still injective. Since the roots of a polynomial are smooth functions of its coefficients, this is a local diffeomorphism, and therefore its linearization has rank $2n$ near p. If we restrict to the affine subspace $\sum_\ell d^\ell_1 = \frac{n(n+1)}{2}$ (corresponding to the affine subspace $b_1 = \frac{n(n+1)}{2}$ in the target), we get a local diffeomorphism from some neighborhood U of p in $\mathbb{R}^n \times \mathbb{R}^{n-1}$ to some neighborhood of $\Psi_|(p)$ in $\mathbb{R}^{2n-1}$.

To finish the argument that $\Psi_|$ is a smooth embedding, we need to prove that it is a diffeomorphism onto its image. Since it is already an injective immersion, we simply have to show that the inverse map is continuous. This is again the statement that roots of a polynomial are continuous functions of its coefficients.

The statement that $\Psi_| : \mathbb{R}^n \times \partial\mathbb{R}^{n-1}_+ \to \mathbb{R}^{2n-1}$ is proper is automatic since it is an embedding with closed image $\{\bar{\Sigma} = 0\}$, where $\bar{\Sigma}$ is the function defined in Equation (13.10).

(3) **Extendable:** Finally, we have to extend $\Psi_|$ to a proper smooth embedding $\Psi_n\colon \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^{2n-1}$ with image $\psi_n\big((\mathbb{R}^n \times \Pi_n)/S_n\big)$. Let $F\colon \mathbb{R}^n \times \mathbb{R}_+^{n-1} \to \mathbb{R}_+$ be the function

$$(s_1, \dots, s_n, t_1, \dots, t_{n-1}) \mapsto t_1 t_2 \cdots t_{n-1}.$$

Let $\mathbb{K}^{n-1} = \mathbb{R}^{n-1} \setminus \mathbb{R}_*^{n-1}$ be the union of the coordinate axes. Since $\Psi_|$ is a proper embedding, by the collar neighborhood theorem, $\Psi_|$ extends to an embedding—call it $\widetilde{\Psi}$—of some neighborhood U of $\mathbb{R}^n \times \partial\mathbb{R}_+^{n-1}$ inside $\mathbb{R}^n \times \mathbb{R}^{n-1}$. (Such an extension can be obtaining by patching together the local extensions—as constructed during the immersion proof earlier—using partitions of unity.) By rescaling if necessary, we may further assume U contains the subspace $F^{-1}([0, \epsilon])$ for some small $\epsilon > 0$, and $\widetilde{\Psi}$ restricts to a proper smooth embedding on that subspace. See Figure 27 for the case $n = 3$, which shows $\widetilde{\Psi}(U)$ (inside $\mathbb{R}^5$) in gray, $\widetilde{\Psi}(U \cap \big(\mathbb{R}^3 \times \mathbb{K}^2\big))$ as black lines (which are solid or dashed depending on whether they are in $\Psi_|(\mathbb{R}^3 \times \partial\mathbb{R}_+^2)$ or not), and $\widetilde{\Psi}(F^{-1}([0, \epsilon]))$ striped. Then $\widetilde{\Psi}(F^{-1}(\epsilon/2))$ is a smoothly properly embedded $\mathbb{R}^{2n-2}$ inside $\mathbb{R}^{2n-1}$; let A denote the component of its complement that *does not* contain $\widetilde{\Psi}(U \cap \big(\mathbb{R}^n \times \mathbb{K}^{n-1}\big))$. Using Propositions 13.2 and 13.3 with $m = 2n-2$, we get a proper smooth embedding $F^{-1}([\epsilon/2, \infty)) \to \mathbb{R}^{2n-1}$ with image A, which agrees with $\widetilde{\Psi}$ on some neighborhood of $F^{-1}(\epsilon/2)$ inside $F^{-1}([\epsilon/2, \epsilon])$; therefore, it glues with $\widetilde{\Psi}$ on $F^{-1}([0, \epsilon/2])$ and produces a proper smooth embedding $\Psi_n\colon \mathbb{R}^n \times \mathbb{R}_+^{n-1} \to \mathbb{R}^{2n-1}$.

The only thing left to check is that Ψ_n has the correct image, namely the subspace

$$\psi_n\big((\mathbb{R}^n \times \Pi_n)/S_n\big) = \{\Sigma_1 \geq 1, \dots, \Sigma_k \geq k(k+1)/2, \dots, \Sigma_{n-1} \geq (n-1)n/2\} = \{\bar{\Sigma} \geq 0\} \subset \mathbb{R}^{2n-1},$$

where $\bar{\Sigma}$ is the function defined in Equation (13.10). By the Jordan-Brouwer theorem, $\mathbb{R}^{2n-1} \setminus \Psi_|(\mathbb{R}^n \times \partial\mathbb{R}_+^{n-1})$ has two components. Let B be the component containing $\widetilde{\Psi}(F^{-1}([0, \epsilon]))$, and therefore, $B = \Psi_n(\mathbb{R}^n \times \mathbb{R}_+^{n-1})$. Let $C = \mathbb{R}^{2n-1} \setminus \bar{B}$. The construction of $\Psi_|$ ensures that $\bar{\Sigma} = 0$ precisely on $\Psi_|(\mathbb{R}^n \times \mathbb{R}_+^{n-1})$, hence $\bar{\Sigma} \neq 0$ on $B \cup C$. So we have to show $\bar{\Sigma} > 0$ somewhere on B and $\bar{\Sigma} < 0$ somewhere on C. Consider the point

$$p = ((0, \dots, 0), (1, 2, \dots, n)) \in \mathbb{R}^n \times \Pi_n \subset \mathbb{R}^n \times \mathbb{A}^{n-1}.$$

It is contained in the affine subspaces $\mathbb{R}^n \times \partial\mathbb{H}_S \subset \mathbb{R}^n \times \mathbb{A}^{n-1}$ from Item (Π-3) for $S = \{1, 2, \dots, k\}$, $1 \leq k < n$. Consider a small open ball V around p in $\mathbb{R}^n \times \mathbb{A}^{n-1}$. These $(n-1)$ hyperplanes cut V into 2^{n-1} regions, which are distinguished by the signs of the following $(n-1)$ functions:

$$\beta_1 - 1, \beta_1 + \beta_2 - 3, \dots, \sum_{j=1}^{k} \beta_j - \frac{k(k+1)}{2}, \dots, \sum_{j=1}^{n-1} \beta_j - \frac{(n-1)n}{2}.$$

The unique region where all the signs are positive is the one that contains $\mathbb{R}^n \times \mathring{\Pi}_{n-1}$; therefore, the image of that region under the map $\psi_n \circ \pi$ (from Equation (13.6)) has $\bar{\Sigma} > 0$, and the image of the other $2^{n-1} - 1$ regions has $\bar{\Sigma} < 0$. From the immersion proof from before, the map $\widetilde{\Psi}^{-1} \circ \psi_n \circ \pi$ is a diffeomorphism from the small open ball V around p to a (not necessarily round) small open ball W around $((0, \dots, 0), (0, \dots, 0))$ inside U inside $\mathbb{R}^n \times \mathbb{R}^{n-1}$. Under this local map, the union of the hyperplanes

$$\bigcup_{\substack{S=\{1,\dots,k\} \\ 1 \leq k < n}} V \cap \big(\mathbb{R}^n \times \partial\mathbb{H}_S\big)$$

maps to $W \cap (\mathbb{R}^n \times \mathbb{K}^{n-1})$. Therefore, one of the 2^{n-1} regions maps into $\widetilde{\Psi}^{-1}(B)$, while the other $2^{n-1}-1$ regions map into $\widetilde{\Psi}^{-1}(C)$. Since $\bar{\Sigma}$ has the same sign on C, on these latter $2^{n-1}-1$ regions, $\bar{\Sigma}$ must have the same sign, which then must be negative. (We are using $n \geq 3$, so we can distinguish the numbers $2^{n-1}-1$ and 1.) Consequently on B, the function $\bar{\Sigma}$ must be positive, thus completing the proof. See also Figure 27. □

13.4. **Embedding and framing the moduli spaces.** In this section, we will smoothly embed and frame the moduli spaces X_I, which were defined to be permutohedra in Equation (13.3). As in Equation (13.1), let $I = \{p_1 < \cdots < p_n\}$. Consider the smooth embedding

$$\jmath_I \colon X_I \hookrightarrow \mathbb{R}^n \times \Pi_n, \qquad x \mapsto ((p_1, p_2, \dots, p_n), x), \tag{13.17}$$

which also respects the $\langle n-1\rangle$-manifold structure. As in the previous sections, using the linear ordering of the elements of I, it will useful to identify the first factor $\mathbb{R}^n$ with $\prod_I \mathbb{R}$ and to treat the second factor Π_n as embedded in $\prod_I \mathbb{R}$.

Compose with the map $\Psi_n^{-1} \circ \psi_n \circ \pi$ from Proposition 13.1 to get a smooth map of $\langle n-1\rangle$-manifolds

$$\Psi_n^{-1} \circ \psi_n \circ \pi \circ \jmath_I \colon X_I \to \mathbb{R}^n \times \mathbb{R}_+^{n-1}. \tag{13.18}$$

Since the points p_i are distinct, every non-trivial element of the symmetric group S_n sends the subset $\jmath_I(X_I) \subset \mathbb{R}^n \times \Pi_n$ to a disjoint subset, and therefore, the map from Equation (13.18) is a smooth embedding. To fit our requirements regarding embeddings of moduli spaces, we need to replace $\mathbb{R}$ with $\mathring{\mathbb{R}}_+$. Fix a diffeomorphism $f \colon \mathbb{R} \to \mathring{\mathbb{R}}_+$, and consider the diffeomorphism

$$F \colon \mathbb{R}^n \times \mathbb{R}_+^{n-1} \to \mathring{\mathbb{R}}_+^n \times \mathbb{R}_+^{n-1}, \quad (s_1, \dots, s_n, t_1, \dots, t_{n-1}) \mapsto (f(s_1), \dots, f(s_n), t_1, \dots, t_{n-1}).$$

Then we embed the moduli spaces by the map

$$\iota_I := F \circ \Psi_n^{-1} \circ \psi_n \circ \pi \circ \jmath_I \colon X_I \hookrightarrow \mathring{\mathbb{R}}_+^n \times \mathbb{R}_+^{n-1}. \tag{13.19}$$

We will also need to choose a framing of the normal bundle of this embedding. At each point $x \in X_I$, this will consist of an internal frame with n vectors which will span a complement to the tangent space to $\iota_I(X_I)$ in $\mathring{\mathbb{R}}_+^n \times \mathbb{R}_+^{n-1}$ at $\iota_I(x)$. The vectors in the frame are indexed by the set I; so we will let $(v_{p_1}(x), \dots, v_{p_n}(x))$ denote the frame at x so that the vector $v_{p_i}(x)$ corresponds to the point $p_i \in I$. Let $(e_{p_1}, \dots, e_{p_n})$ be the unit vectors in $\prod_I \mathbb{R}$; they frame the normal bundle of the embedding $\jmath_I$ from Equation (13.17). Then define

$$v_{p_i}(x) = (d\widehat{\iota})_x(e_{p_i}) \qquad 1 \leq i \leq n, x \in X_I, \tag{13.20}$$

where $\widehat{\iota} = F \circ \Psi_n^{-1} \circ \psi_n \circ \pi$.

All that remains is to check that these embeddings and framings satisfy the required coherence conditions on their lower-dimensional strata. Fix any non-empty proper subset $J \subset I$ of cardinality k. Then there is a corresponding facet in $\partial_k X_I$ which is identified with $X_J \times X_{I \setminus J}$. To show that the embeddings are coherent, we need to check the following diagram commutes,

$$\begin{array}{ccc} X_J \times X_{I\setminus J} & \xhookrightarrow{\quad(\iota_J,\iota_{I\setminus J})\quad} & \mathring{\mathbb{R}}^k_+ \times \mathbb{R}^{k-1}_+ \times \mathring{\mathbb{R}}^{n-k}_+ \times \mathbb{R}^{n-k-1}_+ \\ \big\downarrow & & \big\downarrow \cong \\ \partial_k X_I & & \mathring{\mathbb{R}}^k_+ \times \mathring{\mathbb{R}}^{n-k}_+ \times \mathbb{R}^{k-1}_+ \times \mathbb{R}^{n-k-1}_+ \\ \iota_I \big\downarrow & & \big\downarrow \cong \\ \mathring{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1} & \xhookleftarrow{\qquad\qquad} & \mathring{\mathbb{R}}^n_+ \times \mathbb{R}^{k-1}_+ \times \{0\} \times \mathbb{R}^{n-k-1}_+, \end{array} \tag{13.21}$$

where the identifications on the rightmost column are the usual ones by rearranging the factors. This follows from the commutativity of the following diagram:

$$\begin{array}{ccccc} X_J \times X_{I\setminus J} & & \xhookrightarrow{\qquad\qquad} & & X_I \\ {\scriptstyle (\jmath_J,\jmath_{I\setminus J})} \big\downarrow & & & & \big\downarrow {\scriptstyle \jmath_I} \\ (\prod_J \mathbb{R} \times X_J) \times (\prod_{I\setminus J} \mathbb{R} \times X_{I\setminus J}) & & \xhookrightarrow{\qquad\qquad} & & \prod_I \mathbb{R} \times X_I \\ {\scriptstyle (\Psi_k^{-1}\psi_k\pi,\, \Psi_{n-k}^{-1}\psi_{n-k}\pi)} \big\downarrow & & & & \big\downarrow {\scriptstyle \Psi_n^{-1}\psi_n\pi} \\ (\mathbb{R}^k \times \mathbb{R}^{k-1}_+) \times (\mathbb{R}^{n-k} \times \mathbb{R}^{n-k-1}_+) & \xrightarrow{\cong} & \mathbb{R}^n \times (\mathbb{R}^{k-1}_+ \times \{0\} \times \mathbb{R}^{n-k-1}_+) & \hookrightarrow & \mathbb{R}^n \times \mathbb{R}^{n-1}_+ \\ {\scriptstyle (F,F)} \big\downarrow & & {\scriptstyle F} \big\downarrow & & \big\downarrow {\scriptstyle F} \\ (\mathring{\mathbb{R}}^k_+ \times \mathbb{R}^{k-1}_+) \times (\mathring{\mathbb{R}}^{n-k}_+ \times \mathbb{R}^{n-k-1}_+) & \xrightarrow{\cong} & \mathring{\mathbb{R}}^n_+ \times (\mathbb{R}^{k-1}_+ \times \{0\} \times \mathbb{R}^{n-k-1}_+) & \hookrightarrow & \mathring{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1}_+ \end{array}$$

The central pentagon commutes by Equation (13.7); the top rectangle commutes by definition (Equation (13.17)); and the bottom two rectangles commute since the map F was defined using the map f on each $\mathbb{R}$ component.

To see that the framings are coherent, we have to show that the normal framing of ι_I from Equation (13.20) agrees with the product framing on the subspace $X_J \times X_{I\setminus J}$. However, the normal framings of the embeddings $\jmath_I$ from Equation (13.17) are given by the unit vectors, and so they are indeed coherent. (Recall that the vectors in the frames are indexed by the elements of I.) Since the normal framings for ι_I are defined using those unit vectors via Equation (13.20), the commutativity of Diagram (13.21) implies that the framings are coherent.

Example 13.4. Let us fix the diffeomorphism $f\colon \mathbb{R} \to \mathring{\mathbb{R}}_+$ to be the exponential map. For $n = 1$, the map ι_I embeds the point X_I as $e^{p_1} \in \mathring{\mathbb{R}}_+$, with normal frame $v_{p_1} = e^{p_1}\vec{e}_1^{\,+}$, where $\vec{e}_1^{\,+}$ is the unit vector in $\mathring{\mathbb{R}}_+$ (reusing the notation from Section 12.2).

For $n = 2$, let $(\alpha_1, \alpha_2, \beta_1)$ be the coordinates on $\mathbb{R}^2 \times \Pi_2$ (with $\beta_2 = 3 - \beta_1$), let (a_1, a_2, b_2) be the coordinates on $\mathbb{R}^3$, and let (s_1, s_2, t) be the coordinates on $\mathbb{R}^2 \times \mathbb{R}_+$, as in the proof of Proposition 13.1. The map $\jmath_I$ embeds the interval X_I in $\mathbb{R}^2 \times \Pi_2$ by the map

$$x \mapsto (\alpha_1 = p_1, \alpha_2 = p_2, \beta_1 = 1 + x), \qquad x \in [0, 1],$$

with normal frame given by $(e_{p_1} = \frac{\partial}{\partial \alpha_1}, e_{p_2} = \frac{\partial}{\partial \alpha_2})$. The map $\psi_2 \circ \pi\colon \mathbb{R}^2 \times \Pi_2 \to \mathbb{R}^3$ is given by

$$(\alpha_1, \alpha_2, \beta_1) \mapsto (a_1 = \alpha_1 + \alpha_2, a_2 = \alpha_1\alpha_2 - \beta_1(3 - \beta_1), b_2 = \alpha_1(3 - \beta_1) + \alpha_2\beta_1),$$

and the map $\Psi_2\colon \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}^3$ from Equation (13.9) is given by

$$(s_1, s_2, t) \mapsto (a_1 = s_1 + s_2, a_2 = s_1 s_2 - 2 - t, b_2 = 2s_1 + s_2).$$

Therefore, the composition $\widehat{\iota} = F \circ \Psi_2^{-1} \circ \psi_2 \circ \pi\colon \mathbb{R}^2 \times \Pi_2 \to \mathring{\mathbb{R}}^2_+ \times \mathbb{R}_+$ is given by

$$(\alpha_1, \alpha_2, \beta_1) \mapsto \big(e^{(2-\beta_1)\alpha_1+(\beta_1-1)\alpha_2}, e^{(\beta_1-1)\alpha_1+(2-\beta_1)\alpha_2}, (\beta_1-1)(2-\beta_1)((\alpha_1-\alpha_2)^2+1)\big),$$

and hence the embedding $\iota_I = \widehat{\iota} \circ \jmath_I$ of the interval $X_I \cong [0,1]$ in $\mathring{\mathbb{R}}^2_+ \times \mathbb{R}_+$ is given by

$$x \mapsto \big(e^{(1-x)p_1+xp_2}, e^{xp_1+(1-x)p_2}, x(1-x)((p_1-p_2)^2+1)\big), \qquad x \in [0,1].$$

Let $(\vec{e}_1^{\,+}, \vec{e}_3^{\,+}, \vec{e}_2^{\,+})$ denote the standard frame in $\mathring{\mathbb{R}}^2_+ \times \mathbb{R}_+$ (as in Section 12.2). Then the normal framing to ι_I is given by

$$\begin{aligned} v_{p_1}(x) &= (d\widehat{\iota})_x(\tfrac{\partial}{\partial \alpha_1}) = (1-x)e^{(1-x)p_1+xp_2}\vec{e}_1^{\,+} + xe^{xp_1+(1-x)p_2}\vec{e}_3^{\,+} + 2x(1-x)(p_1-p_2)\vec{e}_2^{\,+},\\ v_{p_2}(x) &= (d\widehat{\iota})_x(\tfrac{\partial}{\partial \alpha_2}) = xe^{(1-x)p_1+xp_2}\vec{e}_1^{\,+} + (1-x)e^{xp_1+(1-x)p_2}\vec{e}_3^{\,+} + 2x(1-x)(p_2-p_1)\vec{e}_2^{\,+}. \end{aligned}$$

Note that the endpoints $x = 0$ and $x = 1$ produce the product embeddings (and product framings) of $X_{\{p_1\}} \times X_{\{p_2\}}$ and $X_{\{p_2\}} \times X_{\{p_1\}}$, respectively, in $\mathring{\mathbb{R}}^2_+ \cong \mathring{\mathbb{R}}^2_+ \times \{0\} \subset \mathring{\mathbb{R}}^2_+ \times \mathbb{R}_+$.

14. The Cohen-Jones-Segal construction

We have now constructed all moduli spaces $\overline{\mathcal{M}}([D])$, as l-dimensional $\langle l \rangle$-manifolds, along with neat embeddings in $\mathbb{E}^d_l$, as well as their their (external) framings. It remains to put them together into a framed flow category, and then run the Cohen-Jones-Segal construction to obtain the knot Floer stable homotopy types, following the set-up in [26].

14.1. **Framed flow categories.**

We review here some definitions from [26, Section 3.2]. One slight difference is that [26] uses the composition order, while we are using the concatenation order, and therefore, our factors will be ordered and indexed differently. The other main difference is that we allow our categories to have infinitely many objects.

