Ratings, certifications and grades: dynamic signaling and market breakdown

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Abstract

We consider the effect a public revelation of information (e.g. rating, grade) has on signaling and trading in a dynamic model. Competing buyers offer prices to a privately informed seller who can reject these offers and delay trade. This delay is costly and the seller has no commitment to the duration of the delay. We show how the external public information allows for signaling in equilibrium. More interestingly, we characterize the dynamics of trade and prices. If the signal is not fully revealing, then there is no trade just before the revelation of external information. A lemons market develops endogenously over time and prevents any trade close to the release of the public announcement. On the other hand, if the external signal is fully revealing, then trade occurs even close to the final period; however, in this case there is a discontinuity in prices.

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1 Introduction

Many real-world markets, such as labor markets and financial markets, feature a privately informed seller who faces a pool of uninformed buyers. A growing literature, starting with Spence's (1973, 1974) seminal work on education, explores how the seller can signal his type in this wide range of markets. In Spence's original work, education is modeled as a static choice, so that a more able seller/worker could commit to more 'units' of education. In practice, many markets are dynamic and often the signaling variable is the time to agreement. Moreover, commitment is rare so agents make a new decision in every period; in particular, the seller can quit signaling at any time. This lack of commitment is likely to disturb signaling, as pointed out by Weiss (1983), Admati and Perry (1987), and Swinkels (1999).

In this paper, we follow the above literature in examining a fully dynamic game (à la Swinkels (1999)), but incorporate another important feature of many markets, the release of external information. In particular, we introduce a public signal that is not fully controlled by the seller or the buyers. Some examples are: a grade that partly reflects the ability of a student, an earnings announcement, an FDA approval, or a court decision that may affect a firm's value. More generally, this external information can be any realization of some uncertainty. As we show, the release of this information enables signaling despite the lack of commitment. However, our focus is on how it affects the resulting trading patterns.

We consider a generic trading model in which an informed seller faces competitive uninformed buyers. The seller owns an indivisible asset and he is privately informed about its value. With the asset as labor, the model includes educational signaling as a special case. The model is dynamic and trade can take place at any point in time (formally we divide time into many small periods). In each period, buyers make simultaneous sealed offers to the seller. The seller decides whether to accept or reject. If he accepts any offer, the game ends; if he rejects all offers, the game continues. Since the cost of delay can depend on the seller's privately known value for the good, rejection can signal information about his type. The new feature of the model is that after a fixed time (at time $t = 1$) an external signal (a grade) is publicly revealed.[1]

A crucial assumption is that if trade takes place before $t = 1$, the terms of trade cannot be made contingent on future realization of the final signal. Such restriction arises naturally in markets where the signal is not generated if the trade takes place before $t = 1$, or is observable but not verifiable in court. The feasibility of such contingent payments may also be affected by the seller's limited liability, preventing him from paying any money back after a low signal realization, or by possible manipulation by the buyer. In other markets, some limited contingent contracts are possible and are used in practice. Examples of such contracts include product warranties or collars in financial markets.

Our equilibria do not rely on the offers becoming public. Hence, off-equilibrium beliefs do not provide commitment to the seller (see Noldeke and Van Damme (1990)). We follow Swinkels (1999) in assuming the opposite extreme. In each period there are two new buyers who replace the two (old) incumbent buyers whose offers were rejected in the previous period. This replacement assumption provides tractability to a model of private offers.

We examine the conditions under which there is signaling in equilibrium. More importantly, we analyze how the external information affects the resulting trading patterns and prices. Our main results can be summarized as follows:

- With the release of public information, the seller can employ costly signaling, as long as waiting costs are not too high. This contrasts a dynamic signaling model without external information, in which the unique sequential equilibrium involves full pooling and trade in the first period whenever delay is unproductive (Swinkels (1999)).
- When the external signal is noisy, there is a discontinuity in trade, i.e. no trade takes place close to the announcement. The intuition

[1] We focus on the case in which the timing of the external signal is fixed and commonly known, which is a reasonable assumption in some markets. We also discuss the effects of relaxing this assumption in the last part of the paper.

is that a lemons market develops endogenously. For example, with a noisy grade and two types of sellers, the low type is overpaid in the last period and the high type is underpaid, but still paid more than the average. As the announcement approaches, waiting costs become small and the reservation price of each type approaches his payoff in the last period. Buyers would hence have to pay more than the value of the average type who is accepting an offer if trade occurred just before graduation (importantly, any price that the high type accepts, the low type does as well).

- If the external signal is fully revealing, then there is no market breakdown. In contrast to the noisy signal case, there is trade just before the release of the external information. However, there is a discontinuity in prices around that time. In particular upon the release of the external signal, average prices are strictly higher than prior prices. The intuition can be illustrated by the following example. Suppose that the distribution of *graduating* types (i.e. types that do not trade before the grade is revealed) is uniform between $[10, 20]$. The average price upon graduation is therefore 15. Just before graduation the highest offer that can be made is 15 since the agent's type is unknown. However, since waiting costs are minimal, types above 15 would reject such an offer. For the buyers not to lose money, trade just before graduation occurs only with types lower than 15, so the highest price they can offer is 12.5 or less. This implies a strict jump in prices.

It is interesting to point out that we do not need to assume that costs of delay are strictly decreasing in value. As we later discuss, this is very useful in financial applications where the seller tries to sell a company he owns. In this case, costs follow from discounting and do not vary by type. In our setup, signaling is still possible as the high types expect to receive better offers upon the realization of the external signal.