Definition 14.1. A *flow category* $\mathscr{C}$ is a category with objects $\mathscr{C} = \mathrm{Ob}(\mathscr{C})$ equipped with a function $\mathrm{gr}\colon \mathscr{C} \to \mathbb{Z}$ (called the grading), such that:

- $\mathrm{Hom}(x, x) = \{\mathrm{Id}\}$ for all $x \in \mathscr{C}$;
- For all distinct $x, y \in \mathscr{C}$ with $\mathrm{gr}(x) - \mathrm{gr}(y) = k$, the morphism space $\mathrm{Hom}(x, y)$ is a compact $(k-1)$-dimensional $\langle k-1 \rangle$-manifold; in particular, it is empty for $\mathrm{gr}(x) \leq \mathrm{gr}(y)$;
- For distinct $x, y, z \in \mathscr{C}$ with $\mathrm{gr}(x) - \mathrm{gr}(y) = m$, the composition
$$\circ\colon\ \mathrm{Hom}(x, y) \times \mathrm{Hom}(y, z) \to \mathrm{Hom}(x, z)$$
is an embedding into $\partial_m \mathrm{Hom}(x, z)$. Moreover,
$$\circ^{-1}(\partial_i \mathrm{Hom}(x, z)) = \begin{cases} \partial_i \mathrm{Hom}(x, y) \times \mathrm{Hom}(y, z) & \text{for } i < m \\ \mathrm{Hom}(x, y) \times \partial_{i-m} \mathrm{Hom}(y, z) & \text{for } i > m; \end{cases}$$
(Since we are using the concatenation order, for $p \in \mathrm{Hom}(x, y), q \in \mathrm{Hom}(y, z)$, we will denote their composition to be $p * q$, in a similar vein to Item (G-10) from Section 2.1.)
- For distinct $x, z \in \mathscr{C}$, the composition $\circ$ induces a diffeomorphism
$$\partial_m \mathrm{Hom}(x, z) \cong \coprod_{\{y \mid \mathrm{gr}(y) = \mathrm{gr}(x) - i\}} \mathrm{Hom}(x, y) \times \mathrm{Hom}(y, z).$$

Given a flow category $\mathscr{C}$ and $x, y \in \mathscr{C}$, we define the *compactified moduli space from x to y* as

$$\overline{\mathcal{M}}(x,y) = \begin{cases} \varnothing & \text{if } x = y, \\ \mathrm{Hom}(x,y) & \text{if } x \neq y. \end{cases}$$

Furthermore, for $i \in \mathbb{Z}$, we let $\mathscr{C}_i = \{x \in \mathscr{C} \mid \mathrm{gr}(x) = i\}$, topologized as a discrete space. Then, for $i, j \in \mathbb{Z}$, we define

$$\overline{\mathcal{M}}(i,j) = \coprod_{x \in \mathscr{C}_i, y \in \mathscr{C}_j} \overline{\mathcal{M}}(x,y).$$

We call a flow category *compact* if $\overline{\mathcal{M}}(i,j)$ is compact for all i, j.

Next we will define neat embeddings of flow categories. For this, recall the space $\mathbb{E}^d_l = \mathbb{R}^d \times \mathbb{R}_+ \times \cdots \times \mathbb{R}_+ \times \mathbb{R}^d \cong \mathbb{R}^l_+ \times \mathbb{R}^{d(l+1)}$ from Section 10. It is convenient to set

$$\mathbb{E}^d_{i,j} = \mathbb{E}^d_{i-j-1}$$

for any pair of integers $i > j$.

Definition 14.2. A *neat embedding* ι of a flow category $\mathscr{C}$ relative $d \in \mathbb{N}$ is a collection of neat embeddings

$$\iota_{x,y} \colon \overline{\mathcal{M}}(x,y) \hookrightarrow \mathbb{E}^d_{\mathrm{gr}(x),\mathrm{gr}(y)},$$

defined for every $x, y \in \mathscr{C}$, such that

- For all $i, j \in \mathbb{Z}$, the union of all $\iota_{x,y}$ for $x \in \mathscr{C}_i, y \in \mathscr{C}_j$ induces a neat embedding of $\overline{\mathcal{M}}(i,j)$;
- For all $x, y, z \in \mathscr{C}$ and for all $(p,q) \in \overline{\mathcal{M}}(x,y) \times \overline{\mathcal{M}}(y,z)$, we have

$$\iota_{x,z}(p * q) = (\iota_{x,y}(p), 0, \iota_{y,z}(q)) \in \mathbb{E}^d_{\mathrm{gr}(x),\mathrm{gr}(y)} \times \mathbb{R}_+ \times \mathbb{E}^d_{\mathrm{gr}(y),\mathrm{gr}(z)} = \mathbb{E}^d_{\mathrm{gr}(x),\mathrm{gr}(z)}.$$

Given a neat embedding ι of $\mathscr{C}$, and objects $x, y \in \mathscr{C}$, we let $\nu_{x,y}$ denote the normal bundle to $\overline{\mathcal{M}}(x,y)$ under the embedding $\iota_{x,y}$.

Definition 14.3. A *framed flow category* is a flow category $\mathscr{C}$ together with a neat embedding ι (relative some d), and also equipped with framings for the normal bundles $\nu_{x,y}$ for all $x, y \in \mathscr{C}$, such that the product framing of $\nu_{x,y} \times \nu_{y,z}$ equals the pullback framing of $\circ^* \nu_{x,z}$ for all x, y, z.

We introduce the following notion of a framed flow subcategory, which is analogous to a subcomplex of a chain complex.

Definition 14.4. Let $\mathscr{C}$ be a framed flow category. Suppose we have a full subcategory $\mathscr{D} \subseteq \mathscr{C}$ such that $\mathrm{Hom}(x,y) = \varnothing$ for all $x \in \mathscr{D}$ and $y \in \mathscr{C} \setminus \mathscr{D}$. We let the embedding and framings of the morphism sets in $\mathscr{D}$ be obtained by restriction from $\mathscr{C}$. We then call $\mathscr{D}$ a *framed flow subcategory* of $\mathscr{C}$.

We can similarly define a *framed flow quotient category* of $\mathscr{C}$, starting from some $\mathscr{D} \subseteq \mathscr{C}$ such that $\mathrm{Hom}(x,y) = \varnothing$ for all $x \in \mathscr{C} \setminus \mathscr{D}$ and $y \in \mathscr{D}$.

Remark 14.5. In [26, Definition 3.29], framed flow subcategories and quotient categories are called downward closed and upward closed subcategories, respectively.

14.2. **Spectra.** In this section, we review notions related to classical (sequential) spectra. For details, see [4, 8, 3].

A *spectrum* X is a collection of based spaces $\{X_n\}_{n\geq 0}$ along with structure maps $\sigma_n^X = \sigma_n\colon X_n \wedge S^1 \to X_{n+1}$. The map σ_n induces maps $\widetilde{H}_k(X_n) \to \widetilde{H}_{k+1}(X_{n+1})$ and $\pi_k(X_n) \to \pi_{k+1}(X_{n+1})$; the homology and homotopy groups of a spectrum X are defined to be

$$H_k(X) = \mathrm{colim}_n\, \widetilde{H}_{k+n}(X_n) \qquad \pi_k(X) = \mathrm{colim}_n\, \pi_{k+n}(X_n).$$

There are shift functors on spectra, given by $\mathrm{sh}(X)_n = X_{n+1}$ and $\mathrm{sh}^{-1}(X)_n = X_{n-1}$ if $n > 0$, and the basepoint if $n = 0$.

Given a spectrum X and a based space S, the spectrum $S \wedge X$ is defined as

$$(S \wedge X)_n \coloneqq S \wedge X_n, \qquad \sigma_n^{S\wedge X} \coloneqq \mathrm{id}_S \wedge \sigma_n^X \colon S \wedge X_n \wedge S^1 \to S \wedge X_{n+1};$$

and the spectrum $\mathrm{Hom}(S, X)$ is defined as

$$\mathrm{Hom}(S,X)_n \coloneqq \mathrm{Hom}(S, X_n), \text{ the based mapping space, and}$$
$$\sigma_n^{\mathrm{Hom}(S,X)}\colon \mathrm{Hom}(S,X_n) \wedge S^1 \to \mathrm{Hom}(S, X_{n+1}), \text{ given by } f \wedge p \mapsto \big(s \mapsto \sigma_n^X(f(s)\wedge p)\big).$$

If $S = S^k$, the functor $S \wedge \cdot$ is called the k^{th} suspension functor Σ^k, and the the functor $\mathrm{Hom}(S, \cdot)$ is called the k^{th} loop functor Ω^k. The simplest example of a spectrum is the sphere spectrum $\mathbb{S}$ with

$$\mathbb{S}_n \coloneqq S^n = \underbrace{S^1 \wedge \ldots \wedge S^1}_{n}, \qquad \sigma_n \coloneqq \mathrm{Id}\colon S^n \wedge S^1 \to S^{n+1}.$$

More generally, the *suspension spectrum* of a based space X is the spectrum $X \wedge \mathbb{S}$; its homology and homotopy groups are simply the reduced homology and stable homotopy groups of the original space X:

$$H_*(X \wedge \mathbb{S}) \cong \widetilde{H}_*(X) \qquad \pi_*(X \wedge \mathbb{S}) \cong \pi_*^s(X).$$

For any $k \in \mathbb{Z}$, define the spectrum $\mathbb{S}^k$ as $\Sigma^k\mathbb{S}$ if $k \geq 0$, and $\Omega^{-k}\mathbb{S}$ if $k \leq 0$.

A *map of spectra*, $f\colon X \to Y$, is a collection of maps $f_n\colon X_n \to Y_n$ such that $f_{n+1} \circ \sigma_n^X = \sigma_{n+1}^Y \circ (f_n \wedge S^1)\colon X_n \wedge S^1 \to Y_{n+1}$. The map is called a *weak equivalence* if it induces an isomorphism $\pi_*(X) \to \pi_*(Y)$; we will usually suppress the adjective 'weak'. If both X and Y are n-connected (that is, have $\pi_k = 0$ for all $k \leq n$) for some $n \in \mathbb{Z}$, then this is equivalent to inducing isomorphism on homology $H_*(X) \to H_*(Y)$. The stable homotopy category of spectra is obtained by inverting all the equivalences. The functors H_* and π_* factor through the stable homotopy category. Spectra also have a notion of homotopy colimits, defined levelwise; and homotopy colimits also preserve equivalences. In the stable homotopy category, there are isomorphisms

$$\Sigma^k X \simeq \mathrm{sh}^k(X) \qquad \Omega^k X \simeq \mathrm{sh}^{-k}(X).$$

(Note that these isomorphisms are not induced by the natural-looking levelwise maps $(S^k \wedge X)_n = S^k \wedge X_n \to (\mathrm{sh}^k(X))_n = X_{n+k}$ and $(\mathrm{sh}^{-k}(X))_n = X_{n-k} \to (\Omega^k X)_n = \Omega^k X_n$, since they do not induce maps on spectra.) In particular, the k^{th} suspension functor Σ^k and loop functor Ω^k are inverses to one another: $X \cong \Omega^k\Sigma^k X$. This isomorphism in the stable homotopy category is realized by the equivalence $X \to \Omega^k\Sigma^k X$, given levelwise $X_n \to \mathrm{Hom}(S^k, S^k \wedge X_n)$ by $x \mapsto (p \mapsto p \wedge x)$; therefore, we will be justified in referring to Ω^k as the k^{th} desuspension functor, and sometimes abusing notation to write Σ^{-k} instead of Ω^k.

14.3. **From framed flow categories to spectra.** We now review how to build a CW complex, and then a spectrum, from a framed flow category (under some mild restrictions). We follow [26, Section 3.3] (with the ordering of factors reversed, and some additional minor modifications), which is in turn inspired from [11].

Let $(\mathscr{C}, \iota, \phi)$ be a compact framed flow category, with ι denoting the neat embedding, and ϕ the normal framings. For now, we assume that the grading function $\mathrm{gr}\colon \mathscr{C} \to \mathbb{Z}$ is bounded, with image in some interval $[B+1, A]$ with $A, B \in \mathbb{Z}$. Let us further impose $B < 0$. Let

$$C_d(A,B) := (A-B)d - B - 1. \tag{14.1}$$

We construct a CW complex $|\mathscr{C}|_{\iota,\phi,A,B}$. We start with a single 0-cell—the basepoint—and then for each $x \in \mathscr{C}$, we attach a cell $\mathcal{C}(x)$, inductively on the grading $\mathrm{gr}(x) = m$. The cell $\mathcal{C}(x)$ will have dimension $C_d(A,B) + m$.

Let us choose $\epsilon > 0$ sufficiently small so that for all i and j, the embedding $\iota_{i,j}$ of $\overline{\mathcal{M}}_{i,j}$ into $\mathbb{E}^d_{i,j}$ extends to an embedding of $\overline{\mathcal{M}}_{i,j} \times [-\epsilon,\epsilon]^{(i-j)d}$ using the normal framings. Choose R sufficiently large so that for all i and j, the image $\iota_{i,j}(\overline{\mathcal{M}}(i,j) \times [-\epsilon,\epsilon]^{(i-j)d})$ lies in

$$[-R,R]^d \times [0,R] \times \cdots \times [0,R] \times [-R,R]^d \subset \mathbb{R}^d \times \mathbb{R}_+ \times \cdots \times \mathbb{R}_+ \times \mathbb{R}^d = \mathbb{E}^d_{i,j}.$$

Let us suppose we attached all the lower dimensional cells and we want to attach $\mathcal{C}(x)$, where $\mathrm{gr}(x) = m$. Define

$$\mathcal{C}(x) = \underbrace{[-\epsilon,\epsilon]^d \times \{0\} \times \cdots \times \{0\} \times [-\epsilon,\epsilon]^d}_{\subset \mathbb{E}^d_{A,m}} \times \{0\} \times \underbrace{[-R,R]^d \times [0,R] \times \cdots \times [0,R] \times [-R,R]^d}_{\subset \mathbb{E}^d_{m,B}} \subset \mathbb{E}^d_{A,B}.$$

(There are $A-m$ instances of $\{0\}$ and $[-\epsilon,\epsilon]^d$, $m-B$ instances of $[-R,R]^d$, and $m-B-1$ instances of $[0,R]$.) For any $1 \leq i < m-B$, define $\partial_i\mathcal{C}(x) \subset \mathcal{C}(x)$ to be the subset where the coordinate in the i^{th} $[0,R]$ factor is 0. (This is a slight abuse of notation since we are not defining $\mathcal{C}(x)$ as a $\langle m-B-1\rangle$-manifold; however, had we replaced all instances of $[-R,R]$, $[0,R]$, and $[-\epsilon,\epsilon]$ with $\mathbb{R}$, $\mathbb{R}_+$, and $(-\epsilon,\epsilon)$, respectively, then $\mathcal{C}(x)$ would have had a natural $\langle m-B-1\rangle$-manifold structure, and $\partial_i\mathcal{C}(x)$ would have been the correct definition.)

To see how to attach $\mathcal{C}(x)$ to a lower cell $\mathcal{C}(y)$ where $\mathrm{gr}(y) = l$, consider the neat embedding $\iota_{x,y}$ (extended using the framing ϕ):

$$\overline{\mathcal{M}}(x,y) \times [-\epsilon,\epsilon]^d \times \{0\} \times \cdots \times \{0\} \times [-\epsilon,\epsilon]^d \hookrightarrow [-R,R]^d \times [0,R] \times \cdots \times [0,R] \times [-R,R]^d \subset \mathbb{E}^d_{m,l}.$$

Extending by Id on $[-\epsilon,\epsilon]^d \times \cdots \times \{0\} \times [-\epsilon,\epsilon]^d \subset \mathbb{E}^d_{A,m}$ and $[-R,R]^d \times \cdots \times [0,R] \times [-R,R]^d \subset \mathbb{E}^d_{l,B}$, we get an embedding of

$$\begin{aligned}\overline{\mathcal{M}}(x,y) \times \mathcal{C}(y) = \overline{\mathcal{M}}(x,y) \times \underbrace{[-\epsilon,\epsilon]^d \times \cdots \times \{0\} \times [-\epsilon,\epsilon]^d}_{\subset \mathbb{E}^d_{A,m}} \times \{0\} \times \underbrace{[-\epsilon,\epsilon]^d \times \cdots \times \{0\} \times [-\epsilon,\epsilon]^d}_{\subset \mathbb{E}^d_{m,l}} \\ \times \{0\} \times \underbrace{[-R,R]^d \times \cdots \times [0,R] \times [-R,R]^d}_{\subset \mathbb{E}^d_{l,B}}\end{aligned}$$

into

$$\begin{aligned}\partial_{m-l}\mathcal{C}(y) = \underbrace{[-\epsilon,\epsilon]^d \times \cdots \times \{0\} \times [-\epsilon,\epsilon]^d}_{\subset \mathbb{E}^d_{A,m}} \times \{0\} \times \underbrace{[-R,R]^d \times \cdots \times [0,R] \times [-R,R]^d}_{\subset \mathbb{E}^d_{m,l}} \\ \times \{0\} \times \underbrace{[-R,R]^d \times \cdots \times [0,R] \times [-R,R]^d}_{\subset \mathbb{E}^d_{l,B}}.\end{aligned}$$

Let $\mathcal{C}_y(x) \subset \partial_{m-l}\mathcal{C}(x)$ be the image of the embedding. We then define the attaching map from $\partial\mathcal{C}(x)$ to the lower skeleton to be the projection to $\mathcal{C}(y)$ on each $\mathcal{C}_y(x) \cong \overline{\mathcal{M}}(x,y) \times \mathcal{C}(y)$, and the basepoint map on $\partial\mathcal{C}(x) \setminus \bigcup_y \mathcal{C}_y(x)$.

After attaching all the cells $\mathcal{C}(x)$, we obtain the desired CW complex $|\mathscr{C}|_{\iota,\phi,A,B}$. Its dependence on A and B is explained in [26, Lemma 3.26]. It is proved there that, if we have $B' \leq B$ and $A' \geq A$, then there is a homotopy equivalence

$$\Sigma^{C_d(A',B')-C_d(A,B)}|\mathscr{C}|_{\iota,\phi,A,B} \xrightarrow{\sim} |\mathscr{C}|_{\iota,\phi,A',B'}. \tag{14.2}$$

In more detail, consider

$$\mathbb{E}^d_{A,B} = \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^d \times \cdots \times \mathbb{R}_+ \times \mathbb{R}^d,$$

and let $T_d(A,B)$ be the set of the $d(A-B)$ $\mathbb{R}$-factors appearing in the above formula, union the set of the $(-B-1)$ $\mathbb{R}_+$-factors appearing in position below 0, that is, in positions $-1,-2,\dots,B+1$. Then $T_d(A,B)$ is a set of cardinality $C_d(A,B)$, and let

$$S_d(A,B) = \bigwedge_{T_d(A,B)} S^1 \tag{14.3}$$

be the corresponding $C_d(A,B)$-dimensional sphere. The homotopy equivalence from Equation (14.2) is actually

$$\Big(\bigwedge_{\substack{T_d(A',B')\\ \setminus T_d(A,B)}} S^1\Big) \wedge |\mathscr{C}|_{\iota,\phi,A,B} \xrightarrow{\sim} |\mathscr{C}|_{\iota,\phi,A',B'}. \tag{14.4}$$

This observation allows us to define the following spectrum canonically:

$$\Omega^{C_d(A,B)}(|\mathscr{C}|_{\iota,\phi,A,B} \wedge \mathbb{S}) = \operatorname{Hom}(S_d(A,B), |\mathscr{C}|_{\iota,\phi,A,B} \wedge \mathbb{S}).$$

Given $B' \leq B$ and $A' \geq A$, we get an equivalence

$$\begin{aligned} \Omega^{C_d(A,B)}|\mathscr{C}|_{\iota,\phi,A,B} \wedge \mathbb{S} &\xrightarrow{\sim} \Omega^{C_d(A,B)}\Omega^{C_d(A',B')-C_d(A,B)}\Sigma^{C_d(A',B')-C_d(A,B)}|\mathscr{C}|_{\iota,\phi,A,B} \wedge \mathbb{S} \\ &\xrightarrow{\sim} \Omega^{C_d(A',B')}|\mathscr{C}|_{\iota,\phi,A',B'} \wedge \mathbb{S}. \end{aligned} \tag{14.5}$$

Here, the first map is induced from the map $X \to \Omega^k\Sigma^k X$ from Section 14.2, with $k = C_d(A',B') - C_d(A,B)$, and Ω^k, Σ^k being the functors $\operatorname{Hom}(S,\cdot)$, and $S \wedge \cdot$, with $S = \bigwedge_{T_d(A',B')\setminus T_d(A,B)} S^1$, and the second map is induced by Equation (14.4). Now define the spectrum

$$S(\mathscr{C},\iota,\phi) := \operatorname{hocolim}_{A,B} \Omega^{C_d(A,B)}|\mathscr{C}|_{\iota,\phi,A,B} \wedge \mathbb{S}, \tag{14.6}$$

where the homotopy colimit is taken using the maps from Equation (14.5), as $A \to \infty$ and $B \to -\infty$.

So far we have worked under the assumption that the grading function gr is bounded. Let us relax this assumption by requiring only that gr is bounded below, by some constant $B+1$. For every $K \in \mathbb{Z}$, there is a compact framed flow subcategory $\mathscr{C}_{\leq K}$ of $\mathscr{C}$, whose objects are those $x \in \mathscr{C}$ with $\operatorname{gr}(x) \leq K$. For any $K \leq A$, we get spaces $|\mathscr{C}_{\leq K}|_{\iota,\phi,A,B}$, and for any $K \leq K' \leq A$, we have inclusions

$$|\mathscr{C}_{\leq K}|_{\iota,\phi,A,B} \hookrightarrow |\mathscr{C}_{\leq K'}|_{\iota,\phi,A,B}.$$

Taking the homotopy colimit as $A, K \to \infty$ and $B \to -\infty$ subject to the restriction $K \leq A$, we define

$$S(\mathscr{C},\iota,\phi) := \operatorname{hocolim}_{A,B,K} \Omega^{C_d(A,B)}|\mathscr{C}_{\leq K}|_{\iota,\phi,A,B} \wedge \mathbb{S}. \tag{14.7}$$

Thus, we have spectra associated to compact framed flow categories even when gr is only bounded below.

Given a compact framed flow category $(\mathscr{C}, \iota, \phi)$, there is an *associated chain complex* $C_*(\mathscr{C}, \iota, \phi)$ whose chain group is freely generated by the objects with homological grading given by the gr function, and whose differential is given by

$$\partial x = \sum_{\substack{y \in \mathscr{C} \\ \mathrm{gr}(y) = \mathrm{gr}(x) - 1}} \#\overline{\mathcal{M}}(x, y) y. \tag{14.8}$$

(Here $\#\overline{\mathcal{M}}(x, y)$ is the signed count of points in the compact framed 0-manifold $\overline{\mathcal{M}}(x, y)$.) When gr is bounded below, it follows from [26, Lemma 3.24] that this chain complex is isomorphic to the shifted reduced cellular chain complex of $|\mathscr{C}|_{\iota,\phi,A,B}$,

$$C_*(\mathscr{C}, \iota, \phi) \cong \Sigma^{-C_d(A,B)} \widetilde{C}_*^{\mathrm{CW}}(|\mathscr{C}|_{\iota,\phi,A,B}), \tag{14.9}$$

and therefore, its homology is isomorphic to the homology of the spectrum $S(\mathscr{C}, \iota, \phi)$:

$$H_*(\mathscr{C}, \iota, \phi) \cong H_*(S(\mathscr{C}, \iota, \phi)). \tag{14.10}$$

Remark 14.6. If gr is not bounded below, we have compact framed flow categories $\mathscr{C}_{\geq L}$ with objects those $x \in \mathscr{C}$ with $\mathrm{gr}(x) \geq L$. There are associated spectra $S(\mathscr{C}_{\geq L}, \iota, \phi)$. Projections between CW complexes (collapsing the lower dimensional cells up to some degree) induce maps

$$S(\mathscr{C}_{\geq L'}, \iota, \phi) \leftarrow S(\mathscr{C}_{\geq L}, \iota, \phi)$$

for all $L' \leq L$. This gives an inverse system of spectra, i.e., a pro-spectrum as in [11]. In this case, we may define $S(\mathscr{C}, \iota, \phi)$ to be this pro-spectrum. This would be useful if one were interested in a minus version of the knot Floer spectrum (which will not be discussed in the current paper).

14.4. **Filtrations.** We make the following definition, which is somewhat non-standard, but sufficient for our purposes.

Definition 14.7. Let H be a partially ordered set, and X a spectrum. Let H^+ be H with an additional terminal element ∞. An *H-filtration on X* is a functor from H^+ to the category of spectra, such that ∞ is mapped to X.

More concretely, an H-filtration is a collection of spectra $X(h)$ for $h \in H$, together with maps

$$\psi_{h,h'} \colon X(h) \to X(h') \text{ for all } h \leq h'$$

and

$$\psi_h \colon X(h) \to X \text{ for all } h,$$

satisfying: $\psi_{h,h} = \mathrm{id}$, $\psi_{h',h''} \circ \psi_{h,h'} = \psi_{h,h''}$ and $\psi_h = \psi_{h'} \circ \psi_{h,h'}$, whenever $h \leq h' \leq h''$.

Definition 14.8. Let X and Y be H-filtered spectra with maps $\psi_h, \psi_{h,h'}$ and $\phi_h, \phi_{h,h'}$, respectively.

(a) An *H-filtered map* from X to Y consists of maps of spectra $f \colon X \to Y$ and $f_h \colon X(h) \to Y(h)$ for all $h \in H$, such that $\phi_{h,h'} \circ f_h = f_{h'} \circ \psi_{h,h'}$ and $\phi_h \circ f_h = f \circ \psi_h$ for all $h \leq h'$. (By a slight abuse of notation, we use f to also denote to the whole data of a H-filtered map.)

(b) An H-filtered map $f \colon X \to Y$ is called an *H-filtered equivalence* if f and f_h (for all $h \in H$) are equivalences of spectra.

Filtrations on spectra arise in our setting as follows.

Definition 14.9. Let H be a partially ordered set. An *H-filtered framed flow category* consists of a framed flow category $(\mathscr{C}, \iota, \phi)$ and a collection of framed flow subcategories $(\mathscr{C}(h), \iota, \phi)$ for $h \in H$, with $\mathscr{C}(h) \subseteq \mathscr{C}(h')$ whenever $h \leq h'$. (By a slight abuse of notation, we still use ι and ϕ to denote their restrictions.)

Suppose we have a compact H-filtered framed flow category, with gr bounded below. Then, we obtain inclusions of the form

$$|\mathscr{C}(h)_{\leq K}|_{\iota,\phi,A,B} \hookrightarrow |\mathscr{C}_{\leq K}|_{\iota,\phi,A,B}.$$

Taking homotopy colimits, we arrive at an H-filtration on $S(\mathscr{C},\iota,\phi)$ made of the spectra $S(\mathscr{C}(h),\iota,\phi)$ for $h \in H$.

Furthermore, given an H-filtered framed flow category and $h \in H$, we can define the associated graded framed flow category $g\mathscr{C}(h)$ using objects that are in $\mathscr{C}(h)$ but not in any $\mathscr{C}(h')$ for $h' < h$. If compact with grading bounded below, the corresponding spectrum is denoted $gS(\mathscr{C}(h),\iota,\phi)$. The associated graded spectrum for the whole filtration is set to be

$$gS(\mathscr{C},\iota,\phi) = \bigvee_{h\in H} gS(\mathscr{C}(h),\iota,\phi).$$

14.5. **The knot Floer spectrum.** We now define the multi-filtered spectrum $\mathcal{X}(\mathbb{G})$ from a grid diagram $\mathbb{G}$, as advertised in the introduction.

In Section 12.7 we glued together the moduli spaces $\overline{\mathcal{M}}_0(D)$ for D in the same equivalence class, with the result being smooth $\langle k\rangle$-manifolds $\overline{\mathcal{M}}([D])$. Since $\vec{N} = \vec{0}$, the thick dimension of these moduli spaces equals their actual dimension k, so the internal framings are empty. We have neat embeddings of $\overline{\mathcal{M}}([D])$ into $\mathbb{E}^d_l$, as well as normal (external) framings for these embeddings.

Recall from Section 2.2 that the grid complex $GC(\mathbb{G})$ has generators

$$[x, j_2, \dots, j_n] = U_2^{-j_2} \dots U_n^{-j_n} x$$

for $x \in \mathbb{S}$ and $j_2, \dots, j_n \in \mathbb{N}$. The generators also have Alexander gradings $(A_1, \dots, A_\ell) \in (\frac{1}{2}\mathbb{Z})^\ell$, one for each component of the link L represented by $\mathbb{G}$.

We define a framed flow category $\mathscr{C}(\mathbb{G})$ as follows. The objects are the generators of GC, and we let gr be the Maslov grading. Given two objects $[x, j_2, \dots, j_n]$ and $[y, i_2, \dots, i_n]$, consider the equivalence class $[D]$ made of domains $D \in \mathscr{D}^+(x,y)$ with $\mathbb{O}(D) = (j_2 - i_2, \dots, j_n - i_n)$. We let

$$\overline{\mathcal{M}}([x, j_2, \dots, j_n], [y, i_2, \dots, i_n]) = \overline{\mathcal{M}}([D]). \tag{14.11}$$

The enumeration of strata in Section 9.1 ensures that the conditions in the definition of a flow category are satisfied. Furthermore, the neat embeddings and the external framings turn $\mathscr{C}(\mathbb{G})$ into a framed flow category. The compatibility conditions in Definitions 14.2 and 14.3 are satisfied because the moduli spaces (along with their embeddings and framings) in Section 12 were constructed coherently with respect to the lower dimensional strata.

There are finitely many generators in each Maslov grading, so $\mathscr{C}(\mathbb{G})$ is compact, and the generators are bounded below in Maslov grading; therefore, we obtain a *knot Floer spectrum*

$$\mathcal{X}(\mathbb{G}) := S(\mathscr{C}(\mathbb{G})).$$

For each $h = (h_1, \dots, h_\ell) \in (\frac{1}{2}\mathbb{Z})^\ell$, we define a framed flow subcategory $\mathscr{C}(\mathbb{G}, h) \subset \mathscr{C}(\mathbb{G})$, using only the generators of GC with Alexander multi-grading $\leq h$. From here we obtain spectra $\mathcal{X}(\mathbb{G}, h)$, producing a $(\frac{1}{2}\mathbb{Z})^\ell$-filtration on $\mathcal{X}(\mathbb{G})$.

There is an additional structure given by the U_i maps. Consider the framed flow quotient category obtained from $\mathscr{C}(\mathbb{G})$ by removing the objects $[x, j_2, \dots, j_n]$ where $j_i = 0$. By mapping

$$[x, j_2, \dots, j_i, \dots, j_n] \mapsto [x, j_2, \dots, j_i - 1, \dots, j_n]$$

we get an isomorphism between this quotient category and $\Sigma^2\mathscr{C}(\mathbb{G})$, a category which is the same as $\mathscr{C}(\mathbb{G})$ except the Maslov grading is shifted by 2. At the level of the associated CW complexes and then spectra, we obtain a map

$$U_i\colon \mathcal{X}(\mathbb{G}) \to \Sigma^2\mathcal{X}(\mathbb{G})$$

given by collapsing the cells corresponding to generators $[x, j_1, \dots, j_n]$ where $j_i = 0$.

The U_i maps interact with the multi-filtration as follows. Suppose the marking U_i lies on the k^{th} component of the link. Then, we get commutative diagrams

$$\begin{array}{ccc} \mathcal{X}(\mathbb{G}, h) & \xrightarrow{U_i} & \Sigma^2\mathcal{X}(\mathbb{G}, h - \vec{e}_k) \\ \downarrow & & \downarrow \\ \mathcal{X}(\mathbb{G}) & \xrightarrow{U_i} & \Sigma^2\mathcal{X}(\mathbb{G}), \end{array}$$

where $\vec{e}_k$ is the unit vector in the k^{th} coordinate.

14.6. **Other versions.** Two other variants of grid complexes, $\widehat{GC}$ and $\widetilde{GC}$, were mentioned at the end of Section 2.2. We construct knot Floer spectra $\widehat{\mathcal{X}}(\mathbb{G})$ and $\widetilde{\mathcal{X}}(\mathbb{G})$ in the same way as we did for GC, but using fewer generators to define the framed Floer categories. In the case of $\widehat{GC}$, we only use those $[x, j_1, \dots, j_n]$ where $j_i = 0$ for one index i chosen from the O-markings on each link component. In the case of $\widetilde{GC}$, we only use the generators where $j_i = 0$ for all i. We then take the framed flow subcategories of $\mathscr{C}(\mathbb{G})$ with those generators as objects.

We also have associated graded framed flow categories, with the resulting spectra denoted $g\mathcal{X}(\mathbb{G})$, $g\widehat{\mathcal{X}}(\mathbb{G})$ and $g\widetilde{\mathcal{X}}(\mathbb{G})$. In particular, we have a decomposition

$$g\widehat{\mathcal{X}}(\mathbb{G}) = \bigvee_{h \in (\frac{1}{2}\mathbb{Z})^\ell} g\widehat{\mathcal{X}}(\mathbb{G}, h).$$

By construction, the (reduced) homology of $g\widehat{\mathcal{X}}(\mathbb{G}, h)$ is the link Floer homology in Alexander multi-grading h:

$$H_i(g\widehat{\mathcal{X}}(\mathbb{G}, h); \mathbb{Z}) = \widehat{HFL}_i(L, h).$$

Remark 14.10. If O_i and O_j are two markings on the same link component, one can show that the U_i and U_j actions on a grid complex are filtered chain homotopic. (See Lemma 2.11 in [32] for a proof for the minus version.) Although we will not pursue it here, one should be able to adapt the proof to the Floer spectra. This would imply that it suffices to consider one U_i map on $\mathcal{X}(\mathbb{G})$ for each link component not containing O_1. For the link component containing O_1, the corresponding map is trivial. In particular, in the case of knots, all the U_i maps should be trivial. Another consequence is that the tilde version $\widetilde{\mathcal{X}}(\mathbb{G})$ should be equivalent to the wedge sum of several (shifted) copies of the hat version $\widehat{\mathcal{X}}(\mathbb{G})$.

15. Invariance

In this section we prove Theorem 1.1. That is, we fix the grid $\mathbb{G}$ and the marking O_1, and prove that the filtered stable homotopy type of $\mathcal{X}(\mathbb{G})$ is independent of the further auxiliary choices. Indeed, we will prove that framed flow categories associated to two sets of choices are related by a map, which induces an equivalence between the filtered spectra. Towards this end, we will discuss the notion of maps of framed flow categories in Section 15.1, mention the relevant modifications in the definition for grid diagrams when we study more general stratified spaces in the presence of bubbles in Section 15.2, and finally list all the auxiliary choices and prove invariance in Section 15.3.

15.1. **Bimodules.** Morphisms of framed flow categories are called bimodules, and they induce maps of spectra associated to the the two framed flow categories. Different versions of this notion have appeared in the literature; see [22, Def. 3.6], [2, Section 4], [13, Def. 3.1] and [6, Def. 6.1]. (It also appears implicitly in [26, Lemma 3.32].) Many of these definitions work with flow categories where the moduli spaces are stably tangentially framed; this is different from the notion that we have used in this paper—our moduli spaces were embedded in specific Euclidean spaces $\mathbb{E}^d_{i,j}$ and normally framed. Therefore, we will give a different definition of bimodules of framed flow categories, modeled on normally framed moduli spaces; this definition will also be more amenable to the modifications in presence of bubbles (in Section 15.2).

Towards this end, we need a mild generalization of our notion of $\langle n\rangle$-manifolds from Section 5. For any poset P, a $\langle P\rangle$-manifold is a multifaceted manifold X, together with multifacets $\partial_i X$ for all $i \in P$ such that

- $\bigcup_{i\in P} \partial_i X = \partial X$;
- $\partial_i X \cap \partial_j X$ is a multifacet of both $\partial_i X$ and $\partial_j X$ for all $i \neq j$;
- $\partial_i X \cap \partial_j X = \varnothing$ if i, j are not comparable in the poset P.

Reusing notation from Equation (5.1), we will let $\partial_I X \coloneqq \bigcap_{i\in I} \partial_i X$ for any subset $I \subset P$; unless I is a chain (i.e., a totally ordered subset) of P, the multifacet $\partial_I X$ is empty.

The special case when $P = n \coloneqq \{1 < 2 < \cdots < n\}$ produces $\langle n\rangle$-manifolds; we will be interested in the case when P is the poset $2 \ltimes n$ whose elements are $\{0\} \times \{1, 2, \dots, n\} \cup \{1\} \times \{0, 1, \dots, n-1\}$, and the partial order is given by $(i_0, i_1) \leq (j_0, j_1)$ if and only if $i_0 \leq j_0, i_1 \leq j_1$.

We are now ready to present the definition of flow bimodules $\mathscr{B}\colon \mathscr{C} \to \mathscr{C}'$, which is essentially [13, Def. 3.1]. Let $\overline{\mathcal{M}}(x, y)$ and $\overline{\mathcal{M}}'(x, y)$ denote the compactified moduli spaces in the flow categories $\mathscr{C}$ and $\mathscr{C}'$, respectively.

The central idea is that to each $x \in \mathscr{C}, y \in \mathscr{C}'$, a flow bimodule will associate a multifaceted manifold $\overline{\mathcal{N}}(x, y)$, as well as maps $\overline{\mathcal{M}}(x, y) \times \overline{\mathcal{N}}(y, z) \to \partial\overline{\mathcal{N}}(x, z)$ (for any $x, y \in \mathscr{C}, z \in \mathscr{C}'$) and $\overline{\mathcal{N}}(x, y) \times \overline{\mathcal{M}}'(y, z) \to \partial\overline{\mathcal{N}}(x, z)$ (for any $x \in \mathscr{C}, y, z \in \mathscr{C}'$). If $\mathrm{gr}(x) - \mathrm{gr}(z) = k$, then $\overline{\mathcal{N}}(x, z)$ will be k-dimensional and its boundary will consist of $2k$ multifacets, which are naturally indexed by the poset $2 \ltimes k$: for $m = 1, \dots, k$, the multifacet $\partial_{0,m}\overline{\mathcal{N}}(x, z)$ is the image of the maps $\overline{\mathcal{M}}(x, y) \times \overline{\mathcal{N}}(y, z)$ with $\mathrm{gr}(y) = \mathrm{gr}(x) - m$, and for $m = 0, \dots, k-1$, the multifacet $\partial_{1,m}\overline{\mathcal{N}}(x, z)$ is the image of the maps $\overline{\mathcal{N}}(x, y) \times \overline{\mathcal{M}}'(y, z)$ with $\mathrm{gr}(y) = \mathrm{gr}(x) - m$. See the left half of Figure 31.