Literature Review

This paper is related to the literature on dynamic signaling that started with Spence's (1973, 1974) papers on educational signaling. In these papers, he suggests that workers may over-invest in education in order to signal their productivity. Cho and Kreps (1987) formalized this and showed that if the worker can commit to the length of education, then, in the unique equilibrium satisfying standard refinements, the agent obtains least-costly full separation (known as the 'Riley outcome', Riley (1979)).

Weiss (1983) and Admati and Perry (1987) pointed out that long-term commitment is difficult in many markets and may disturb signaling. Swinkels (1999) formalized this argument in a paper that is the most related to ours. He showed that if offers are private, then the unique sequential equilibrium is full pooling. Before school starts, the firms offer wages equal to the average productivity and all types accept. Therefore, the lack of commitment makes it very difficult to use non-productive education for signaling. We differ from Swinkels (1999) in that we introduce the exogenous release of information; this exogenous information is the focus of this paper. Another difference is that rather than the often-used framework of education, we describe a generic dynamic trading model. As we discussed above, this generality is needed for some of the applications we consider in which there are no explicit differences in delay costs across types.

Noldeke and Van Damme (1990) show how signaling can be restored as off-equilibrium beliefs may provide commitment for the worker to continue signaling. In their model, offers are public and there are many sequential equilibria. They employ a standard refinement (Never a Weak Best Response from Kohlberg and Mertens (1986)) to select a unique one. In that equilibrium, firms believe that a worker is the highest type whenever she deviates from the equilibrium by rejecting an offer. As a result, the firms have no incentive to approach the worker publicly and the worker signals her type over time. As the length of the periods converges to zero, the equilibrium outcome essentially converges to the 'Riley outcome': in the limit, the workers separate at the beginning of the game. Swinkels (1999) shows that for this result, it is critical that all offers are made public, given that firms have incentives to approach the worker privately.

Weiss (1983) was the first to examine the role of grades in an education model. Similar to our paper, he observes that signaling may occur even if costs do not vary by type. His model differs from ours in two important dimensions. First, in his model, there is full commitment, as agents choose the duration of their education and firms are not able to approach them until graduation.[2] As a result, there are no trade patterns before graduation. In contrast, these trade patterns are the main focus of our paper. Second, in his model, grades provide information not available to any agent ex-ante. Therefore, grades are productive as they yield new information and serve an important allocative role. In fact, he motivates his assumption that the firms do not offer wages before graduation by assuming that firms care not only about a worker's own assessment of his ability, but also about the grades per se. In our model, grades may or may not provide new information; more importantly, if there was no information asymmetry between the seller and the buyers, then trade would occur immediately and efficiently.

There are also a few papers that examine static models in which the external release of information plays a role. Teoh and Hwang (1991) study a model in which a firm can control the release of some information. The firm can choose not to release information, but rather wait for it to become public. They show that the firm may actually decide to disclose negative information rather than positive. Feltovich, Harbaugh and To (2002) examine a model in which there is an external noisy signal. They show that with more than two types, high types may choose not to signal (or countersignal) to ensure they are separated from medium types.

Finally, Kojarczyk, Lucas, and McDonald (1991, 1992) consider a model in which a seller times the issuance of equity in a market with external information release. In this model, buyers are not aware that the seller exists until he decides to issue the equity. Therefore, they cannot approach him in the interim period. In other words, this is a model of pure signaling. In contrast, we allow the buyers and the seller to be active in the market, allowing for both signaling and screening.

The paper is organized as follows: in the next section we present the details of the model. In Section 3 we present general analysis of all sequen-

[2] Weiss (1983) also assumes that the timing of the grade is chosen by the seller, while we assume it is fixed.

tial equilibria in the general model. We then specialize the model to two cases: noisy grades (Section 4) and fully revealing grades (Section 5) and prove the main theorems about the dynamic patterns of trade. Section 6 contains examples of equilibria in a binary model for both noisy and fully revealing grades. Section 7 considers a case of productive signaling. Section 8 concludes, and proofs appear in the Appendix.

2 The Model

We analyze dynamic trade with signaling and no commitment. In our model, an informed seller is facing a pool of uninformed buyers. Buyers decide what offers to make and when; the seller decides which offer to accept and when. As we have both sides of the market active, our model involves both signaling by the seller as well as screening by the buyers.[3]

Our model is based on that of Swinkels (1999). The two key features are the impossibility of commitment and private offers. Thus, we do not rely on offers being public and on off-equilibrium beliefs to provide commitment (see Noldeke and Van Damme (1990)). Similar to Swinkels (1999), for tractability we assume the opposite extreme. We assume no recall of past offers (by having new buyers every period) so that the strategies of buyers do not depend on the levels of offers rejected in the past. We differ from Swinkels (1999) in that we introduce the exogenous release of information; this is the focus of our paper.

Another difference is that we describe a generic dynamic trading model rather than the usual framework of education. The difference is more than just reinterpretation of an existing model. To see why, consider first the often-studied case of education signaling in which an agent sells his labor. The cost of education is usually assumed to depend on the agent's productivity. In a dynamic framework the cost for a type v of staying in school from zero to t takes the form of $t\widetilde{c}(v)$ for some decreasing function $\widetilde{c}(v)$. Thus, education is less costly for the high-productivity workers than it is for

[3]Each buyer is active in only one period, so the screening is by the buyers as a group rather than by individual buyers.

low-productivity workers.

Now consider an example of an entrepreneur who sells his company. The motive for trade follows from the fact that the entrepreneur has a preference for getting money earlier rather than later. This is described by his discount factor $\beta \in (0,1)$, while the market is normalized to have a zero discount rate. This implies that the seller's cost of waiting depends on the amount he receives and the delay, but not his type. He values \$$w$ in period t only as $\beta^t w$ and hence his waiting costs are given by $\left(1-\beta^t\right) w$. Nevertheless, as we shall see, future realization of uncertainty still enables signaling in equilibrium.

We incorporate both cases by employing an abstract cost function $c(t,v,w)$ that represents the loss for a type v if he receives \$$w$ in period t rather than today. This cost function plays a key role in our model, which we describe now formally.

Types and timing

A seller owns a single indivisible asset that has value to the potential buyers given by $v \in V \subseteq [\underline{v}, \bar{v}]$, where $\underline{v} > 0$.[4] The seller knows this value, while potential buyers share only a commonly known prior, μ_0. The market belief (i.e. a probability distribution over V) regarding the value at time t is denoted by μ_t.

For simplicity we focus on the case in which V is finite and values are given by $\underline{v} = v_1 < v_2 ... < v_N = \bar{v}$. Most of our conclusions do not depend on this assumption, and in some cases we consider explicitly the continuous case where $V = [\underline{v}, \bar{v}]$.

We consider a discrete time model in which we divide the time of the game $[0,1]$ into T intervals, each of length $\Delta = \frac{1}{T}$. This defines a grid $t = \{0, \Delta, ...1\}$ in which offers are made. In general, taking Δ to be small corresponds to less commitment by the seller as he may change his mind in a short period of time. Since we are interested in the seller being unable to commit, we focus on Δ being small.

Following Swinkels (1999), for tractability we assume that at the begin-

[4] One could expect a normalization of types to $[0,1]$; however, a type zero has some unique implications. This is related to what is known in the literature as the gap case.

ning of each period there are two new buyers that make simultaneous offers to the seller; in total there are $2(T+1)$ buyers in the game. The seller observes offers privately and he decides whether to accept or reject. If he rejects, he waits for another period Δ and two new buyers enter next period. These buyers replace the two "old" buyers that exit the game. If the seller accepts an offer, the game is over. Hence, we describe a competitive market in which buyers are unable to recall the history of price offers.[5]

Payoffs

We assume that all the agents are risk neutral with respect to payment. The payoff to a buyer that pays w and in return gets v is

$$v - w$$

The payoff to the seller of an asset with value v that sells at time t for w is

$$w - c(t, v, w)$$

where $c(t, v, w)$ denotes the waiting cost. We assume the following:

Assumption 1:

- $c(t, v, w)$ is non-negative and $c(0, v, w) = 0$.
- $c(t, v, w)$ is continuously differentiable and there exists $\alpha > 0$ so that $\frac{\partial c(t,v,w)}{\partial t} > \alpha$ for any v and $w \geq \underline{v}$.
- $c(t, v, w)$ is non-increasing in value v, non-decreasing in w, and $(w - c(t, v, w))$ is strictly increasing in w.
- $\frac{\partial^2 c(t,v,w)}{\partial t \partial v} \leq 0$ so that the incremental delay costs are weakly decreasing in value.
- $c(t, v, w)$ is linear in w, so that the seller is risk neutral.

[5] The game is very difficult to analyze if the offers are private but the same buyers continue to make offers, since over time the buyers will have different private beliefs. The concluding section elaborates on this.

This structure allows us to embed a model of discounting as well as an education model. As we discussed before, assuming $c(t, v, w) = \left(1 - \beta^t\right) w$ lets us describe financial markets, while assuming $c(t, v, w) = t\widetilde{c}(v)$ fits a traditional education-signaling model.

Grades and final offers

If the game is not over by $t = 1$, then a public signal regarding the value of the asset is revealed and becomes common knowledge.[6] We borrow the education terminology and name this public signal a *grade.* We denote it by g. We say in this case that the seller *graduated.* We assume that

Assumption 2: *g and v are strictly affiliated.*

The above assumption, which is standard in the literature, implies that expected type, conditional on the grade, is increasing in it. We specialize our analysis to one of the following two cases:

- Case 1- The grade is noisy. In this case we assume that the signal is represented by a continuous random variable g with a distribution conditional on the type being v given by a density function $f(g|v)$. To guarantee that all grades are noisy, we assume that there exists some $\gamma > 0$ so that $f(g|v) > \gamma$ for any v and g.
- Case 2- The grade is fully revealing, that is $g = v$.

After the announcement of the grade, the final offers are made. Without loss of generality, we assume that the game ends after the grade is revealed. As long as no other external information is expected after $t = 1$, Swinkels (1999) has shown that for a sufficiently small Δ in the unique (continuation) sequential equilibrium, trade takes place immediately at a price equal to the expected value.