Definition 15.1. Let $\mathscr{C}$ and $\mathscr{C}'$ be two flow categories. A *flow bimodule* $\mathscr{B}\colon \mathscr{C} \to \mathscr{C}'$ consists of the following data:

- For all $x \in \mathscr{C}, y \in \mathscr{C}'$ with $\mathrm{gr}(x) - \mathrm{gr}(y) = k$, a compact k-dimensional $\langle 2 \ltimes k\rangle$-manifold $\overline{\mathcal{N}}(x, y)$;
- For all $x, y \in \mathscr{C}, z \in \mathscr{C}'$ with $\mathrm{gr}(x) - \mathrm{gr}(y) = m$, a map
$$\circ\colon \overline{\mathcal{M}}(x, y) \times \overline{\mathcal{N}}(y, z) \to \overline{\mathcal{N}}(x, z),$$
which is an embedding into $\partial_{0,m}\overline{\mathcal{N}}(x, z)$, satisfying
$$\circ^{-1}(\partial_{i_0,i_1}\overline{\mathcal{N}}(x, z)) = \begin{cases} \partial_{i_1}\overline{\mathcal{M}}(x, y) \times \overline{\mathcal{N}}(y, z) & \text{for } i_0 = 0, i_1 < m \\ \overline{\mathcal{M}}(x, y) \times \partial_{0,i_1-m}\overline{\mathcal{N}}(y, z) & \text{for } i_0 = 0, i_1 > m \\ \overline{\mathcal{M}}(x, y) \times \partial_{1,i_1-m}\overline{\mathcal{N}}(y, z) & \text{for } i_0 = 1, i_1 \geq m; \end{cases}$$
moreover, these maps induce a diffeomorphism
$$\partial_{0,m}\overline{\mathcal{N}}(x, z) \cong \coprod_{\{y|\mathrm{gr}(y)=\mathrm{gr}(x)-m\}} \overline{\mathcal{M}}(x, y) \times \overline{\mathcal{N}}(y, z);$$

- For all $x \in \mathscr{C}, y, z \in \mathscr{C}'$ with $\mathrm{gr}(x) - \mathrm{gr}(y) = m$, a map
$$\circ\colon \overline{\mathcal{N}}(x,y) \times \overline{\mathcal{M}}'(y,z) \to \overline{\mathcal{N}}(x,z),$$
which is an embedding into $\partial_{1,m}\overline{\mathcal{N}}(x,z)$, satisfying
$$\circ^{-1}(\partial_{i_0,i_1}\overline{\mathcal{N}}(x,z)) = \begin{cases} \partial_{0,i_1}\overline{\mathcal{N}}(x,y) \times \overline{\mathcal{M}}'(y,z) & \text{for } i_0 = 0, i_1 \leq m \\ \partial_{1,i_1}\overline{\mathcal{N}}(x,y) \times \overline{\mathcal{M}}'(y,z) & \text{for } i_0 = 1, i_1 < m \\ \overline{\mathcal{N}}(x,y) \times \partial_{i_1-m}\overline{\mathcal{M}}'(y,z) & \text{for } i_0 = 1, i_1 > m; \end{cases}$$
moreover, these maps induce a diffeomorphism
$$\partial_{1,m}\overline{\mathcal{N}}(x,z) \cong \coprod_{\{y \mid \mathrm{gr}(y) = \mathrm{gr}(x) - m\}} \overline{\mathcal{N}}(x,y) \times \overline{\mathcal{M}}'(y,z).$$

Note that we may view the $\langle 2 \ltimes k\rangle$-manifold $\overline{\mathcal{N}}(x,y)$ as a $\langle k\rangle$-manifold by declaring $\partial_i\overline{\mathcal{N}}(x,y) = \partial_{0,i}\overline{\mathcal{N}}(x,y) \amalg \partial_{1,i-1}\overline{\mathcal{N}}(x,y)$. This forgets a little bit of its structure, but after that, a flow bimodule is simply a flow category $\mathscr{B}$ with $\mathscr{C}'$ as a subcategory and $\Sigma\mathscr{C}$ as the corresponding quotient category (where $\Sigma\mathscr{C}$ is the flow category obtained from $\mathscr{C}$ by increasing the grading of each object by 1).

Recall from Definitions 14.2 and 14.3 that we defined framed flow categories by coherently embedding and framing the moduli spaces $\overline{\mathcal{M}}(i,j)$ in the spaces $\mathbb{E}^d_{i,j}$. These spaces $\mathbb{E}^d_{i,j}$ had the salient feautures that they were also $\langle i-j-1\rangle$-manifolds, and had maps—indeed diffeomorphisms—of the form
$$\mathbb{E}^d_{i,j} \times \mathbb{E}^d_{j,k} \to \partial_{i-j}\mathbb{E}^d_{i,k} \subset \mathbb{E}^d_{i,k}.$$
While not necessary, it was also useful that the spaces $\mathbb{E}^d_{i,j}$ were polyhedra, that is, intersections of finitely many half-spaces in some $\mathbb{R}^n$.

In order to define framed flow bimodules connecting framed flow categories, we need to embed and frame these spaces $\overline{\mathcal{N}}(x,y)$ in certain new spaces, which we now describe. As before, let
$$\overline{\mathcal{N}}(i,j) = \coprod_{x \in \mathscr{C}_i, y \in \mathscr{C}'_j} \overline{\mathcal{N}}(x,y),$$
and call a flow bimodule *compact* if $\overline{\mathcal{N}}(i,j)$ is compact for all i,j. Fix integers d,e, which we may assume are even in order to make sign calculations easier. We will now construct polyhedra $\mathbb{F}^{d,e}_{i,j}$ (with similar properties as the spaces $\mathbb{E}^d_{i,j}$) where the spaces $\overline{\mathcal{N}}(i,j)$ will be embedded. Indeed, we will define spaces $\mathbb{F}^{d,e}_n$, and set $\mathbb{F}^{d,e}_{i,j} = \mathbb{F}^{d,e}_{i-j}$.

Recall the n-dimensional permutohedron $\Pi_{n+1} \subset \mathbb{A}^n \subset \mathbb{R}^{n+1}$ from Section 13.1. Consider $(n+1)$ of its vertices
$$\begin{aligned} w_0 &= v_{[(n+1)12\cdots n]} = (2,3,\dots,n+1,1),\\ w_1 &= v_{[1(n+1)2\cdots n]} = (1,3,\dots,n+1,2),\\ &\cdots\\ w_n &= v_{[12\cdots n(n+1)]} = (1,2,\dots,n,n+1).\end{aligned}$$
(The left half of Figure 31 sheds light on our motive for considering these vertices.) Consider the facets F_S that contain at least one of these vertices. From Item (Π-4), there are exactly $2n$ such facets $F_{\{1\}}$, $F_{\{n+1\}}$, $F_{\{1,2\}}$, $F_{\{1,n+1\}}$, $F_{\{1,2,3\}}$, $F_{\{1,2,n+1\}}, \dots$; that is, S is either of the form $\{1,2,\dots,k\}$ or $\{1,2,\dots,k-1,n+1\}$. Consider the corresponding half-spaces $\mathbb{H}_S \subset \mathbb{A}^n$, and define $\widetilde{\Delta}^n$ to be

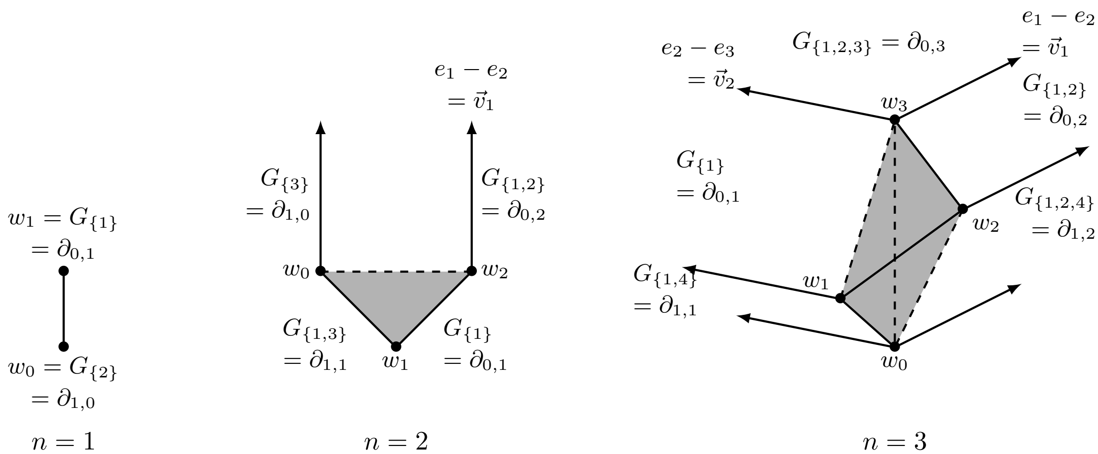


FIGURE 28. The polyhedra $\widetilde{\Delta}^n$ for small values of n. (We have skipped the $n = 0$ case of a point.) We have also shown the vertices w_i and the facets $G_S = \partial_{i_0,i_1}$ (except the facet $G_{\{4\}} = \partial_{1,0}$ for $n = 3$, which is at the bottom and hence not labeled). The vectors $\vec{v}_1, \dots, \vec{v}_{n-1}$ are labeled and the simplex Δ^n is shaded and its edges are dashed (the dashed lines are not edges of $\widetilde{\Delta}^n$).

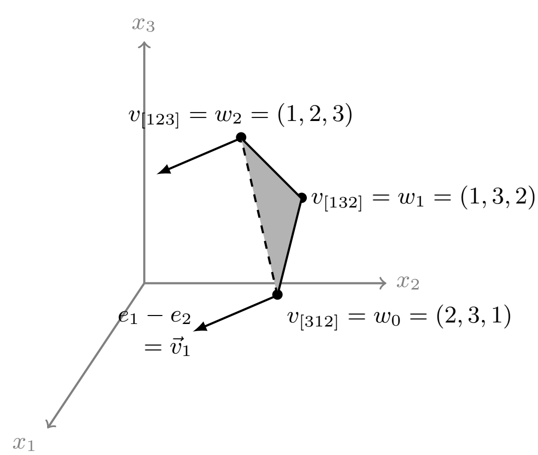
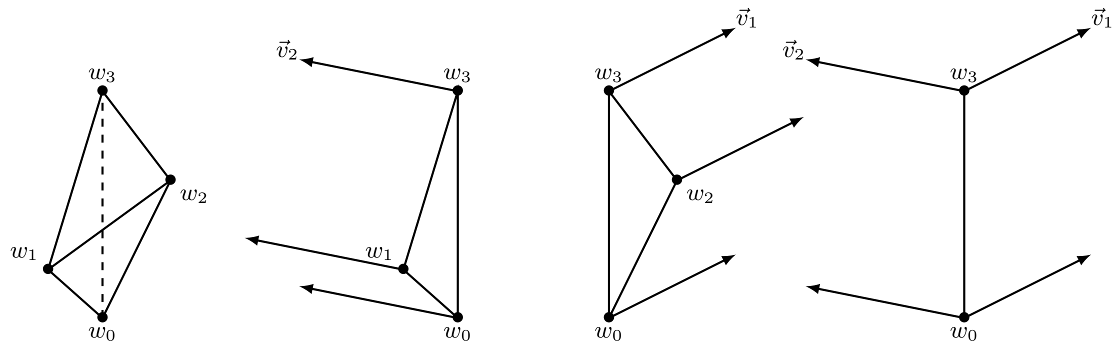


FIGURE 29. Continuing from Figure 25, we have shown the (non-compact) polyhedron $\widetilde{\Delta}^2$ sitting in the ambient $\mathbb{R}^3$, and have labeled the vertices. The other conventions are same as in Figure 28.

the intersection of those half-spaces. The polyhedron $\widetilde{\Delta}^n$ has vertices $w_0, \dots, w_n$, and has $2n$ facets, which we denote $G_S = \widetilde{\Delta}^n \cap \partial\mathbb{H}_S$. See Figures 28 and 29.

It is also useful to note that this polyhedron $\widetilde{\Delta}^n$ is the Minkowski sum of Δ^n and $\mathbb{R}_+^{n-1}$. Specifically, the n-simplex $\Delta^n \subset \mathbb{A}^n$ is the convex hull $[w_0, \dots, w_n]$ of the vertices $w_0, \dots, w_n$, and $\mathbb{R}_+^{n-1}$ is the positive cone $\mathbb{R}_+\langle \vec{v}_1, \dots, \vec{v}_{n-1} \rangle$ of the vectors $\vec{v}_1 = e_1 - e_2, \vec{v}_2 = e_2 - e_3, \dots, \vec{v}_{n-1} = e_{n-1} - e_n$



$\Delta^3_{\{1,2\}} \times \partial_{\{1,2\}}\mathbb{R}^2_+$ $[w_0, w_1, w_2, w_3] \times \mathbb{R}_+\langle\rangle$

$\Delta^3_{\{1\}} \times \partial_{\{1\}}\mathbb{R}^2_+$ $[w_0, w_1, w_3] \times \mathbb{R}_+\langle\vec{v}_2\rangle$

$\Delta^3_{\{2\}} \times \partial_{\{2\}}\mathbb{R}^2_+$ $[w_0, w_2, w_3] \times \mathbb{R}_+\langle\vec{v}_1\rangle$

$\Delta^3_{\varnothing} \times \partial_{\varnothing}\mathbb{R}^2_+$ $[w_0, w_3] \times \mathbb{R}_+\langle\vec{v}_1, \vec{v}_2\rangle$

FIGURE 30. Decomposition of $\widetilde{\Delta}^3$ into four pieces.

(where e_i are the standard unit vectors of the ambient $\mathbb{R}^{n+1}$). Then $\widetilde{\Delta}^n$ is the Minkowski sum

$$\widetilde{\Delta}^n = \Delta^n + \mathbb{R}^{n-1}_+ = \{x + y \mid x \in [w_0, \dots, w_n], y \in \mathbb{R}_+\langle\vec{v}_1, \dots, \vec{v}_{n-1}\rangle\}. \tag{15.1}$$

More specifically, consider the 2^{n-1} subsets $I \subset \{1, 2, \dots, n-1\}$, and let $\Delta^n_I = [(w_i)_{i\in I\cup\{0,n\}}]$ be the face of Δ^n which is the convex hull of vertices w_i for $i \in I \cup \{0, n\}$. (These are precisely the 2^{n-1} faces of Δ^n that contain w_0, w_n.) Also let $\partial_I\mathbb{R}^{n-1}_+ = \mathbb{R}_+\langle(\vec{v}_i)_{i\notin I}\rangle$ be the positive cone of the vectors $\vec{v}_i$ for $i \notin I$. Then we have a decomposition into Minkowski sums

$$\widetilde{\Delta}^n = \bigcup_{I\subset\{1,\dots,n-1\}} (\Delta^n_I + \partial_I\mathbb{R}^{n-1}_+).$$

Moreover, Δ^n_I is perpendicular to $\partial_I\mathbb{R}^{n-1}_+$, so the Minkowski sum $\Delta^n_I + \partial_I\mathbb{R}^{n-1}_+$ is diffeomorphic to the product $\Delta^n_I \times \partial_I\mathbb{R}^{n-1}_+$, and we have a diffeomorphism

$$\widetilde{\Delta}^n \cong \bigcup_{I\subset\{1,\dots,n-1\}} (\Delta^n_I \times \partial_I\mathbb{R}^{n-1}_+). \tag{15.2}$$

See again Figures 28 and 29. Figure 30 shows the decomposition of $\widetilde{\Delta}^3$ into these four pieces.

These $\widetilde{\Delta}^n$ are simple polyhedra, that is, each vertex is contained in the minimal number of n facets. Therefore, they are multifaceted manifolds. They can be endowed with the structure of a $\langle 2 \ltimes n\rangle$-manifold by declaring

$$\partial_{i_0,i_1}\widetilde{\Delta}^n = \begin{cases} G_{\{1,2,\dots,i_1\}} & \text{if } i_0 = 0, \\ G_{\{1,2,\dots,i_1-1,n+1\}} & \text{if } i_0 = 1. \end{cases} \tag{15.3}$$

Define

$$\mathbb{F}^{d,e}_n = \widetilde{\Delta}^n \times \mathbb{R}^e \times \mathbb{R}^{dn} \tag{15.4}$$

with the induced $\langle 2 \ltimes n\rangle$-manifold structure, and set

$$\mathbb{F}^{d,e}_{i,j} = \mathbb{F}^{d,e}_{i-j} \tag{15.5}$$

for all $i \geq j$.

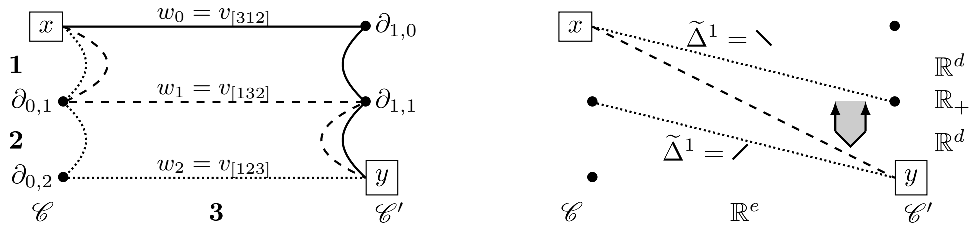


FIGURE 31. Left: We consider $x \in \mathscr{C}$ (left column) and $y \in \mathscr{C}'$ (right column) with $\mathrm{gr}(x) - \mathrm{gr}(y) = 2$. The boundary of $\overline{\mathcal{N}}(x, y)$ corresponds to broken trajectories, so there are four types $\partial_{0,1}, \partial_{0,2}, \partial_{1,0}, \partial_{1,1}$, indexed by $2 \ltimes 2$, depending on where they break (shown by black dots). The completely broken trajectories are ∂_I where I is a maximal chain of $2 \ltimes 2$, so there are exactly three: $\partial_{1,0} \cap \partial_{1,1}$ (solid lines), $\partial_{0,1} \cap \partial_{1,1}$ (dashed lines), and $\partial_{0,1} \cap \partial_{0,2}$ (dotted lines), and they correspond to the vertices w_0, w_1, w_2 of $\widetilde{\Delta}^2$, respectively. (If we number the 'gaps' between gradings as **1**, **2**, and the 'gap' between $\mathscr{C}$ and $\mathscr{C}'$ as **3**, then the permutation σ in $w_i = v_{[\sigma]}$ can easily be determined from the broken trajectory.) Right: The $\langle 2 \ltimes 2 \rangle$-manifold $\overline{\mathcal{N}}(x, y)$ is embedded in $\mathbb{F}_2^{d,e} = \widetilde{\Delta}^2 \times \mathbb{R}^e \times \mathbb{R}^{2d}$, and its multifacets $\partial_{0,1}, \partial_{0,2}, \partial_{1,0}, \partial_{1,1}$ are embedded in $\mathbb{R}^d \times \widetilde{\Delta}^1 \times \mathbb{R}^e \times \mathbb{R}^d$, $\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^e$, $\mathbb{R}^e \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^d$, and $\widetilde{\Delta}^1 \times \mathbb{R}^e \times \mathbb{R}^d$, respectively; here the two copies of $\mathbb{R}^d$ are associated to the two 'gaps' between gradings, $\mathbb{R}^e$ is associated to the 'gap' between $\mathscr{C}$ and $\mathscr{C}'$, the half-line $\mathbb{R}_+$ is associated to the grading between $\mathrm{gr}(x)$ and $\mathrm{gr}(y)$, the two copies of the interval $\widetilde{\Delta}^1$ are associated to the two dotted lines, and $\widetilde{\Delta}^2$ is associated to the dashed line.

Consider the facet $\partial_{0,k}\widetilde{\Delta}^n = G_{\{1,\dots,k\}}$. The map

$$\mathbb{R}^k \times \mathbb{R}^{n+1-k} \xrightarrow{+(1,2,\dots,k,k,k,\dots,k)} \mathbb{R}^{n+1}$$

identifies the product $\mathbb{R}_+\langle \vec{v}_1, \dots, \vec{v}_{k-1}\rangle \times \widetilde{\Delta}^{n-k}$ with the facet $G_{\{1,\dots,k\}}$. (This map is similar to the one from Item (Π-6), except we add $(1, 2, \dots, k)$ instead of $(0, 0, \dots, 0)$ to the first $\mathbb{R}^k$ factor in order to translate the cone $\mathbb{R}_+\langle \vec{v}_1, \dots, \vec{v}_{k-1}\rangle$ and make it start from the point $(1, 2, \dots, k) \in \mathbb{R}^k$.) Similarly, for the facet $\partial_{1,k-1}\widetilde{\Delta}^n = G_{\{1,\dots,k-1,n+1\}}$, we use the map

$$\mathbb{R}^k \times \mathbb{R}^{n+1-k} \xrightarrow{+(0,\dots,0,k+1,k+2,\dots,n+1)} \mathbb{R}^{n+1} \xrightarrow{(x_1,\dots,x_n)\mapsto(x_1,\dots,x_{k-1},x_{k+1},\dots,x_n,x_k)} \mathbb{R}^{n+1}$$

where we have postcomposed with a shuffle that sends the k^{th} $\mathbb{R}$-factor to the end. (This is again similar to what we did in Item (Π-6).) This map identifies $\widetilde{\Delta}^{k-1} \times \mathbb{R}_+\langle \vec{v}_{k+1}, \dots, \vec{v}_n\rangle$ with the facet $G_{\{1,\dots,k-1,n+1\}}$. That is, we get identifications

$$\partial_{0,k}\widetilde{\Delta}^n \cong \mathbb{R}_+^{k-1} \times \widetilde{\Delta}^{n-k}, \qquad \partial_{1,k-1}\widetilde{\Delta}^n \cong \widetilde{\Delta}^{k-1} \times \mathbb{R}_+^{n-k}. \tag{15.6}$$

Then for any $i \geq j \geq k$, we get induced identifications

$$\partial_{0,i-j}\mathbb{F}_{i,k}^{d,e} \cong \mathbb{E}_{i,j}^d \times \mathbb{F}_{j,k}^{d,e}, \qquad \partial_{1,i-j}\mathbb{F}_{i,k}^{d,e} \cong \mathbb{F}_{i,j}^{d,e} \times \mathbb{E}_{j,k}^d. \tag{15.7}$$

by just shuffling the factors.