Bertrand competition among buyers implies that upon graduation offered prices are equal to the conditional type based on the grade g and

[6] See Section 8 for a discussion of a model with more than one signal.

the population of types that graduate, μ_1. That is, $w^{final}(g) = E_{\mu_1}[v|g]$.[7] Hence, upon graduation buyers make zero expected profit while the agent earns $w^{final}(g) - c\left(1, v, w^{final}(g)\right)$. Let $h_\mu(v)$ denote the final price upon graduation that a type v expects, given that the market belief in the final period is $\mu_1 = \mu$:

$$h_\mu(v) = E_\mu\left[w^{final}(g)\,|v\right]$$

In the case where g is fully revealing (case 2), this simplifies to $h_\mu(v) = v$.[8]

It is a crucial assumption that trade before $t = 1$ cannot have prices contingent on the future realization of the signal. Such restriction arises naturally in some markets where the signal does not arrive if trade takes place before graduation, or is observable but not verifiable in court, or is subject to negative manipulation by the buyer. For example, a student may need to drop out of school to take the job (and avoid the cost of signaling); the buyer can settle the court case privately, affecting the observed outcome; or he can delay submission of the documents to receive the FDA approval. In other markets such contingent contracts are possible and are used in practice, for example, as warranties or as collars in financial markets.[9] The feasibility of such contingent contracts can be affected by the information problems listed above, as well as by the seller's limited liability that would prevent him from paying any money back after a low-signal realization. Admittedly, in many markets the situation is more complicated than our model depicts, but we believe that the assumption of non-contractible grades allows us to isolate the relevant effects of non-commitment and dynamic signaling from static signaling through warranties.[10]

[7] Our assumption that competition eliminates the buyers' surplus is not crucial and is made to simplify the exposition. For example, our analysis holds even if the buyers and the seller split the surplus evenly.

[8] When the grade is fully revealing, then even if $\mu_{1-\Delta}$ assigns probability 0 to the revealed type, the posterior beliefs agree with the grade. This follows from the definition of sequential equilibrium (the consistency requirement for off-equilibrium beliefs).

[9] A different market solution is for the seller to obtain a loan backed by the value of the asset and to sell it only after the realization of the signal. Our model is therefore relevant to cases where the seller faces above-market interest rates (for example, due to limited liability) which may be the main reason he wants to sell the asset.

[10] In case the signal was contractible ex-ante, we conjecture that the results of Swinkels (1999) would generalize, thus, all types would trade at time 0 and sign contracts con-

3 General Analysis

The game we have described has potentially more than a single equilibrium. Our goal is not to describe a specific equilibrium or to provide conditions that guarantee that the equilibrium is unique.[11] Our approach is to provide characterization of the entire equilibria set and to focus on some salient and robust features.

In this section, we characterize the equilibrium strategy of the seller and explain some simple dynamics that it implies. We name this section "General Analysis" since more interesting insights are obtained in the next two sections when we introduce additional assumptions regarding the information content of the grade. This allows us to establish important properties of trade and price dynamics

We first note that the seller follows a reservation-price strategy. At time t, type v accepts an offer w based on whether it exceeds a threshold $R(t, v)$. This is an immediate implication of our no-recall assumption. The level of a rejected offer has no further future consequences. Thus, $R(t, v)$ equals the time t seller's expected continuation payoff (i.e., expected price less the incremental cost of delay) if he declines current offers. We argue that

Lemma 1 *The reservation price,* $R(t, v)$, *is non-decreasing in type, that is* $v' > v$ *implies* $R(t, v') \geq R(t, v)$. *Moreover, if* $\frac{\partial^2 c(t,v,w)}{\partial t \partial v} < 0$ *or if* v *graduates (i.e., reaches the final period) with positive probability then* $R(t, v') > R(t, v)$ *for any* $v' > v$.

The above Lemma implies that the quitting decision is (weakly) monotone in type. If a type v that graduates with positive probability accepts an offer at $t < 1$, then lower types accept this offer with certainty (or they never graduate). As a result we get the following:

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tingent on the grade realization. Therefore, signaling would only be static through the contracts/warranties signed at time 0 and not dynamic through delay of trade.

[11] A two-type case is an exception and we construct the equilibrium in this case, see Section 6.1.

Corollary 1 *If in equilibrium there is a positive probability that the seller graduates, then μ_1 first-order stochastically dominates μ_t for any $t < 1$.*[12]

Without grades, an argument similar to that of Swinkels (1999) shows that for a small Δ, the unique equilibrium is a pooling equilibrium in which all types accept the initial offer. In our model, this would be the unique outcome for very high costs (say, $c(1, \bar{v}, w) > \bar{v} - \underline{v}$), but it is not an equilibrium outcome for lower costs:

Proposition 1 *If $c(1, \underline{v}, \bar{v}) < \frac{1}{2}(h_{\mu_0}(\bar{v}) - E_{\mu_0} v)$, then if (i) the grade is fully revealing $(g = v)$, or if (ii) off-equilibrium beliefs satisfy "Divinity", then the high type $\bar{v}$ graduates with a positive probability, bounded away from zero (for all Δ).*

The reasoning behind the result is quite simple. In a pooling equilibrium, all types accept an average offer. With final grade and low costs, high types prefer to graduate, prove their type, and on average earn more. Formally, in the case of a noisy grade, this claim depends on ruling out non-realistic off-equilibrium beliefs. Specifically, consider the off-equilibrium belief regarding a seller who graduates and assigns probability 1 to the lowest type. Such a belief supports a pooling equilibrium as the grade is ignored and the seller gets the lowest payoff upon graduation. However, such a belief does not survive standard refinement such as "Divinity."[13]

From now on we will maintain the sufficient assumption that

Assumption 3: $c(1, \underline{v}, \bar{v}) < \frac{1}{2}(h_{\mu_0}(\bar{v}) - E_{\mu_0} v)$.[14]

One may wonder if this condition excludes the possibility of a separating equilibrium even under commitment and no grades. However, as we argued

[12] One could expect that μ_t improves stochastically over time, not only compared with $t = 1$. We were not able to establish this without additional assumptions to guarantee that the reservation prices are strictly monotone (see Proposition 2). The problem is that an offer that matches the reservation price of a pool of types can potentially be accepted by high types and rejected by low types.

[13] By "Divinity" we mean divine equilibrium introduced by Banks and Sobel (1987). The stronger D1 (Cho and Kreps (1987)) is obviously sufficient as well.

[14] We use this sufficient condition as it simplifies the proofs. As we show in the next proposition, for a small Δ it is possible to relax this assumption.

before, without loss of generality we can extend the game beyond $t = 1$. This has no effect on our results as there would not be any trade beyond $t = 1$ (following Swinkels (1999)). Extending the model ensures the existence of separating equilibria under commitment (for example, least-costly separating equilibria known as the "Riley Outcome") as long as delay costs are decreasing in type.

So far we were able to provide only a partial characterization of equilibria in the general model: the seller follows a monotone reservation-price strategy, some types signal, graduation is reached with positive probability and graduating types are better than non-graduating. If reservation prices are strictly increasing in type, then we can provide a more complete characterization:

Proposition 2 *Suppose that $R(t,v)$ is strictly increasing in v. Then the following properties hold:*
(i) μ_t improves over time (in the sense of first-order stochastic dominance),
(ii) (zero profit) In equilibrium, every offer made by a buyer earns him zero expected profit. Moreover, if an offer is accepted with positive probability at time t, then it takes the form of $E_{\mu_t}[v|v \leq v^*]$ and if it is higher than the current lowest type, all types of sellers play a pure acceptance strategy (even if the offer is equal to the reservation price of some types),
(iii) (no gaps in trade) If there is a positive probability of trade at some $t < 1$, then there is a positive probability of trade at any $t' < t$,
(iv) If $c(1,\bar{v},\bar{v}) < (h_{\mu_0}(\bar{v}) - E_{\mu_0}v)$, then for small enough Δ the high type, $\bar{v}$, graduates with positive probability.

The proof follows the methods introduced in Swinkels (1999), and we provide it in the Appendix. If the incremental costs do not vary by value, (i.e. $\frac{\partial^2 c(t,v,w)}{\partial t \partial v} = 0$) then the reservation price is the same for all types that never graduate. This, in general, allows for a more complicated evolution of the distribution of types, and we were not able to establish such general characterization in this case.[15]

[15] See the Appendix for an example of an equilibrium with a continuum of types in which this equality of reserve prices plays a role.

When is $R(t,v)$ strictly increasing in v? First, one can use Lemma 1 to conclude that a sufficient condition is $\frac{\partial^2 c(t,v,w)}{\partial t \partial v} < 0$ (or, by a similar argument, that there are only two types). Second, our model can be slightly modified to ensure that $R(t,v)$ would be increasing even if $\frac{\partial^2 c(t,v,w)}{\partial t \partial v} = 0$. For example, in each period we could introduce a small probability that buyers do not show up; alternatively, in each period there could be a small probability that the external signal arrives early. In either case, upon rejection, each type has a positive probability of reaching the revelation of an external signal, and by Lemma 1, reservation prices are strictly increasing.

The last general question we examine about equilibria is whether a fully separating equilibrium exists. Note that in a separating equilibrium, the lowest type gets no benefit from waiting, and hence, he quits at $t = 0$. By induction it then follows that in a separating equilibrium types quit one by one; type v_{i-1} accepts an offer at $t = i\Delta$. Since we focus on the case in which the agent is unable to commit, we consider a small Δ. If Δ is sufficiently small, namely $\Delta < \frac{1}{N}$, then no agent would graduate in a separating equilibrium. This contradicts our previous proposition, and hence, a fully separating equilibrium does not exist.

We now argue that having a small number of possible types (relative to $1/\Delta$) is not crucial for this result. To illustrate this, consider the case of a continuum of types: $V = [\underline{v}, \bar{v}]$. Let $U(v)$ denote the utility of type v in a separating equilibrium. In the Appendix we show using an envelope condition that:

Lemma 2 *$U(v)$ is differentiable at $\underline{v}$ and $\frac{d}{dv}U(\underline{v}) = 0$.*

This allows us to establish the following:

Proposition 3 *Assuming that either (i) the set of types is finite and Δ is sufficiently small, or that (ii) there is a continuum of types, then there does not exist a fully separating equilibrium.*

The proof of part (ii) is as follows. In a fully separating equilibrium, the lowest type not only trades in the beginning but he also gets a price that equals his value (as buyers compete away all profits). Therefore, $U(\underline{v}) = \underline{v}$. Consider now a price offer of $\underline{v} + \varepsilon$ in period zero. This attracts all types v for which $R(0, v) = U(v) \leq \underline{v} + \varepsilon$. As U is continuous and $\frac{d}{dv}U(\underline{v}) = 0$, increasing the price from $\underline{v}$ by a small enough ε attracts a strictly better pool:

$$E[v|U(v) \leq \underline{v} + \varepsilon] > \underline{v} + \varepsilon$$

Hence, this offer yields a profit to the buyer, which leads to a contradiction.