We will require $\overline{\mathcal{N}}(i, j)$ to be neatly embedded in $\mathbb{F}_{i,j}^{d,e}$. The notion of neat embedding is similar to Definition 5.3. Being a polyhedron, $\mathbb{F}_n^{d,e}$ is Thom-Mather stratified (as in Definition 6.8) with

$Y_I := \mathring{\partial}_I \mathbb{F}^{d,e}_n$ as the open strata for $I \subset 2 \ltimes n$; we may further assume that for any open stratum Y_I, the projection map from its tubular neighborhood,

$$\pi_{Y_I} : T_{Y_I} \to Y_I,$$

is the orthogonal projection. A smooth embedding of a $\langle 2 \ltimes n\rangle$-manifold X into $\mathbb{F}^{d,e}_n$ is called *neat* if

- It respects the strata, that is, for every $i \in 2 \ltimes n$, $\partial_i X = X \cap \partial_i \mathbb{F}^{d,e}_n$.
- For every $I \subset 2 \ltimes n$,

$$T_{Y_I} \cap X = \pi^{-1}_{Y_I}(\mathring{\partial}_I X).$$

Now we are all set to define framed flow bimodules; compare Definitions 14.2 and 14.3.

Definition 15.2. Let $\mathscr{C} = (\mathscr{C}, \iota, \phi)$ and $\mathscr{C}' = (\mathscr{C}', \iota', \phi')$ be two framed flow categories, both relative some $d \in \mathbb{N}$. A *framed flow bimodule* $\mathscr{B} = (\mathscr{B}, \jmath, \psi)$ from $\mathscr{C}$ to $\mathscr{C}'$ consists of the following data:

- A flow bimodule $\mathscr{B} \colon \mathscr{C} \to \mathscr{C}'$;
- Neat embeddings $\jmath_{i,j} \colon \overline{\mathcal{N}}(i,j) \to \mathbb{F}^{d,e}_{i,j}$ (for some fixed integer e) for all $i \geq j$;
- Framings $\psi_{i,j}$ of their normal bundles $\mu_{i,j}$;

satisfying, for all $i \geq j \geq k$,

- The following diagrams commute:

$$\begin{array}{ccccc} \overline{\mathcal{M}}(i,j) \times \overline{\mathcal{N}}(j,k) & \longrightarrow & \partial_{0,i-j}\overline{\mathcal{N}}(i,k) & \hookrightarrow & \overline{\mathcal{N}}(i,k) \\ {\scriptstyle \iota_{i,j} \times \jmath_{j,k}} \big\downarrow & & \big\downarrow & & \big\downarrow {\scriptstyle \jmath_{i,k}} \\ \mathbb{E}^d_{i,j} \times \mathbb{F}^{d,e}_{j,k} & \xrightarrow{\cong} & \partial_{0,i-j}\mathbb{F}^{d,e}_{i,k} & \hookrightarrow & \mathbb{F}^{d,e}_{i,k} \end{array} \qquad \begin{array}{ccccc} \overline{\mathcal{N}}(i,j) \times \overline{\mathcal{M}}'(j,k) & \longrightarrow & \partial_{1,i-j}\overline{\mathcal{N}}(i,k) & \hookrightarrow & \overline{\mathcal{N}}(i,k) \\ {\scriptstyle \jmath_{i,j} \times \iota'_{j,k}} \big\downarrow & & \big\downarrow & & \big\downarrow {\scriptstyle \jmath_{i,k}} \\ \mathbb{F}^{d,e}_{i,j} \times \mathbb{E}^d_{j,k} & \xrightarrow{\cong} & \partial_{1,i-j}\mathbb{F}^{d,e}_{i,k} & \hookrightarrow & \mathbb{F}^{d,e}_{i,k} \end{array}$$

- In the above two cases, the product framing $\phi_{i,j} \times \psi_{j,k}$ on $\nu_{i,j} \times \mu_{j,k}$ and the product framing $\psi_{i,j} \times \phi'_{j,k}$ on $\mu_{i,j} \times \nu'_{j,k}$ agree with the pullback framing of $\psi_{i,k}$ on $\nu_{i,k}$.

See the right half of Figure 31. In line with earlier notation, for $x \in \mathscr{C}_i, y \in \mathscr{C}'_j$, we will denote the restrictions of $\jmath_{i,j}, \mu_{i,j}, \psi_{i,j}$ to $\overline{\mathcal{N}}(x,y)$ as $\jmath_{x,y}, \mu_{x,y}, \psi_{x,y}$.

Example 15.3. The only explicit example of a framed flow bimodule that we will provide is that of the identity bimodule connecting a framed flow category $\mathscr{C} = (\mathscr{C}, \iota, \phi)$ to itself. Assume $\mathscr{C}$ is embedded relative d, and fix $e = 0$.

For the 0-dimensional spaces, if $\mathrm{gr}(x) = \mathrm{gr}(y)$, define $\overline{\mathcal{N}}(x,y)$ to be empty if $x \neq y$, and to be a point p_x embedded in $\mathbb{F}^{d,0}_0 = \mathbb{R}^0$, otherwise.

For the 1-dimensional spaces, if $\mathrm{gr}(x) = \mathrm{gr}(y) + 1$, define $\overline{\mathcal{N}}(x,y) = \Delta^1 \times \overline{\mathcal{M}}(x,y)$, embedded in $\mathbb{F}^{d,0}_1 = \widetilde{\Delta}^1 \times \mathbb{R}^d = \Delta^1 \times \mathbb{R}^d$ by the product embedding $(\mathrm{Id}, \iota_{x,y})$; the normal bundle $\mu_{x,y}$ is the pullback of $\nu_{x,y}$ under the projection map $\Delta^1 \times \overline{\mathcal{M}}(x,y) \to \overline{\mathcal{M}}(x,y)$, and framing $\psi_{x,y}$ is given by $\phi_{x,y}$. Its boundary consists of two pieces: $\partial_{0,1} = \overline{\mathcal{M}}(x,y) \times p_y$ and $\partial_{1,0} = p_x \times \overline{\mathcal{M}}(x,y)$, and they are embedded in $\partial_{0,1}\mathbb{F}^{d,0}_1$ and $\partial_{1,0}\mathbb{F}^{d,0}_1$ by the product embedding as well.

In general, if $\mathrm{gr}(x) = \mathrm{gr}(y) + k$, the space $\overline{\mathcal{M}}(x,y)$ is a $\langle k-1\rangle$-manifold, so it has closed strata $\partial_I \overline{\mathcal{M}}(x,y)$ for $I \subset \{1, \dots, k-1\}$, which are embedded and framed in $\partial_I \mathbb{R}^{k-1}_+ \times \mathbb{R}^{kd}$. Equation (15.2) produces a decomposition

$$\mathbb{F}^{d,0}_k \cong \bigcup_{I \subset \{1,\dots,k-1\}} (\Delta^k_I \times \partial_I \mathbb{R}^{k-1}_+ \times \mathbb{R}^{kd}),$$

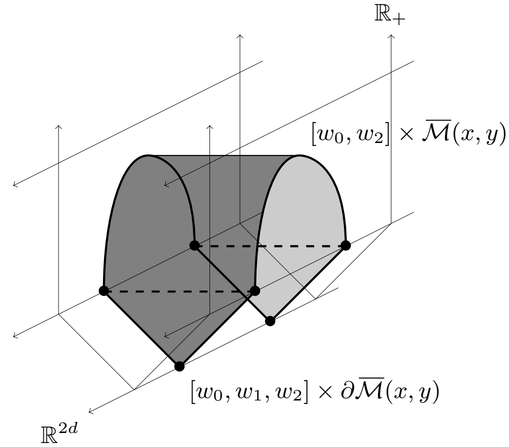


FIGURE 32. If $\mathrm{gr}(x) - \mathrm{gr}(y) = 2$, the space $\overline{\mathcal{N}}(x, y)$ is the union of two pieces $[w_0, w_2] \times \overline{\mathcal{M}}(x, y)$ and $[w_0, w_1, w_2] \times \partial\overline{\mathcal{M}}(x, y)$ (separated by dashed lines), and is embedded in $\mathbb{F}_2^{d,0} = \big([w_0, w_2] \times \mathbb{R}_+ \times \mathbb{R}^{2d}\big) \cup \big([w_0, w_1, w_2] \times \mathbb{R}^{2d}\big)$. We drew here the case where $\overline{\mathcal{M}}(x, y)$ is an interval and hence $\overline{\mathcal{N}}(x, y)$ is a hexagon.

where $\Delta_I^k = [(w_i)_{i \in I \cup \{0,k\}}]$ is the convex hull of vertices w_i fo $i \in I \cup \{0, k\}$ as before. Define $\overline{\mathcal{N}}(x, y)$ to be corresponding union

$$\overline{\mathcal{N}}(x, y) := \bigcup_{I \subset \{1,\dots,k-1\}} (\Delta_I^k \times \partial_I \overline{\mathcal{M}}(x, y)),$$

with the subspace $\Delta_I^k \times \partial_I \overline{\mathcal{M}}(x, y)$ embedded in the subspace $\Delta_I^k \times \partial_I \mathbb{R}_+^{k-1} \times \mathbb{R}^{kd} \subset \mathbb{F}_k^{d,0}$ by the product embedding $(\mathrm{Id}, \iota_{x,y})$ and framed by $\phi_{x,y}$. The smooth structure and the $\langle 2 \ltimes k\rangle$-manifold structure on $\overline{\mathcal{N}}(x, y)$ are induced as a subspace of $\mathbb{F}_k^{d,0}$. See Figure 32.

If $\mathscr{B} = (\mathscr{B}, \jmath, \psi)$ is a compact framed flow bimodule connecting compact framed flow categories $\mathscr{C} = (\mathscr{C}, \iota, \phi)$ and $\mathscr{C}' = (\mathscr{C}', \iota', \phi')$, then there is an associated chain map $f_{\mathscr{B}} \colon C_*(\mathscr{C}) \to C_*(\mathscr{C}')$, defined on generators $x \in \mathscr{C}$ by

$$f_{\mathscr{B}}(x) = \sum_{\substack{y \in \mathscr{C}' \\ \mathrm{gr}(y) = \mathrm{gr}(x)}} \#\overline{\mathcal{N}}(x, y) y. \tag{15.8}$$

Compare Equation (14.9).

Proposition 15.4. *Let $\mathscr{C} = (\mathscr{C}, \iota, \phi)$ and $\mathscr{C}' = (\mathscr{C}', \iota', \phi')$ be two compact framed flow categories with gradings bounded below. A compact framed flow bimodule $\mathscr{B} = (\mathscr{B}, \jmath, \psi)$ connecting them induces a map of spectra $F_{\mathscr{B}} \colon S(\mathscr{C}) \to S(\mathscr{C}')$ so that the induced maps on homology $H(F_{\mathscr{B}}) \colon H(S(\mathscr{C})) \to H(S(\mathscr{C}'))$ and $H(f_{\mathscr{B}}) \colon H(\mathscr{C}) \to H(\mathscr{C}')$ agree.*

Proof. Mimicking the construction from Section 14.3, we will associate a CW complex to $\mathscr{B}$. Again, first assume both $\mathscr{C}, \mathscr{C}'$ have gradings supported in some interval $[B + 1, A]$, with $A, B \in \mathbb{Z}$ and $B < 0$, and set $D_{d,e}(A, B) = e + (A - B)d - B - 1$. Our CW complex $|\mathscr{B}|_{\jmath,\psi,A,B}$ again has a single 0-cell, and for each $x \in \mathscr{C} \amalg \mathscr{C}'$, a cell $\mathcal{D}(x)$ of dimension $\mathrm{gr}(x) + D_d(A, B)$ if $x \in \mathscr{C}'$, or $1 + \mathrm{gr}(x) + D_d(A, B)$ if $x \in \mathscr{C}$.

As before, choose small $\epsilon > 0$ and large R such that $\iota_{i,j}$ (respectively $\iota'_{i,j}$) extends via the normal framings to embeddings of $\overline{\mathcal{M}}(i,j) \times [-\epsilon,\epsilon]^{(i-j)d}$ (respectively, $\overline{\mathcal{M}}'(i,j) \times [-\epsilon,\epsilon]^{(i-j)d}$) inside $[0,R]^{i-j-1} \times [-R,R]^{(i-j)d} \subset \mathbb{R}_+^{i-j-1} \times \mathbb{R}^{(i-j)d} \cong \mathbb{E}^d_{i,j}$. Just as $[0,R]^{i-j-1} \times [-R,R]^{(i-j)d}$ plays the role of a large compact subset of $\mathbb{E}^d_{i,j}$, we will pick a suitably large compact subset of $\mathbb{F}^{d,e}_{i,j}$ as follows: Consider $\mathbb{R}_+^{n-1} = \mathbb{R}_+\langle \vec{v}_1, \dots, \vec{v}_{n-1}\rangle$ from Equation (15.1), and let

$$[0,R]^{n-1} = [0,R]\langle \vec{v}_1, \dots, \vec{v}_{n-1}\rangle = \Big\{ \sum_i t_i \vec{v}_i \mid 0 \le t_i \le R, \forall i \Big\} \subset \mathbb{R}_+^{n-1}.$$

Let

$$\widetilde{\Delta}^n_R = \Delta^n + [0,R]^{n-1} \subset \Delta^n + \mathbb{R}_+^{n-1} = \widetilde{\Delta}^n$$

be the corresponding compact subset; the corresponding compact subset of $\mathbb{F}^{d,e}_{i,j}$ (from Equations (15.4) and (15.5)) is

$$\widetilde{\Delta}^{i-j}_R \times [-R,R]^e \times [-R,R]^{(i-j)d} \subset \widetilde{\Delta}^{i-j} \times \mathbb{R}^e \times \mathbb{R}^{(i-j)d} = \mathbb{F}^{d,e}_{i,j}.$$

We will assume that the embeddings $\jmath_{i,j}$ extend via the normal framing to embeddings of $\overline{\mathcal{N}}(i,j) \times [-\epsilon,\epsilon]^e \times [-\epsilon,\epsilon]^{(i-j)d}$ inside $\widetilde{\Delta}^{i-j}_R \times [-R,R]^e \times [-R,R]^{(i-j)d}$.

Let T_e be the set of e $\mathbb{R}$-factors appearing in $\mathbb{R}^e$, and let $S^e = \bigwedge_{T_e} S^1$ denote the corresponding e-dimensional sphere. Then $T_e \amalg T_d(A,B)$ is a set of cardinality $D_{d,e}(A,B)$, and $S^e \wedge S_d(A,B)$ is the corresponding $D_{d,e}(A,B)$-dimensional sphere.

The CW complex $|\mathscr{B}|_{\jmath,\psi,A,B}$ has $\Sigma^e|\mathscr{C}'|_{\iota',\phi',A,B} = S^e \wedge |\mathscr{C}'|_{\iota',\phi',A,B}$ as a subcomplex. In more detail, for any $x \in \mathscr{C}'$ with $\mathrm{gr}(x) = m$, define the cell

$$\mathcal{D}(x) = [-\epsilon,\epsilon]^{(A-m)d} \times [0,R]^{m-B-1} \times [-\epsilon,\epsilon]^e \times [-R,R]^{(m-B)d};$$

letting $\partial_i \mathcal{D}(x)$ denote the subset where the coordinate in the i^{th} $[0,R]$-factor is 0, we get an embedding

$$\overline{\mathcal{M}}'(x,y) \times \mathcal{D}(y) \hookrightarrow \partial_{m-l}\mathcal{D}(x)$$

for any $y \in \mathscr{C}'$ with $\mathrm{gr}(y) = l < m$. The attaching map on $\partial\mathcal{D}(x)$ maps the image $\mathcal{D}_y(x)$ of this embedding to $\mathcal{D}(y)$ and $\partial\mathcal{D}(x) \setminus \bigcup_y \mathcal{D}_y(x)$ to the basepoint. (This construction is exactly as described in Section 14.3, but with an extra factor of $[-\epsilon,\epsilon]^e$.)

Now for $x \in \mathscr{C}$ with $\mathrm{gr}(x) = m$, define the cell

$$\mathcal{D}(x) = \underbrace{[-\epsilon,\epsilon]^{(A-m)d}}_{\subset \mathbb{E}^d_{A,m}} \times \underbrace{\widetilde{\Delta}^{m-B}_R \times [-R,R]^e \times [-R,R]^{(m-B)d}}_{\subset \mathbb{F}^{d,e}_{m,B}} \subset \mathbb{F}^{d,e}_{A,B}; \tag{15.9}$$

see Figure 33. There is a map $\mathcal{D}(x) \to \widetilde{\Delta}^{m-B}$ by first projecting to the $\widetilde{\Delta}^{m-B}_R$ factor, followed by the inclusion $\widetilde{\Delta}^{m-B}_R \hookrightarrow \widetilde{\Delta}^{m-B}$. By an abuse of notation, for any $(i_0,i_1) \in 2 \ltimes (m-B)$, let $\partial_{i_0,i_1}\mathcal{D}(x)$ denote the preimage of $\partial_{i_0,i_1}\widetilde{\Delta}^{m-B}$ under this map.

For $y \in \mathscr{C}$ with $\mathrm{gr}(y) = l < m$, we have an embedding

$$\overline{\mathcal{M}}(x,y) \times [-\epsilon,\epsilon]^{(m-l)d} \hookrightarrow [0,R]^{m-l-1} \times [-R,R]^{(m-l)d} \subset \mathbb{E}^d_{m,l};$$

extending by identity, we get an embedding

$$\begin{aligned}\overline{\mathcal{M}}(x,y) \times \mathcal{D}(y) &\cong [-\epsilon,\epsilon]^{(A-m)d} \times \big(\overline{\mathcal{M}}(x,y) \times [-\epsilon,\epsilon]^{(m-l)d}\big) \times \big(\widetilde{\Delta}^{l-B}_R \times [-R,R]^e \times [-R,R]^{(l-B)d}\big)\\ &\hookrightarrow \underbrace{[-\epsilon,\epsilon]^{(A-m)d}}_{\subset \mathbb{E}^d_{A,m}} \times \underbrace{[0,R]^{m-l-1} \times [-R,R]^{(m-l)d}}_{\subset \mathbb{E}^d_{m,l}} \times \underbrace{\widetilde{\Delta}^{l-B}_R \times [-R,R]^e \times [-R,R]^{(l-B)d}}_{\subset \mathbb{F}^{d,e}_{l,B}} = \partial_{0,m-l}\mathcal{D}(x),\end{aligned}$$

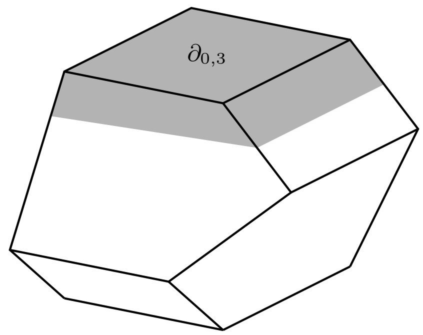


FIGURE 33. The cell $\mathcal{D}(x)$ from Equation (15.9); here $n = m - B = 3$, and we have only shown the $\widetilde{\Delta}^3_R$ factor. (Compare the $n = 3$ case in Figure 28.) The top facet is $\partial_{0,3}\mathcal{D}(x)$, and its δ-neighborhood $\widetilde{\mathcal{D}}(x) = N_\delta \partial_{0,3}\mathcal{D}(x)$ is shown in gray.

and let $\mathcal{D}_y(x)$ be its image. Similarly, for $y \in \mathscr{C}'$ with $\mathrm{gr}(y) = l \leq m$, we have an embedding

$$\overline{\mathcal{N}}(x,y) \times [-\epsilon,\epsilon]^e \times [-\epsilon,\epsilon]^{(m-l)d} \hookrightarrow \widetilde{\Delta}^{m-l}_R \times [-R,R]^e \times [-R,R]^{(m-l)d} \subset \mathbb{F}^{d,e}_{m,l};$$

once again, extending by identity, we get an embedding

$$\begin{aligned}\overline{\mathcal{N}}(x,y) \times \mathcal{D}(y) &\cong [-\epsilon,\epsilon]^{(A-m)d} \times \big(\overline{\mathcal{N}}(x,y) \times [-\epsilon,\epsilon]^e \times [-\epsilon,\epsilon]^{(m-l)d}\big) \times \big([0,R]^{l-B-1} \times [-R,R]^{(l-B)d}\big)\\ &\hookrightarrow \underbrace{[-\epsilon,\epsilon]^{(A-m)d}}_{\subset \mathbb{E}^d_{A,m}} \times \underbrace{\widetilde{\Delta}^{m-l}_R \times [-R,R]^e \times [-R,R]^{(m-l)d}}_{\subset \mathbb{F}^{d,e}_{m,l}} \times \underbrace{[0,R]^{l-B-1} \times [-R,R]^{(l-B)d}}_{\subset \mathbb{E}^d_{l,B}} = \partial_{1,m-l}\mathcal{D}(x),\end{aligned}$$

and let $\mathcal{D}_y(x)$ again denote its image. As before, the attaching map on $\partial\mathcal{D}(x)$ maps $\mathcal{D}_y(x)$ to $\mathcal{D}(y)$ (for all $y \in \mathscr{C} \amalg \mathscr{C}'$), and maps $\partial\mathcal{D}(x) \setminus \bigcup_y \mathcal{D}_y(x)$ to the basepoint.

By construction, $\Sigma^e|\mathscr{C}'|_{\iota',\phi',A,B}$ is a subcomplex of $|\mathscr{B}|_{\jmath,\psi,A,B}$, and let $|\widetilde{\mathscr{C}}|_{\iota,\phi,A,B}$ be the corresponding quotient complex. Therefore, we have a homotopy equivalence

$$|\widetilde{\mathscr{C}}|_{\iota,\phi,A,B} \simeq \mathrm{Cone}\big(\Sigma^e|\mathscr{C}'|_{\iota',\phi',A,B} \to |\mathscr{B}|_{\jmath,\psi,A,B}\big). \tag{15.10}$$

The quotient complex also has a cell $\mathcal{D}(x)$ for each $x \in \mathscr{C}$, with attaching map sending $\mathcal{D}_y(x)$ to $\mathcal{D}(y)$ for all $y \in \mathscr{C}$, and the rest $\partial\mathcal{D}(x) \setminus \bigcup_y \mathcal{D}_y(x)$ to the basepoint. Consider the subspace $\partial_{0,m-B}\mathcal{D}(x) \cong [0,R]^{m-B-1}$, and let $\widetilde{\mathcal{D}}(x) = N_\delta\partial_{0,m-B}\mathcal{D}(x) \cong [0,\delta] \times \partial_{0,m-B}\mathcal{D}(x)$ be a small (closed) δ-neighborhood, see Figure 33. Comparing with Equation (15.9), we get

$$\widetilde{\mathcal{D}}(x) \cong [-\epsilon,\epsilon]^{(A-m)d} \times [0,\delta] \times [0,R]^{m-B-1} \times [-R,R]^e \times [-R,R]^{(m-B)d}.$$

These δ-neighborhoods are constructed inductively. We construct δ-neighborhoods $N_\delta\partial_{0,m-B}\widetilde{\Delta}^{m-B}_R \cong [0,\delta]\times\partial_{0,m-B}\widetilde{\Delta}^{m-B}_R$ of $\partial_{0,m-B}\widetilde{\Delta}^{m-B}_R$ inside $\widetilde{\Delta}^{m-B}_R$, such that, restricted to any facet $\partial_{0,m-l}\widetilde{\Delta}^{m-B}_R \cong [0,R]^{m-l-1} \times \widetilde{\Delta}^{l-B}_R$, it is $[0,R]^{m-l-1}$ times the δ-neighorbood $N_\delta\partial_{0,l-B}\widetilde{\Delta}^{l-B}_R \cong [0,\delta] \times \partial_{0,l-B}\widetilde{\Delta}^{l-B}_R$ of $\partial_{0,l-B}\widetilde{\Delta}^{l-B}_R$ inside $\widetilde{\Delta}^{l-B}_R$; and then define the δ-neighborhood $N_\delta\partial_{0,m-B}\mathcal{D}(x)$ to be this δ-neighborhood $N_\delta\partial_{0,m-B}\widetilde{\Delta}^{m-B}_R$ times $[-\epsilon,\epsilon]^{(A-m)d} \times [-R,R]^e \times [-R,R]^{(m-B)d}$. The embedding $\overline{\mathcal{M}}(x,y) \times \mathcal{D}(y) \hookrightarrow \partial_{0,m-l}\mathcal{D}(x)$ restricts to give an embedding

$$\overline{\mathcal{M}}(x,y) \times \widetilde{\mathcal{D}}(y) \hookrightarrow \partial_{0,m-l}\widetilde{\mathcal{D}}(x),$$

with image say $\widetilde{\mathcal{D}}_y(x)$. Therefore, we can define a new CW complex with cells $\widetilde{\mathcal{D}}(x)$ for $x \in \mathscr{C}$, and attaching maps sending $\widetilde{\mathcal{D}}_y(x)$ to $\widetilde{\mathcal{D}}(y)$ and the rest $\partial\widetilde{\mathcal{D}}(x) \setminus \bigcup_y \widetilde{\mathcal{D}}_y(x)$ to the basepoint. This gives precisely the construction from Section 14.3, but with an extra factor of $[0,\delta] \times [-\epsilon,\epsilon]^e$, and therefore, this new CW complex is $\Sigma\Sigma^e|\mathscr{C}|_{\iota,\phi,A,B}$, where the first suspension corresponds to the $[0,\delta]$-factor, and the second suspension is smash product with S^e from earlier. On the other hand, we have a quotient map

$$|\widetilde{\mathscr{C}}|_{\iota,\phi,A,B} \xrightarrow{\sim} \Sigma\Sigma^e|\mathscr{C}|_{\iota,\phi,A,B}, \tag{15.11}$$

that maps each $\mathcal{D}(x)$ to $\widetilde{\mathcal{D}}(x)$ by quotienting $\mathcal{D}(x) \setminus \widetilde{\mathcal{D}}(x)$ to the basepoint. This map is identity on the cellular chain complex, and hence a homotopy equivalence.

Combining Equations (15.10) and (15.11), we get a homotopy equivalence

$$\Sigma\Sigma^e|\mathscr{C}|_{\iota,\phi,A,B} \simeq \mathrm{Cone}\big(\Sigma^e|\mathscr{C}'|_{\iota',\phi',A,B} \to |\mathscr{B}|_{\jmath,\psi,A,B}\big).$$

Quotienting $|\mathscr{B}|_{\jmath,\psi,A,B}$ to the basepoint, we get a map $\mathrm{Cone}\big(\Sigma^e|\mathscr{C}'|_{\iota',\phi',A,B} \to |\mathscr{B}|_{\jmath,\psi,A,B}\big) \to \Sigma\Sigma^e|\mathscr{C}'|_{\iota',\phi',A,B}$; composing, we get our required map

$$F_{\mathscr{B}} \colon \Sigma\Sigma^e|\mathscr{C}|_{\iota,\phi,A,B} \to \Sigma\Sigma^e|\mathscr{C}'|_{\iota',\phi',A,B}.$$

(This is the standard Puppe construction.) As in Equation (14.9), the shifted reduced cellular chain complex of $|\mathscr{B}|_{\jmath,\psi,A,B}$ is isomorphic to the mapping cone of the map $f_{\mathscr{B}}$ from Equation (15.8)

$$\mathrm{Cone}(f_{\mathscr{B}}) \cong \Sigma^{-D_d(A,B)}\widetilde{C}^{\mathrm{CW}}_*(|\mathscr{B}|_{\jmath,\psi,A,B});$$

Therefore, $F_{\mathscr{B}}$ induces the chain map $f_{\mathscr{B}} \colon C_*(\mathscr{C}) \to C_*(\mathscr{C}')$ (after shifting gradings by $1+D_d(A,B)$).

We pass to spectra as before. (The cone appearing in Equation (15.10) is a special case of homotopy colimit, so is taken levelwise when constructing spectra.) By taking the suspension spectra, and desuspending $1 + D_d(A,B) = 1 + e + C_d(A,B)$ times, we get induced maps (still denoted $F_{\mathscr{B}}$),

$$F_{\mathscr{B}} \colon \Omega\Omega^e\Sigma\Sigma^e\Omega^{C_d(A,B)}|\mathscr{C}|_{\iota,\phi,A,B} \wedge \mathbb{S} \to \Omega\Omega^e\Sigma\Sigma^e\Omega^{C_d(A,B)}|\mathscr{C}'|_{\iota',\phi',A,B} \wedge \mathbb{S}.$$

Via the equivalence between X and $\Omega\Sigma X$ for any spectrum X, we get the map

$$F_{\mathscr{B}} \colon \Omega^{C_d(A,B)}|\mathscr{C}|_{\iota,\phi,A,B} \wedge \mathbb{S} \to \Omega^{C_d(A,B)}|\mathscr{C}'|_{\iota',\phi',A,B} \wedge \mathbb{S}.$$

Taking homotopy colimits as $A \to \infty$ and $B \to -\infty$, we get our map on spectra

$$\begin{aligned} F_{\mathscr{B}} = S(\mathscr{C},\iota,\phi) &= \mathrm{hocolim}_{A,B}\,\Omega^{C_d(A,B)}|\mathscr{C}|_{\iota,\phi,A,B} \wedge \mathbb{S} \\ &\to S(\mathscr{C}',\iota',\phi') = \mathrm{hocolim}_{A,B}\,\Omega^{C_d(A,B)}|\mathscr{C}'|_{\iota',\phi',A,B} \wedge \mathbb{S}. \end{aligned}$$

Finally, if the gradings of $\mathscr{C}$ and $\mathscr{C}'$ were only bounded below, we modify the construction as in Equation (14.7). We truncate $\mathscr{C},\mathscr{C}',\mathscr{B}$ to only objects with $\mathrm{gr} \leq K$, and only consider $A \geq K$, and then take homotopy colimits:

$$\begin{aligned} F_{\mathscr{B}} = S(\mathscr{C},\iota,\phi) &= \mathrm{hocolim}_{A,B,K}\,\Omega^{C_d(A,B)}|\mathscr{C}_{\leq K}|_{\iota,\phi,A,B} \wedge \mathbb{S} \\ &\to S(\mathscr{C}',\iota',\phi') = \mathrm{hocolim}_{A,B,K}\,\Omega^{C_d(A,B)}|\mathscr{C}'_{\leq K}|_{\iota',\phi',A,B} \wedge \mathbb{S}. \end{aligned}$$ □

Mimicking Definition 14.4, a *framed flow sub-bimodule* of a framed flow bimodule $\mathscr{B} \colon \mathscr{C} \to \mathscr{C}'$ is a pair of framed flow subcategories $\mathscr{D} \subset \mathscr{C}$, $\mathscr{D}' \subset \mathscr{C}'$, such that $\overline{\mathcal{N}}(x,y) = \varnothing$ for all $x \in \mathscr{D}$ and $y \in \mathscr{C}' \setminus \mathscr{D}'$. (The embeddings and framings on the framed flow sub-bimodule are induced from those on $\mathscr{B}$.)

We can now extend the notion of framed flow bimodules to the H-filtered setting. Specifically, if $\mathscr{C}$ and $\mathscr{C}'$ are H-filtered framed flow categories (as in Definition 14.9), an H-filtered framed flow bimodule $\mathscr{B}$ between them is a framed flow bimodule $\mathscr{B}\colon \mathscr{C} \to \mathscr{C}'$ which restricts on the objects of $\mathscr{C}(h)$ and $\mathscr{C}'(h)$ to give framed flow sub-bimodules $\mathscr{B}(h)\colon \mathscr{C}(h) \to \mathscr{C}'(h)$, for all $h \in H$. In this case, Proposition 15.4 produces an H-filtered map of spectra.

15.2. **Bimodules from grid diagrams.** Assume $(\mathscr{C}, \iota, \phi)$ and $(\mathscr{C}', \iota', \phi')$ are two framed flow categories associated to some fixed grid diagram $\mathbb{G}$, both constructed following the steps in Section 12; let $\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D)$ and $\overline{\mathcal{M}}'_{\vec{N},\vec{\lambda}}(D)$ be the moduli spaces used in the two constructions. In this section, we outline the steps that will be needed in order to construct a framed flow bimodule $(\mathscr{B}, \jmath, \psi)$ connecting them. The construction is similar, so we will only describe the major waypoints, and will skip the details.

For each triple $(D, \vec{N}, \vec{\lambda}) \in \mathscr{T}$, there is a new "interpolating" moduli space $\mathcal{N}_{\vec{N},\vec{\lambda}}(D)$, with compactification $\overline{\mathcal{N}}_{\vec{N},\vec{\lambda}}(D)$. This plays the role of the moduli space of continuation maps in Floer theory with domain D, and $|\vec{N}|$ marked points on the boundary, with the N_j points corresponding to H_j or V_j bubbles, which have been grouped according to the partition λ_j. There is no $\mathbb{R}$ translation for continuation maps, so compared to Equation (8.1), the dimension is given by

$$\dim \mathcal{N}_{\vec{N},\vec{\lambda}}(D) = \mathrm{gr}(D, \vec{N}, \vec{\lambda}) = \mu(D) + |\vec{\lambda}|.$$

Recall from Section 8 that in the case of ordinary moduli spaces, we did not consider the trivial case $(D, \vec{N}, \vec{\lambda}) = (c_x, \vec{0}, \vec{0})$. This has $\mathrm{gr}(c_x, \vec{0}, \vec{0}) = 0$ and therefore the formal dimension of $\mathcal{M}_{\vec{0},\vec{0}}(c_x)$ should be -1. The interpolating moduli space $\mathcal{N}_{\vec{0},\vec{0}}(c_x)$ has dimension zero and we no longer ignore it. Rather, we define it to be a point p_x.

These new spaces are stratified, and similar to Equation (9.1), the strata of $\overline{\mathcal{N}}_{\vec{N},\vec{\lambda}}(D)$ are given by

$$\prod_{j=1}^{i-1} \mathcal{M}_{\vec{N}^j+\mathbb{O}(E^j)+\mathbb{O}(F^j),\vec{\lambda}^j}(D^j) \times \mathcal{N}_{\vec{N}^i+\mathbb{O}(E^i)+\mathbb{O}(F^i),\vec{\lambda}^i}(D^i) \times \prod_{j=i+1}^{r} \mathcal{M}'_{\vec{N}^j+\mathbb{O}(E^j)+\mathbb{O}(F^j),\vec{\lambda}^j}(D^j) \tag{15.12}$$

for $1 \le i \le r$, subject to similar conditions. Similarly to Section 9.1, each $\overline{\mathcal{N}}_{\vec{N},\vec{\lambda}}(D)$ has a single codimension zero stratum, namely $\mathcal{N}_{\vec{N},\vec{\lambda}}(D)$; and similarly to Section 9.2, the codimension-one strata correspond to trajectory breaking into two pieces

$$\mathcal{M}_{\vec{N}^1,\vec{\lambda}^1}(D^1) \times \mathcal{N}_{\vec{N}^2,\vec{\lambda}^2}(D^2) \qquad \text{or} \qquad \mathcal{N}_{\vec{N}^1,\vec{\lambda}^1}(D^1) \times \overline{\mathcal{M}}'_{\vec{N}^2,\vec{\lambda}^2}(D^2)$$

with $D^1 * D^2 = D$, $\vec{N}^1 + \vec{N}^2 = \vec{N}$, and $\lambda^1_j * \lambda^2_j = \lambda_j$ for all $j = 2, \dots, n$; or a single boundary degeneration (Type II)

$$\mathcal{N}_{\vec{N}+\vec{e}_j,\vec{\lambda}'}(D^1), \text{ with } D^1 + H_j = D \text{ or } D^1 + V_j = D \tag{15.13}$$

and $\lambda'_s = \lambda_s$ for all $s \neq j$ and $\lambda'_j \in \mathrm{UE}(\lambda_j)$; or an elementary coarsening of partitions (Type III)

$$\mathcal{N}_{\vec{N},\vec{\lambda}'}(D) \tag{15.14}$$

with $\lambda'_s = \lambda_s$ for all $s \neq j$ and $\lambda'_j \in \mathrm{EC}(\lambda_j)$. Specifically, the codimension-one strata where the $\overline{\mathcal{M}}$ or $\overline{\mathcal{M}}'$ factor is zero-dimensional is either a Type I product

$$\mathcal{M}_0(R) \times \mathcal{N}_{\vec{N},\vec{\lambda}}(E) \qquad \text{or} \qquad \mathcal{N}_{\vec{N},\vec{\lambda}}(E) \times \overline{\mathcal{M}}'_0(R) \tag{15.15}$$

where R is a rectangle and $R * E = D$, resp. $E * R = D$; or a Type IV product

$$\mathcal{M}_{N\vec{e}_j,(N)_j}(c_x) \times \mathcal{N}_{\vec{M},\vec{\mu}}(D) \qquad \text{or} \qquad \mathcal{N}_{\vec{M},\vec{\mu}}(D) \times \overline{\mathcal{M}}'_{N\vec{e}_j,(N)_j}(c_y) \tag{15.16}$$

where $M_s = N_s$ and $\mu_j = \lambda_j$ for $j \neq s$, $N + M_j = N_j$, and $(N) * \mu_j = \lambda_j$, resp. $\mu_j * (N) = \lambda_j$. Note that the terms appearing in $\delta^{\mathrm{II}}(D, \vec{N}, \vec{\lambda})$, $\delta^{\mathrm{III}}(D, \vec{N}, \vec{\lambda})$, $\delta^{\mathrm{I}}(D, \vec{N}, \vec{\lambda})$, and $\delta^{\mathrm{IV}}(D, \vec{N}, \vec{\lambda})$ (from Formula (4.5)) correspond to the strata from Formulas (15.13)–(15.16).

Similarly to Section 9.3, each $\overline{\mathcal{N}}_{\vec{N},\vec{\lambda}}(D)$ is a closed stratum of $\overline{\mathcal{N}}_0(\widetilde{D})$ with $\widetilde{D} = D + \widetilde{E} + \widetilde{F}$; we define $\overline{\mathcal{N}}([\widetilde{D}]) = \bigcup \overline{\mathcal{N}}_0(\widetilde{D})$, which are $2 \ltimes (\mathrm{gr}(x) - \mathrm{gr}(y))$-manifolds. The local models will be based on the stratified space $\widetilde{Z}_N$ with strata $\widetilde{Z}(p^-, p^0, p^+; \lambda)$ from Equation (7.7); the more general local model is $\widetilde{Z}_{\vec{N}} = \prod_j \mathrm{Sym}^{N_j}(\mathbb{C})$ (compare Equation (7.9))) with strata $\widetilde{Z}(\vec{p}^-, \vec{p}^0, \vec{p}^+; \vec{\lambda})$; and the most general local model is of the form $\mathbb{R}^a \times \mathbb{R}_+^{r-1} \times \prod_{j=1}^{i-1} Z_{\vec{N}^i} \times \widetilde{Z}_{\vec{N}^i} \times \prod_{j=i+1}^{r} Z_{\vec{N}^r}$ (compare Equation (7.10)). The standard frame of any stratum inside this space is same as the standard frame from Definition 7.12. We require the local model of the stratum from Equation (15.12) inside $\overline{\mathcal{N}}([\widetilde{D}])$ to be same as the local model of

$$\mathbb{R}^a \times \{0\} \times \prod_{j=1}^{i-1} Z(0, \vec{N}^j + \mathbb{O}(E^j) + \mathbb{O}(F^j), 0; \vec{\lambda}^j) \times \widetilde{Z}(0, \vec{N}^i + \mathbb{O}(E^i) + \mathbb{O}(F^i), 0; \vec{\lambda}^i)$$
$$\times \prod_{j=i+1}^{r} Z(0, \vec{N}^j + \mathbb{O}(E^j) + \mathbb{O}(F^j), 0; \vec{\lambda}^j)$$

inside

$$\mathbb{R}^a \times \mathbb{R}_+^{r-1} \times \prod_{j=1}^{i-1} Z_{\vec{N}^j + \mathbb{O}(E^j) + \mathbb{O}(F^j)} \times \widetilde{Z}_{\vec{N}^i + \mathbb{O}(E^i) + \mathbb{O}(F^i)} \times \prod_{j=i+1}^{r} Z_{\vec{N}^j + \mathbb{O}(E^j) + \mathbb{O}(F^j)}.$$

Similarly to Section 10, $\overline{\mathcal{N}}_{\vec{N},\vec{\lambda}}(D)$ is neatly embedded and framed inside $\mathbb{F}_l^{d,e}$ where $l = \mu(D) + 2|\vec{N}|$ is the thick dimension. The data of the neat embedding consists of an l-dimensional thickening such that the local model of each stratum inside the thickening is identified with the local model of that stratum inside $\overline{\mathcal{N}}([\widetilde{D}])$; this is recorded via the internal frame, which is the image of the standard frame under this identification. The external frame is a framing of the thickening inside the ambient space $\mathbb{F}_l^{d,e}$. We require the neat embeddings and framings to be coherent with respect to the lower dimensional strata (cf. Definitions 10.3 and 10.5). (For the stratum from Equation (15.12), the neat embeddings and framings on the $\mathcal{M}$ and $\mathcal{M}'$ factors are induced from (ι, ϕ) and (ι', ϕ'), respectively.)