4 Noisy Grades and Market Breakdown

To further characterize trade patterns, we now specialize the model and assume that grades are only partly revealing; in the next section we consider the case of fully revealing grades. Noise reflects the fact that in many cases the grade is not fully informative. For example, a pass/fail grade or a credit rating provides only partial information. Similarly, an earnings announcement is a noisy predictor of future performance of a company. In this case, the seller has additional private information even when the grade is revealed. Our main finding is that this implies a market breakdown. Close to $t = 1$, a severe lemons market develops endogenously, preventing any trade.[16]

We consider the case in which the cost of signaling is not too high so that by Proposition 1 some types graduate with positive probability. In fact, using Assumption 3, one can strengthen Proposition 1 and conclude that

Lemma 3 *In equilibrium the probability that the second-highest type graduates is uniformly (for all Δ) bounded away from zero.*

As a result, average prices received by low types (i.e., below some threshold) upon graduation are higher than their average type:

Lemma 4 *Upon graduation, low types tend to be overpaid, that is, $E_{\mu_1}[h_{\mu_1}(v)|v \leq v^*] > E_{\mu_1}[v|v \leq v^*] + \eta$ for some $\eta > 0$ and any $v^* < \bar{v}$.*

[16] Clearly, the market breaks down only close to the announcement, and the trade takes place either before or after the signal arrives.

At a time close to graduation that creates a severe lemons market. If the reservation price $R(t, v)$ is increasing in v, buyers can attract type v only if they attract all types below it. However, types lower than v^* expect to be overpaid upon graduation and hence reject offers that do not cause buyers to lose money. Attracting all types is not an option either, since to attract $\bar{v}$, the buyers have to offer more than average value, which again would make them lose money. Formally we argue that

Theorem 1 *In any equilibrium there exists* $t^* < 1$ *so that there is no trade in* $[t^*, 1]$.

To illustrate the intuition behind this theorem, consider a game with two types. At $t = 1 - \Delta$ both types are present with positive probability (otherwise no type would graduate). Any price the high type would accept with positive probability will definitely be accepted by the low type (at $t = 1 - \Delta$ the reservation prices $R(t, v)$ are strictly increasing since upon rejection the seller is certain to graduate). Upon graduation, the low type expects a price above his value and the high type expects a price above average. Therefore, if the price is accepted by the low type only, it must be above his value. If a price is accepted by the high type, it has to be above the average value, but it will be accepted by both types. In either case the trade would make the buyers lose money. We can extend the reasoning to time $t^* \leq 1$, showing that if there is no trade after t^* then there will not be any trade at t^* either, unless t^* is sufficiently less than 1. Lemma 4 allows us to generalize this argument to N types. This result stands in contrast to the case in which grades are fully informative. As we shall see later, in this case there is trade even immediately before graduation.

In Section 6.2 we construct an example of equilibrium with noisy grades. It is important to note that the strategy space in this game is not finite since prices are not restricted to a finite set. Hence, standard arguments do not guarantee that equilibrium exists. Accordingly, the explicit example establishes existence. It also illustrates possible equilibrium dynamics (for example, Figure 2 in Section 6.2 illustrates the "silent period" before graduation).

5 Fully revealing grades and graduation premium

We now turn to the case of fully revealing grades. We assume that the value becomes public at time $t = 1$. Since agents are risk-neutral, this setup also has a different interpretation. The seller may not know the exact value of his asset and the grade is more informative than his own private information.

We argue that in this case the market is active even close to graduation. This stands in contrast to the case of noisy grades. However, while there is continuity in trade, the price pattern is discontinuous. Specifically, there is a graduation premium so that the average accepted price at $t = 1 - \Delta$ is strictly lower than the average price upon graduation ($t = 1$). This difference is bounded away from zero when we take the limit as $\Delta \to 0$.

As in the previous section, we focus on the case in which the costs are small enough so that by Proposition 1 some types graduate. We strengthen this assumption slightly by assuming throughout this section that

Assumption 4: *The cost function satisfies* $c\left(1, \underline{v}, \bar{v}\right) < \frac{1}{2}\left(v_{N-1} - E_{\mu_0} v\right).$

Note that since costs are positive, Assumption 4 can be satisfied only if there are at least three types, that is, $N \geq 3$. We first argue that

Lemma 5 *The two highest types v_N and v_{N-1} graduate with probability that is bounded away from zero.*

The proof is almost identical to the proof of Proposition 1. If type v_{N-1} does not graduate, all worse types can mimic him. Given Assumption 4, this would make buyers overpay on average and lead to a contradiction. We continue by arguing that even at $1-\Delta$ there is a strictly positive probability of trade. Formally,

Lemma 6 *The probability of trade at $t = 1 - \Delta$ is bounded away from zero.*

The proof uses the following reasoning: there exists a worst type at $t = 1 - \Delta$. This type has no benefit from graduating and therefore trades with certainty. Note that the bound is not uniform for all Δ, since when Δ

gets small, the probability of trade in almost any period converges to zero. However, the probability that there is trade in time interval $t \in [1-\varepsilon, 1)$ for any $\varepsilon > 0$ is bounded away from zero as $\Delta \to 0$. Next, the average price upon graduation is not lower than any prior price:

Lemma 7 *In any equilibrium, the average transaction price at* $t = 1$ *is not lower than the highest price offered at any other period.*

The reasoning behind this lemma is that the pool of types that graduate is better than any earlier pool.

Based on the above short lemmas, we argue that there is a jump in the prices upon graduation, i.e., a graduation premium. The average price at $t = 1$ is strictly higher than at $t = 1 - \Delta$; there is a gap that is bounded away from zero uniformly for all small Δ (i.e., even as $\Delta \to 0$ the gap does not disappear). The intuition is that the non-trivial presence of v_{N-1} at $1 - \Delta$ prevents buyers from making an offer that can attract the highest type, v_N. This is because the highest type's reservation price, $R(1-\Delta, v_N)$, approaches his value, v_N, when Δ is close to zero, and any w that attracts him attracts all lower types. As a result, the selection of types that trade at $t = 1 - \Delta$ is strictly worse than at $t = 1$.

Theorem 2 *The average price upon graduation is strictly higher than the average accepted price at* $t = 1 - \Delta$*. This graduation premium is bounded away from zero when we take* $\Delta \to 0$*.*

Proof. Consider $t = 1 - \Delta$. By Lemma 7, the highest possible offer is $E_{\mu_t} v$. However, since type v_{N-1} has a strictly positive probability of not quitting before t, we conclude that $E_{\mu_t} v$ is bounded away from v_N. Hence, for a small Δ, the highest type, v_N, will reject this offer. This implies that such an offer would result in losses to the buyers. The upper bound on offers that would not make the buyers lose money is

$$w_t \leq E_{\mu_t}[v | v < v_N] < E_{\mu_1} v$$

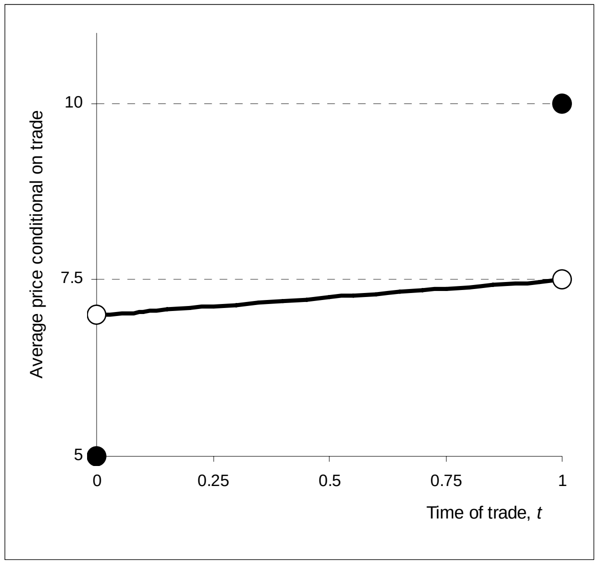

Figure 1: Average price conditional on trade

The strict inequality comes both from conditioning and from μ_1 stochastically dominating μ_t. The right-hand side is the average price upon graduation. The left hand side is an upper bound on offered prices at $t = 1 - \Delta$. Given there is trade with positive probability (by Lemma 6), there is strict inequality in accepted prices at $t = 1 - \Delta$ and $t = 1$, even in the limit as $\Delta \to 0$. ■

The result is illustrated in Figure 1. It shows average price conditional on trade in equilibrium. The example equilibrium is constructed in Section 6.1. As we can see, there is discontinuity at $t = 1$. It is interesting to note that for Theorem 2 we require at least three types. However, in Section 6.1 we establish directly that the discontinuity in transaction prices is the unique outcome, even with only two types.

6 Examples of equilibria

In this section we construct equilibria in a game with two types of sellers for both fully revealing and noisy grades. This allows us to establish the existence of equilibria and illustrate possible equilibrium dynamics. The equilibria are similar, so we start with the simpler one when grades are fully revealing.

6.1 Example with fully revealing grades

Consider a case with two types of sellers $V = \{v_l, v_h\}$. For simplicity we assume that $c(t, v, w) = t$, and that $v_h - E_{\mu_0} v > 1$.[17]

We slightly abuse the notation and let μ_t denote the probability assigned by the buyers to type v_h at time t. The equilibrium strategies are as follows. At time $t = 0$ buyers offer price $w_0 = v_l$. After that, buyers randomize between $w_t = v_l$ and a higher price (which is offered with probability ρ_t).[18] The higher price is equal to the reservation price of the v_h type as well as to the current average type:

$$w_t = E_{\mu_t} v = R(t, v_h) = v_h - (1 - t)$$

Conditional on being offered $w_t = v_l$, the high type, v_h, rejects and the low type, v_l, accepts with probability σ_t. Conditional on being offered the higher price both types accept with probability 1. The particular probabilities are derived from the Bayes rule and indifference conditions (that the v_l type has to be indifferent between accepting and rejecting $w_t = v_l$, and that buyers

[17] In words, we assume delay costs are the same for both types and are proportional to time. We normalize the cost of graduating to one, and we assume the difference in values relative to delay costs is large enough so that v_h prefers to graduate rather than immediately receive a price equal to average type. See the Appendix for a more general example with different delay costs for the two types as well as for an example with a continuum of types, which illustrates that in general the equilibria are quite complicated and non-unique.

[18] Any randomization strategies of the individual buyers that give the same distribution of the highest price form a payoff-equivalent equilibrium. As a result, the equilibrium is *essentially* unique.

make zero profit from either offer) that yield a system of equations:

$$\mu_{t+\Delta} = \frac{\mu_t}{\mu_t + (1-\mu_t)(1-\sigma_t)} \tag{1a}$$

$$R(t, v_h) = v_h - (1-t) \tag{1b}$$

$$\Delta = \rho_t (R(t, v_h) - v_l) \tag{1c}$$

For $t > 0$ these probabilities are[19]

$$\mu_t = 1 - \frac{1-t}{v_h - v_l}, \ \sigma_t = \frac{\mu_{t+\Delta} - \mu_t}{\mu_{t+\Delta}(1-\mu_t)}, \ \rho_t = \frac{\Delta}{v_h - v_l - (1-t)} \tag{2}$$