Finally, for grid generators $[x, j_2, \dots, j_n]$ and $[y, i_2, \dots, i_n]$, we set

$$\overline{\mathcal{N}}([x, j_2, \dots, j_n], [y, i_2, \dots, i_n]) = \overline{\mathcal{N}}([D])$$

with induced embedding $\jmath$ and framing ψ, similarly to Equation (14.11).

15.3. **Invariance under auxiliary choices.** Finally we are ready to prove Theorem 1.1. After fixing the grid $\mathbb{G}$ (with numbered markings $O_1, \dots, O_n, X_1, \dots, X_n$) and and designating the marking O_1 as distinguished, the further auxiliary choices in the construction of the filtered Floer spectrum $\mathcal{X}(\mathbb{G})$ are listed below, reusing the notaion from the steps in Section 12.

(I-1) The (even) dimension $d \gg 0$ chosen at the the start of Section 10;
(I-2) The diffeomorphism Ψ_n from (13.5), which is used to embed the permutohedron in Section 13. In turn, to construct Ψ_n, we needed:

(a) The proper smooth embedding in Proposition 13.2. Any two such differ by an element in the relative diffeomorphism group $\mathrm{Diff}_\partial(\mathbb{R}^m \times \mathbb{R}_+)$;
(b) The embedding i' in Proposition 13.3;
(c) The collar neighborhood in part (3) of the construction of Ψ_n, where we extended $\Psi_|$.

(I-3) The construction of the neighborhoods $U(\partial\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D))$ and $V(\partial\overline{\mathcal{M}}_{\vec{N},\vec{\lambda}}(D))$ in Step (C-2), involving choice of small $\epsilon_Y > 0$;
(I-4) The smoothing in Step (C-3), involving choice of different small $\epsilon_Y > 0$ (from Lemma 6.17) and $\delta > 0$ (from Definition 5.8);
(I-5) The element $\mathfrak{b} \in \mathrm{Hom}(CDP'_k, \tilde{\Omega}^{k-1}_{\mathrm{fr}})$ in Step (C-7);
(I-6) The framed manifold $M(F, \vec{P}, \vec{\nu})$ representing $-\mathfrak{b}(F, \vec{P}, \vec{\nu})$, also in Step (C-7), along with its thickening;
(I-7) The filling $\overline{\mathcal{M}}'_{\vec{N},\vec{\lambda}}(D)$ (along with its embedding and framing) in Step (C-8);
(I-8) The thickening in Step (C-9);
(I-9) The values of ϵ, R, A and B in the construction of a spectrum from a flow category in Section 14.3.

We seek to prove that making different choices in each of these cases produces equivalent (filtered) spectra.

Lemma 15.5. *The filtered spectrum $\mathcal{X}(\mathbb{G})$ is independent of the choices in (I-1) and (I-9), up to filtered equivalence.*

Proof. For (I-1), we can turn a construction for d to one for a higher d' by a stabilization move in the sense of [27, Section 3.1]. Given a framed flow category $\mathscr{C}$ with a neat embedding relative d, we get one relative d' by post-composing each embedding

$$\iota_{x,y} \colon \overline{\mathcal{M}}(x,y) \hookrightarrow \mathbb{E}^d_{\mathrm{gr}(x),\mathrm{gr}(y)}$$

with the standard inclusion of $\mathbb{E}^d_{\mathrm{gr}(x),\mathrm{gr}(y)}$ into $\mathbb{E}^{d'}_{\mathrm{gr}(x),\mathrm{gr}(y)}$. We also let the new framings be obtained from the old by letting them be trivial in the new directions. It is proved in [26, Lemma 3.26] that the resulting CW complex is obtained from the original one by suspending $C_{d'}(A,B) - C_d(A,B) = (d'-d)(A-B)$ times, (in the notation of Equation (14.1)), or more specifically, by smash product with $\bigwedge_{T_{d'}(A,B)\setminus T_d(A,B)} S^1$ (in notation of Equation (14.3)). Since in the definition of the spectrum in Equation (14.6) we de-suspended $C_d(A,B)$ times, or more specifically, we looped with respect to $S_d(A,B) = \bigwedge_{T_d(A,B)} S^1$, it follows that the two spectra are equivalent. The same argument applies to their filtered pieces.

Independence of the quantities in (I-9) was proved (for general framed flow categories) in Lemmas 3.25 and 3.26 in [26]. □

Lemma 15.6. *The filtered spectrum $\mathcal{X}(\mathbb{G})$ is independent of the choices in (I-2), (I-3), and (I-4), up to filtered equivalence.*

Proof. We claim that all of these choices are taken from a contractible set. This is clearly the case for (I-3) and (I-4). It is also the case for (b) and (c) from (I-2), because the space of collars is contractible by [10]. As for part (a) from (I-2), we will prove in Lemma 15.7 below that $\mathrm{Diff}_\partial(\mathbb{R}^m \times \mathbb{R}_+)$ is contractible.

In all these cases, if we make two choices we can relate the respective framed flow categories $(\mathscr{C}_0, \iota_0, \phi_0)$ and $(\mathscr{C}_1, \iota_1, \phi_1)$ by a smooth 1-parameter family $(\mathscr{C}_t, \iota_t, \phi_t)_{t\in[0,1]}$. The flow categories have the same objects, the morphism spaces are diffeomorphic (in a way compatible with gluing), and the neat embeddings and framings vary smoothly. This yields isomorphic CW complexes $|\mathscr{C}_t|_{\iota(t),\phi(t),B,A}$, and hence equivalent spectra; compare [26, Lemma 3.25]. □

To complete the proof of Lemma 15.6, it remains to study $\mathrm{Diff}_\partial(\mathbb{R}^m \times \mathbb{R}_+)$. We will be using the weak topology on this group (i.e., that of uniform C^∞ convergence on compact subsets). The weak topology is sufficient for our purposes because in the end the diffeomorphism of $\mathbb{R}^m \times \mathbb{R}_+$ is used to construct an embedding of the permutohedron into a Euclidean space (with corners). The permutohedron is compact, and hence when we relate two elements of $\mathrm{Diff}_\partial(\mathbb{R}^m \times \mathbb{R}_+)$ by a path, we only have to show that their restriction to a compact subset is varying smoothly.

Lemma 15.7. *The space* $\mathrm{Diff}_\partial(\mathbb{R}^m \times \mathbb{R}_+)$ *(in the weak topology) is contractible.*

Proof. We will relate an arbitrary element $f \in \mathrm{Diff}_\partial(\mathbb{R}^m \times \mathbb{R}_+)$ to the identity by a continuous path, which is canonical up to a contractible set of choices.

We first apply a theorem of Kirby and Siebenmann [20, Theorem A.2] about the uniqueness of collars to map $f(\mathbb{R}^m \times [0,\epsilon))$ into $\mathbb{R}^m \times [0,\epsilon)$. This gives an isotopy f_t of $\mathbb{R}^m \times \mathbb{R}_+$ so that $f_0 = f$ and f_1 is the identity on $\mathbb{R}^m \times [0,\epsilon)$. (Also, f_t is constantly the identity on the boundary.)

Next, we relate f_1 to the identity by making the collar $\mathbb{R}^m \times [0,\epsilon)$ bigger and bigger. Precisely, consider the re-scaling map

$$\psi_t \colon \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}^m \times \mathbb{R}_+,\ \psi_t(x,s) = (x,ts)$$

and let $h\colon [1,2) \to [1,\infty)$ be an increasing homeomorphism. Then, for $t \in [1,2]$, the diffeomorphism

$$f_t \colon \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}^m \times \mathbb{R}_+, \quad f_t = \psi_{h(t)} \circ f_1 \circ \psi_{h(t)}^{-1}$$

is the identity on the collar $\mathbb{R}^m \times [0, h(t)\epsilon)$. Furthermore, in the limit as $t \to 2$, we have that f_t converges to the identity (which we denote by f_2) in the weak topology.

Altogether, the family $(f_t)_{t\in[0,2]}$ gives a path of diffeomorphisms from f_0 to the identity. □

Lemma 15.8. *The filtered spectrum $\mathcal{X}(\mathbb{G})$ is independent of the choices in (I-5), (I-6), (I-7), and (I-8), up to filtered equivalence.*

Proof. Suppose $(\mathscr{C},\iota,\phi)$ and $(\mathscr{C}',\iota',\phi')$ are two framed flow categories for the grid that were constructed by making different choices in (I-5), (I-6), (I-7), or (I-8). We will construct a compact framed flow bimodule $(\mathscr{B},\jmath,\psi)$ (as in Definition 15.2) connecting them, and show that the induced map (from Proposition 15.4) is a filtered equivalence.

Since we already have invariance under (I-2), we may assume that the two framed flow categories agree on $\mathscr{T}^\dagger$. The 0-dimensional moduli spaces were constructed in Section 12.2 and the only choice was the choice of embedding of points in contractible spaces (the frames were chosen explicitly); therefore, we may assume that the two framed flow categories agree on $\mathscr{T}_0'$ as well.

As discussed in Section 15.2, in order to construct the framed flow bimodule $(\mathscr{B},\jmath,\psi)$, we need to construct the stratified spaces $\overline{\mathcal{N}}_{\vec{N},\vec{\lambda}}(D)$, along with coherent neat embeddings and framings. The construction follows the same steps as in Section 12. For Step (C-1), we will construct $\overline{\mathcal{N}}_{\vec{N},\vec{\lambda}}(D)$ for all triples $(D,\vec{N},\vec{\lambda}) \in \mathscr{T}^\dagger \cup \mathscr{T}_0'$. Since $(\mathscr{C},\iota,\phi)$ and $(\mathscr{C}',\iota',\phi')$ agree on $\mathscr{T}^\dagger \cup \mathscr{T}_0'$, we set $e = 0$ for the moment, and define $\overline{\mathcal{N}}_{\vec{N},\vec{\lambda}}(D)$ as in Example 15.3. In particular, the 0-dimensional spaces $\overline{\mathcal{N}}_0(c_x)$ for $(c_x,0,0) \in \mathscr{T}_{-1}$ are points p_x, embedded in $\mathbb{R}^0$. This ensures that the chain map $f_{\mathscr{B}}\colon C_*(\mathscr{C}) \to C_*(\mathscr{C}')$ (from Equation (15.8)) is the identity.

When we go to the next steps, we may need to use a larger value of e. Starting from the above construction with $e = 0$, we post-compose by the standard inclusion $\mathbb{F}^{d,0}_{i,j} \hookrightarrow \mathbb{F}^{d,e}_{i,j}$, and define the new framing to be the old one, plus the standard frame in the new $\mathbb{R}^e$ direction.

The rest of the steps are almost identical to what was done in Section 12; we summarize them below. Assume $\overline{\mathcal{N}}_{\vec{N},\vec{\lambda}}(D)$ have been constructed (with coherent neat embeddings and coherent external framings) for all $(D,\vec{N},\vec{\lambda}) \in \mathscr{T}^\dagger \cup \mathscr{T}'_{\leq k-1}$. For any $(D,\vec{N},\vec{\lambda}) \in \mathscr{T}'_k$, its k-dimensional boundary $\partial\overline{\mathcal{N}}_{\vec{N},\vec{\lambda}}(D)$ is already constructed. By smoothing, we get an element $[\partial'\overline{\mathcal{N}}_{\vec{N},\vec{\lambda}}(D)] \in \tilde{\Omega}^k_{\mathrm{fr}}$. Together, these produce a cochain (indeed a cocycle) $\mathfrak{o}_k \in \mathrm{Hom}(CDP'_{k+1}, \tilde{\Omega}^k_{\mathrm{fr}})$. Therefore, $\mathfrak{o}_k$ is the coboundary of some $\mathfrak{b} \in \mathrm{Hom}(CDP'_k, \tilde{\Omega}^k_{\mathrm{fr}})$; change each k-dimensional spaces $\overline{\mathcal{N}}_{\vec{M},\vec{\mu}}(E)$ for $(E,\vec{M},\vec{\mu}) \in \mathscr{T}'_{k-1}$ by $-\mathfrak{b}(E,\vec{M},\vec{\mu})$, and then proceed inductively to construct the spaces $\overline{\mathcal{N}}_{\vec{N},\vec{\lambda}}(D)$ for all $(D,\vec{N},\vec{\lambda}) \in \mathscr{T}'_k$.

In the end, we obtain the framed flow bimodule $(\mathscr{B}, \jmath, \psi)$ connecting $(\mathscr{C}, \iota, \phi)$ to $(\mathscr{C}', \iota', \phi')$. As discussed above, the induced chain map $f_{\mathscr{B}}$ is identity, so the induced map on spectra is an equivalence. The same goes for the maps on the filtration levels of the spectra. □

Remark 15.9. The method in the proof of Lemma 15.8 can also provide alternate proofs of invariance under (I-3) and (I-4).

Proof of Theorem 1.1. This is a combination of Lemmas 15.5, 15.6 and 15.8. □

16. Examples

In this section we will compute the knot Floer spectra in several examples. In all the cases we considered, the spectra are determined by the information already existing in the knot Floer complexes. At the end we will discuss how one may look for an example where this is not the case.

16.1. **Generalities.** We will make repeated use of the following well-known result.

Proposition 16.1. *Suppose that the (reduced) homology of a spectrum $\mathcal{X}$ is finitely generated free Abelian and supported in at most two consecutive gradings; i.e. it is isomorphic to $\mathbb{Z}^k \oplus \mathbb{Z}^l$, with $\mathbb{Z}^k$ in homological grading d and $\mathbb{Z}^l$ in homological grading $d+1$. Then $\mathcal{X}$ is homotopy equivalent to the wedge sum of k copies of $\mathbb{S}^d$ and l copies of $\mathbb{S}^{d+1}$.*

Proof. By the Hurewicz theorem we have $\pi_d(\mathcal{X}) \cong \mathbb{Z}^k$, so there is a map $f\colon \bigvee^k \mathbb{S}^d \to \mathcal{X}$ inducing an isomorphism on H_d. The cone of this map has homology $\mathbb{Z}^l$ supported in degree $d+1$; applying the Hurewicz theorem again, together with Whitehead's theorem, tells us that this cone is equivalent to $\bigvee^l \mathbb{S}^{d+1}$. From the coexact sequence of f it follows that $\mathcal{X}$ is equivalent to the cone of a map $\bigvee^l \mathbb{S}^d \to \bigvee^k \mathbb{S}^d$. This map is zero on homology, and therefore zero by the Hopf theorem. The conclusion follows. □

Consider now a grid diagram $\mathbb{G}$, and suppose it represents a knot K (rather than a link with more components). In that case the spectra $\mathcal{X}(\mathbb{G})$ and $\widehat{\mathcal{X}}(\mathbb{G})$ coincide. Furthermore, in view of Remark 14.10, we expect the U_i actions on the spectrum $\mathcal{X}(\mathbb{G})$ to be trivial. Thus, we will just focus on $\mathcal{X}(\mathbb{G})$ as a filtered spectrum. If we ignore the filtration, the homology of $\mathcal{X}(\mathbb{G})$ is just $\widehat{HF}(S^3) = \mathbb{Z}$, so by Proposition 16.1 we have that $\mathcal{X}(\mathbb{G})$ is equivalent to $\mathbb{S}^0$. Thus, the interesting information is in the filtered pieces $\mathcal{X}(\mathbb{G}, h)$ for $h \in \mathbb{Z}$, and the maps $\mathcal{X}(\mathbb{G}, h-1) \to \mathcal{X}(\mathbb{G}, h)$. Note also that the cone of the map $\mathcal{X}(\mathbb{G}, h-1) \to \mathcal{X}(\mathbb{G}, h)$ is equivalent to the associated graded spectrum $g\mathcal{X}(\mathbb{G}, h) = g\widehat{\mathcal{X}}(\mathbb{G}, h)$.

The filtered chain complex underlying $\mathcal{X}(\mathbb{G})$ is $GC = \widehat{GC}$, which is filtered chain homotopy equivalent to $\widehat{CFK}(K)$, the hat version of the knot Floer complex in the notation of [25, Section

11.3] and [29].[(i)] The Alexander filtration on $\widehat{CFK}(K)$ is denoted

$$\dots \subseteq \mathcal{F}_{h-1}\widehat{CFK}(K) \subseteq \mathcal{F}_h\widehat{CFK}(K) \subseteq \cdots$$

We will use the notation $HFK(K,h)$ for the homology $H_*(\mathcal{F}_h\widehat{CFK}(K)) = H_*(\mathcal{X}(\mathbb{G},h))$. The associated graded of the filtered complex $\widehat{CFK}(K)$ is denoted $g\widehat{CFK}(K)$, with homology $\widehat{HFK}_*(K,h) = H_*(g\mathcal{X}(\mathbb{G},h))$.

The filtered complex $\widehat{CFK}(K)$ can be defined from any Heegaard diagram for the knot K, not necessarily a grid diagram; for example, we can use a doubly-pointed diagram and ask for disks to not pass through one basepoint. As such, we can appeal to the computations of knot Floer complexes from the literature, or to the computer program [50]. It turns out that in many cases, the information in the filtered complex $\widehat{CFK}(K)$ determines the filtered equivalence class of the spectrum $\mathcal{X}(\mathbb{G})$. To formulate a general condition that makes this hold, consider the inclusion $\mathcal{F}_h\widehat{CFK}(K) \to \widehat{CFK}(K)$ which induces on homology a map $HFK(K,h) \to \widehat{HF}(S^3) \cong \mathbb{Z}$. Let

$$HFK^{\diamond}(K,h) = \ker(HFK(K,h) \to \mathbb{Z}).$$

Following [38], we let $\tau(K)$ be the minimum value of h such that the map $HFK(K,h) \to \widehat{HF}(S^3) \cong \mathbb{Z}$ is nontrivial. It follows that

$$HFK(K,h) \cong \begin{cases} HFK^{\diamond}(K,h) & \text{if } h < \tau(K), \\ HFK^{\diamond}(K,h) \oplus \mathbb{Z} & \text{if } h \geq \tau(K). \end{cases} \tag{16.1}$$

(In the second case, the isomorphism $HFK(K,h) \cong HFK^{\diamond}(K,h) \oplus \mathbb{Z}$ is not canonical; we only have a canonical short exact sequence $0 \to HFK^{\diamond}(K,h) \to HFK(K,h) \to \mathbb{Z} \to 0$.)

Definition 16.2. A knot $K \subset S^3$ is called *Floer constrained* if the following conditions are satisfied:

(1) For every $h \in \mathbb{Z}$, the homology $HFK(K,h)$ is torsion-free;[(ii)]
(2) For any fixed $h \in \mathbb{Z}$, the homology groups $HFK^{\diamond}(K,h)$ are supported in at most two homological degrees and, if there are two, then these degrees are consecutive;
(3) (Monotonicity) There are no integers $h < h'$ and $k > k'$ such that $\widehat{HFK}_k(K,h)$ and $\widehat{HFK}_{k'}(K,h')$ are both non-zero.

Proposition 16.3. *For any Floer constrained knot, the filtered homotopy type of $\mathcal{X}(\mathbb{G})$ is determined by the filtered chain homotopy type of $\widehat{CFK}(K)$.*

Proof. The filtration on $\widehat{CFK}(K)$ induces a spectral sequence from $\bigoplus_{i \leq h} \widehat{HFK}(K,i)$ to $HFK(K,h)$, so $HFK(K,h)$ is supported (in homological grading) on a subset of the union of the supports of $\widehat{HFK}(K,i)$ for $i \leq h$. Using this fact, we deduce from Condition (3) in Definition 16.2 that monotonicity also holds for $CFK(K,h)$: there are no $h < h'$ and $k > k'$ such that $HFK_k(K,h)$ and $HFK_{k'}(K,h')$ are both non-zero.

(i) Warning: In the original paper [39] where they defined knot Floer homology, Ozsváth and Szabó used the notation $\widehat{CFK}(K)$ for the associated graded complex that is denoted $g\widehat{CFK}(K)$ here, and whose homology is $\widehat{HFK}(K)$. Compare [29, Remark 3.17].

(ii) This condition holds in practice. To date, no torsion has been observed in the knot Floer homologies of knots in S^3: neither in $\widehat{HFK}(K)$ nor in $HFK(K,h)$.

At the spectrum level, the map $HFK(K,h) \to \mathbb{Z}$ is realized by the inclusion $\mathcal{X}(\mathbb{G},h) \to \mathcal{X}(\mathbb{G}) \simeq \mathbb{S}^0$. Let us define a spectrum $\mathcal{X}^\diamond(\mathbb{G},h)$ by

$$\mathcal{X}^\diamond(\mathbb{G},h) = \begin{cases} \mathcal{X}(\mathbb{G},h) & \text{if } h < \tau(K), \\ \Omega\,\mathrm{Cone}(\mathcal{X}(\mathbb{G},h) \to \mathbb{S}^0) & \text{if } h \geq \tau(K). \end{cases}$$

The homology of $\mathcal{X}^\diamond(\mathbb{G},h)$ is $HFK^\diamond(K,h)$. Condition (1) implies that $HFK^\diamond(K,h)$ is torsion-free. Using Proposition 16.1 and Condition (2), we see that $\mathcal{X}^\diamond(\mathbb{G},h)$ must be a wedge of spheres, of at most two (consecutive) dimensions.

We claim that each $\mathcal{X}(\mathbb{G},h)$ is also a wedge of spheres. For $h < \tau(K)$, this is clearly true, because $\mathcal{X}(\mathbb{G},h) \simeq \mathcal{X}^\diamond(\mathbb{G},h)$. For $h \geq \tau(K)$, notice first that, since $HFK(K,\tau(K)-1)$ maps trivially to $\widehat{HF}(S^3) \cong \mathbb{Z}$ but $HFK(K,\tau(K))$ does not, the associated graded $\widehat{HFK}(K,\tau(K))$ must have some generator in degree 0. By monotonicity, we have that $\widehat{HFK}(K,h)$ is supported in degrees ≥ 0 for $h > \tau(K)$. Since every element of $HFK^\diamond(K,h)$ is eventually mapped to zero in $\widehat{HFK}(K,h') \cong \mathbb{Z}$ (for $h' \gg 0$), it must be cancelled by an element of $\widehat{HFK}$ that lies in homological degree one higher (and also in a higher Alexander grading). Therefore, for $h \geq \tau(K)$ we have $HFK^\diamond(K,h)$ must be supported in degrees ≥ -1. Since $\Sigma\mathcal{X}^\diamond(\mathbb{G},h) = \Sigma\Omega\,\mathrm{Cone}(\mathcal{X}(\mathbb{G},h) \to \mathbb{S}^0)$ is equivalent to $\mathrm{Cone}(\mathcal{X}(\mathbb{G},h) \to \mathbb{S}^0)$, we get a cofibration sequence

$$\mathcal{X}(\mathbb{G},h) \to \mathbb{S}^0 \to \Sigma\mathcal{X}^\diamond(\mathbb{G},h) \to \Sigma\mathcal{X}(\mathbb{G},h) \to \cdots$$

which shows that $\Sigma\mathcal{X}(\mathbb{G},h)$ is equivalent to the cone of a map $\mathbb{S}^0 \to \Sigma\mathcal{X}^\diamond(\mathbb{G},h)$. This map sends $\mathbb{S}^0$ to a wedge of spheres of non-negative dimensions, and therefore it is determined by its effect on homology. Since we know that $HFK(K,h) = H_*(\mathcal{X}(\mathbb{G},h))$ is torsion-free by Condition (1), we deduce that $\mathcal{X}(\mathbb{G},h)$ is a wedge of spheres, and in fact

$$\mathcal{X}(\mathbb{G},h) \simeq \mathcal{X}^\diamond(\mathbb{G},h) \vee \mathbb{S}^0$$

for $h \geq \tau(K)$. Thus, $\mathcal{X}(\mathbb{G},h)$ is determined by its homology.

To fully determine the filtered homotopy type of $\mathcal{X}(\mathbb{G})$, we also need the maps $\mathcal{X}(\mathbb{G},h) \to \mathcal{X}(\mathbb{G},h+1)$. By the monotonicity of $HFK(K,h)$, these are maps between wedges of spheres of non-decreasing dimensions, and therefore they are determined by what they do on homology. □

We proceed to give several examples of Floer constrained knots.

16.2. **The unknot.** The simplest example is when $\mathbb{G}$ represents the unknot U. The knot Floer homology $\widehat{HFK}(U)$ is isomorphic to $\mathbb{Z}$, supported in Maslov and Alexander degrees 0. The spectrum $\mathcal{X}(\mathbb{G},h)$ has homology $HFK(K,h) \cong \mathbb{Z}$ (in degree 0) for $h \geq 0$, and trivial for $h < 0$. We deduce that

$$\mathcal{X}(\mathbb{G},h) \cong \begin{cases} \mathbb{S}^0 & \text{if } h \geq 0, \\ * & \text{otherwise.} \end{cases}$$

Moreover, for $h \geq 0$, the maps $\mathcal{X}(\mathbb{G},h) \to \mathcal{X}(\mathbb{G},h+1)$ are isomorphisms.

16.3. **Thin and almost thin knots.** For a knot $K \subset S^3$, we will represent its knot Floer homology

$$\widehat{HFK}(K) = \bigoplus_{k,h\in\mathbb{Z}} \widehat{HFK}_k(K,h)$$

by a collection of Abelian groups in the plane, with the two coordinate axes being the Alexander grading $A = h$ and the homological (Maslov) grading $\mathrm{gr} = k$.

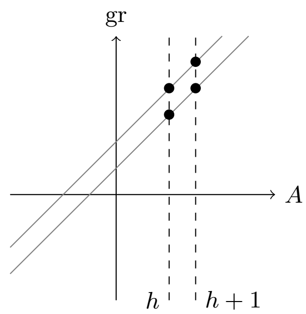


FIGURE 34. The knot Floer homology of an almost thin knot. The two diagonals represent the support of $\widehat{HFK}$. The homology $CFK(K,h)$ is that of the complex in the half-plane $\{A \leq h\}$.

A knot K is called *Floer homologically thin* (or *thin*) if its knot Floer homology is supported in a single diagonal line given by $A - \mathrm{gr} = \delta$, for some $\delta \in \mathbb{Z}$. Thin knots were first studied in [46], [45]. All alternating and quasi-alternating knots are thin [37], [30].

More generally, for any K, we define its *diagonal support* to be

$$\delta\text{-supp}(K) = \{\delta \mid \widehat{HFK}_k(K,h) \neq 0 \text{ for some } k, h \in \mathbb{Z} \text{ with } h - k = \delta\}$$

and its *Floer thickness* as

$$\mathrm{th}(K) = 1 + \max \delta\text{-supp}(K) - \min \delta\text{-supp}(K).$$

Thus, K is thin iff $\mathrm{th}(K) = 1$. We can use thickness to define almost thin knots, see also Figure 34.

Definition 16.4. A knot K is called *almost thin* if $\mathrm{th}(K) = 2$.

Proposition 16.5. *Let K be a knot that is either thin or almost thin. Suppose that the groups $\widehat{HFK}(K,h)$ and $HFK(K,h)$ are torsion-free for all h. Then K is Floer constrained.*

Proof. Condition (1) in Definition 16.2 is satisfied by hypothesis. Moreover, since $\mathrm{th}(K) \leq 2$, we have that in any Alexander grading $A = h$, the homology $\widehat{HFK}(K,h)$ is supported in at most two consecutive gradings $h - \delta - 1$ and $h - \delta$. This implies that Condition (3) is satisfied.

It remains to check Condition (2). We prove by induction on h that $HFK^\diamond(K,h)$ is supported in degrees $h - \delta - 1$ and $h - \delta$. For $h \ll 0$, we have that $HFK^\diamond(K,h) = HFK(K,h) = 0$. For the inductive step, suppose the claim is true for h and let us prove it for $h+1$. From Equation (16.1) we see that $HFK(K,h)$ is supported in degrees $h-\delta-1$, $h-\delta$ and possibly 0 (where the last comes from an element mapping nontrivially to $\widehat{HF}(S^3)$). Then, the long sequence

$$\cdots \to HFK_k(K,h) \to HFK_k(K,h+1) \to \widehat{HFK}_k(K,h+1) \to \cdots,$$

together with the almost thin condition, tells us that that $HFK(K,h+1)$ can only be supported in degrees $h-\delta-1, h-\delta, h+1-\delta$ and 0. Thus, $HFK^\diamond(K,h+1)$ is supported in some of the degrees $h-\delta-1, h-\delta, h+1-\delta$. To reach the conclusion, it remains to exclude degree $h-\delta-1$. If $HFK^\diamond{}_{h-\delta-1}(K,h+1) \neq 0$, then by looking at the same long exact sequence for higher h, and taking into account the almost thin condition on $\widehat{HFK}$, we deduce that the elements of $HFK^\diamond{}_{h-\delta-1}(K,h+$

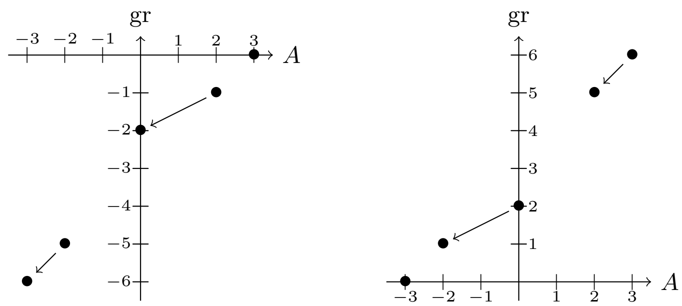


FIGURE 35. The knot Floer homology of $T(3,4)$ (left) and its mirror (right). Each dot is a generator of $\widehat{HFK}$. The arrows represent the extra differentials in $\widehat{CFK}$ going over the z basepoint (or X-markings on the grid).

$1) \subset HFK_{h-\delta-1}(K, h+1)$ survive under the maps to any higher $HFK_{h-\delta-1}(K, h')$. For $h' \gg 0$, we get that they survive to $\widehat{HF}(S^3)$, which contradicts the definition of $HFK^{\diamond}$. The claim is proven. $\square$

16.4. **L-space knots.** Another important class of knots are L-space knots, which were introduced in [40]. These are the knots such that large surgeries on them are L-spaces (manifolds such that $\widehat{HF}$ has rank one in each Spin^c structure). For example, all positive torus knots are L-space knots.

Proposition 16.6 (Ozsváth-Szabó [40])**.** *Given an L-space knot K there exist two increasing sequences of integers*

$$n_{-l} < \cdots < n_l, \quad \delta_{-l} < \cdots < \delta_l = 0$$

such that for all j between $-l$ and l we have

$$\widehat{HFK}_{\delta_j}(K, n_j) \cong \mathbb{Z},$$

and all other groups $\widehat{HFK}_k(K, h)$ are zero.

Example 16.7. See Figure 35 for the knot Floer homology of the torus knot $T(3,4)$ and its mirror.

Proposition 16.8. *L-space knots and their mirrors are Floer constrained.*

Proof. If K is an L-space knot, we use Proposition 16.6. Since $\delta_l = 0$, we must have $\tau(K) = n_l$; that is, in $\widehat{CFK}(K)$ all the elements in Alexander gradings less than n_l cancel in pairs, with the surviving element being in Alexander grading n_l. Furthermore, because of the monotonicity of the sequences (n_j) and (δ_j), the cancellation can only happen between elements in Alexander gradings n_i and n_{i+1}, for various i. It follows that $HFK(K, h)$ is either trivial or a copy of $\mathbb{Z}$ in degree δ_j. This easily implies Conditions (1) and (2) of Definition 16.2. Condition (3) is a direct consequence of Proposition 16.6.

For the mirrors, recall from [39, Proposition 3.7] that

$$\widehat{HFK}_k(K, h) \cong \widehat{HFK}^{-k}(m(K), -h),$$

where on the right hand side we have Floer cohomology. When K is an L-space knot, Proposition 16.6 implies (almost) the same conclusion for the mirror $m(K)$, with n_i and δ_i replaced by $-n_{-i}$ and $-\delta_{-i}$. (We use here the universal coefficients formula for cohomology.) The one difference is that we

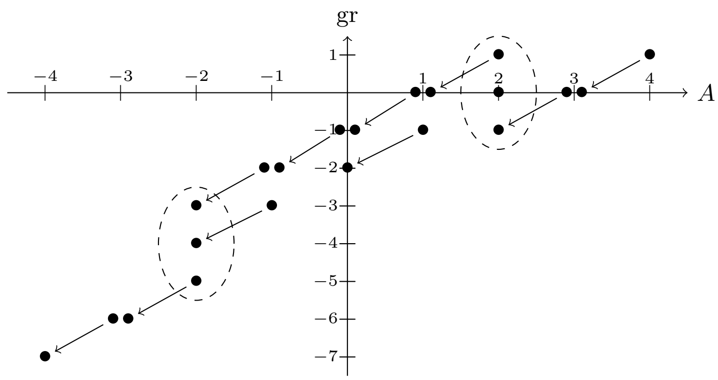


FIGURE 36. The knot Floer homology of $14n26580$. The two ovals are places where it looks like there are two options for the spectrum $g\mathcal{X}(\mathbb{G},h)$. However, only one option remains after taking into account the differentials.

now have $\delta_{-l} = 0$ instead of $\delta_l = 0$. Therefore, in this case the surviving element in $\widehat{HF}(S^3)$ comes at the beginning: $\tau(m(K)) = n_{-l}$. It follows that $HFK(K,h)$ is either trivial or consists of copies of $\mathbb{Z}$ in at most two gradings: δ_j and 0. (The latter appears for example for $h = -2$ on the right hand side of Figure 35.) Nevertheless, we still have that $HFK^{\diamond}(K,h)$ is either trivial or is $\mathbb{Z}$ in degree δ_j. As before, we deduce that the conditions in Definition 16.2 are satisfied. □

16.5. **Another example.** Most small knots are thin, almost thin, or L-space; or, even if they do not fit into any of these categories, they may still be Floer constrained. (An example of this kind is the knot $13n5016$.) One has to go quite far in the list of knots to find one that is not Floer constrained. An example is the 14-crossing knot $14n26580$. Its knot Floer homology, computed using the program [50], is depicted in Figure 36. Observe that the conditions in Definition 16.2 are satisfied, with one exception: the monotonicity of $\widehat{HFK}$ in Alexander gradings 1 through 3.

Interestingly, if we look only at $\widehat{HFK}$, there seems to be some flexibility in the spectrum in two places: Alexander gradings −2 and 2, where the support is in three homological gradings so Proposition 16.1 cannot be applied. When $A = -2$, from knowing only the homology $\widehat{HFK}(K,-2)$, there are two possible choices for the associated graded spectrum $g\mathcal{X}(\mathbb{G},-2)$: it could be $\mathbb{S}^{-5} \vee \mathbb{S}^{-4} \vee \mathbb{S}^{-3}$ or $\Sigma^{-7}\mathbb{CP}^2 \vee \mathbb{S}^{-4}$. (We abuse notation and write $\mathbb{CP}^2$ to denote its suspension spectrum $\mathbb{CP}^2 \wedge \mathbb{S}$.) However, the additional information in $HFK(K,h)$ eliminates the second choice, because the conditions in Definition 16.2 hold in that grading range, and we can use the arguments in the proof of Proposition 16.3. More concretely, we can argue that that the second Steenrod square Sq_2 (viewed as a stable homology operation) cannot map the generator in grading −3 to the generator in grading −5, since it would not commute with the extra differentials (shown by arrows in Figure 36).

In Alexander grading 2, we have two possibilities for the spectrum $g\mathcal{X}(\mathbb{G},2)$: it could be $\mathbb{S}^{-1} \vee \mathbb{S}^0 \vee \mathbb{S}^1$ or $\Sigma^{-3}\mathbb{CP}^2 \vee \mathbb{S}^0$. This time, because of the failure of monotonicity for $\widehat{HFK}$, both possibilities are compatible with the extra arrows. They each give rise to a unique possibility for the filtered spectrum $\mathcal{X}(\mathbb{G})$: the only ambiguity is in the map

$$\mathcal{X}(\mathbb{G},1) \simeq \mathbb{S}^0 \to \mathcal{X}(\mathbb{G},2) \simeq \mathbb{S}^{-1} \vee \mathbb{S}^0$$

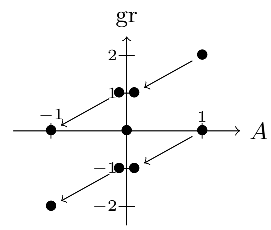


FIGURE 37. A potential knot Floer complex that does not uniquely determine its Floer spectrum.

or, more precisely, in its composition with projection to the $\mathbb{S}^{-1}$ summand. The resulting map $\mathbb{S}^0 \to \mathbb{S}^{-1}$ can be either the zero map (in which case $g\mathcal{X}(\mathbb{G}, 2)$ is a wedge of spheres) or the Hopf map (in which case $g\mathcal{X}(\mathbb{G}, 2)$ has the $\Sigma^{-3}\mathbb{CP}^2$ summand). Our expectation is that the first case occurs. While the Hopf map is not excluded by the structure of the filtration on $\mathcal{X}(\mathbb{G})$, it would be excluded if we could enhance the spectrum to also include domains that go over O_1. Indeed, apart from the arrows in Figure 36 (which come from going over z basepoints, i.e., the X-markings), we would also have arrows from going over w basepoints, i.e., the O-markings. The two kinds of arrows are related by the symmetry $(A, \mathrm{gr}) \to (-A, \mathrm{gr} - 2A)$ in the plane, under which the two ovals in Figure 36 are interchanged. We conjecture that this symmetry lifts to the level of spectra. Assuming that, the fact that $g\mathcal{X}(\mathbb{G}, -2)$ was a wedge of spheres would imply the same about $g\mathcal{X}(\mathbb{G}, 2)$.

16.6. **An algebraic example.** Given the (proved and conjectural) constraints on the filtered spectrum $\mathcal{X}(\mathbb{G})$ from the chain level, the reader may wonder if these always force the knot Floer spectra to be wedges of spheres. We give here a potential counterexample: a model for a chain complex $\widehat{CFK}(K)$ that satisfies all the structural properties of a knot Floer complex, and may produce an interesting Floer spectrum. Of course, the catch is that we have not found any knot K with this knot Floer complex.

Consider the complex shown in Figure 37. Suppose it appears as $\widehat{CFK}(K)$ for a knot K. Then, the homology $\widehat{HFK}(K, -1) = HFK(K, -1)$ is supported in gradings -2 and 0. There are two possibilities for the spectrum $\mathcal{X}(\mathbb{G}, -1)$: it could be either $\mathbb{S}^{-2} \vee \mathbb{S}^0$ or $\Sigma^{-4}\mathbb{CP}^2$, according to whether the attaching map between the two cells is zero or the Hopf map. Because of compatibility with the differentials, each possibility uniquely determines the rest of the filtered spectrum $\mathcal{X}(\mathbb{G})$. In the first case, all the spectra $\mathcal{X}(\mathbb{G}, h)$ are wedges of spheres, whereas in the more interesting second case, we have

$$\mathcal{X}(\mathbb{G}, -1) \simeq \Sigma^{-4}\mathbb{CP}^2, \quad \mathcal{X}(\mathbb{G}, 0) \simeq \mathbb{S}^0 \vee \Sigma^{-3}\mathbb{CP}^2, \quad \mathcal{X}(\mathbb{G}, 1) \simeq \mathbb{S}^0.$$

Further, the map $\mathcal{X}(\mathbb{G}, -1) \to \mathcal{X}(\mathbb{G}, 0)$ is trivial, whereas the map $\mathcal{X}(\mathbb{G}, 0) \to \mathcal{X}(\mathbb{G}, 1)$ maps the summand $\mathbb{S}^0$ isomorphically to $\mathbb{S}^0$, and vanishes on the other summand.

Remark 16.9. The discussion in this section makes clear the difficulties in finding a knot whose Floer spectrum contains more information than the knot Floer complex. Once the functoriality properties of knot Floer spectra (under knot cobordisms) are established, an easier thing to look for would be a non-trivial map between spectra that is trivial on homology. Many such maps exist even between spheres (e.g., the Hopf map).

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