We prove in the Appendix that these strategies form a unique equilibrium:

Proposition 4 *If $v_h - E_{\mu_0} v > 1$, for small Δ the proposed strategies form the (essentially) unique sequential equilibrium.*

The equilibrium dynamics can be easily calculated from the strategies. In particular, the probability of trade at $t \in (0,1)$ (conditional on reaching this period) decreases over time:

$$P_t = \rho_t + \sigma_t (1-\mu_t)(1-\rho_t) = \frac{2\Delta}{v_h - v_l - (1-t-\Delta)}$$

and the average transaction price, conditional on trade, increases over time:

$$E[w_t | accept] = \frac{\rho_t R(t, v_h) + \sigma_t (1-\mu_t)(1-\rho_t) v_l}{P_t} = \frac{1}{2}(v_h + v_l) - \frac{1}{2}(1-t-\Delta)$$

Finally, the average transaction price at $t = 1-\Delta$ (for all Δ) is

$$E[w_{t=1-\Delta} | accept] = \frac{1}{2}(v_h + v_l) << v_h$$

which means that even in the two-type case, average transaction prices jump discontinuously upon graduation (recall that in equilibrium, only the v_h

[19] To assure $\rho_t < 1$ we need to assume that Δ is small.

type graduates). Figure 1 shows an example of that discontinuity in the equilibrium for $v_h = 10$ and $v_l = 5$ (for $\Delta \to 0$).

6.2 Example with noisy grades

The equilibrium we construct now is similar to the one above. The main differences are that close to graduation there is zero probability of trade and that both types graduate with positive probability. As before, we have two types of sellers, $V = \{v_l, v_h\}$, and the simple cost function $c(t, v, w) = t$. Assume there are two possible external signals: $g \in \{H, L\}$ so that $\Pr(L|v = v_l) = \Pr(H|v = v_h) = q > \frac{1}{2}$. Again, we slightly abuse notation and denote by μ_t the probability assigned to type v_h at time t.

If the buyers arrive at the last period with a prior μ^*, then the final prices expected by the two types are given by Bayes rule:

$$h_{\mu^*}(v_h) = v_l + (v_h - v_l)\left[\frac{q^2\mu^*}{\mu^* q + (1-q)(1-\mu^*)} + \frac{(1-q)^2\mu^*}{\mu^*(1-q) + q(1-\mu^*)}\right]$$
$$h_{\mu^*}(v_l) = v_l + (v_h - v_l)\left[\frac{(1-q)q\mu^*}{\mu^* q + (1-q)(1-\mu^*)} + \frac{(1-q)q\mu^*}{\mu^*(1-q) + q(1-\mu^*)}\right]$$

In equilibrium, there exists a time t^* such that in all periods $t \leq t^*$ there is trade with positive probability, and for $t > t^*$ (until graduation) there is no trade. In periods $t \in (0, t^*]$ buyers mix between two offers: $w_t = v_l$ and $w_t = E_{\mu_t} v$. At time 0 they offer a price based on the low type, $w_0 = v_l$.

The reservation price of the low type for $t \leq t^*$ matches his value, $R(t, v_l) = v_l$; for $t > t^*$, his reservation price exceeds his value, $R(t, v_l) = h_{\mu^*}(v_l) - (1-t) > v_l$. When offered $w = v_l$ before t^*, this type mixes between accepting and rejecting the offer. The reservation price of the high type for $t \leq t^*$ is $R(t, v_h) = h_{\mu^*}(v_h) - (1-t) = E_{\mu_t} v$ (equals the payoff he gets from rejecting all offers until graduation *and* the current average value). When offered this price before or at t^*, he (as well as the low type) accepts it with certainty. For $t > t^*$, his reservation price is $R(t, v_h) = h_{\mu^*}(v_h) - (1-t) > E_{\mu^*} v$ so that he prefers to graduate rather than to accept a price equal to the value of the average type.

To find t^* we look for a time such that type v_l is indifferent between graduating and getting $w = v_l$ while type v_h is indifferent between graduating and getting a price equal to the average value. These conditions are

$$\begin{aligned} h_{\mu^*}(v_l) - (1 - t^*) &= v_l \\ h_{\mu^*}(v_h) - (1 - t^*) &= \mu^* v_h + (1 - \mu^*) v_l \end{aligned} \tag{3}$$

Solving them yields $\mu^* = 1 - \frac{\sqrt{q(1-q)}}{2q-1}$, $t^* = 1 - (h_{\mu^*}(v_l) - v_l)$. To ensure that there is some trade before graduation, we assume that the prior belief and q are such that $\mu_0 < \mu^*$ and $t^* > 0$.[20] For now, to simplify exposition, assume that t^* is on the time grid defined by Δ.

The mixing is as follows. At $t = 0$, buyers offer price $w_0 = v_l$ that is accepted by the v_l type with a probability such that the posterior average type, $E_{\mu_\Delta} v$, is equal to $R(\Delta, v_h)$. At $t \in (0, t^*]$, buyers randomize between the low and high offers in such a way that the v_l type is indifferent between accepting and rejecting $w_t = v_l$. The probability ρ_t of the higher offer being offered by at least one of the buyers is pinned down by the indifference condition for type v_l, equation (1c).[21] At $t > t^*$ buyers do not make any serious offers (i.e., with positive probability of acceptance). Finally, denote by σ_t the probability that type v_l accepts $w = v_l$ at $t \in (0, t^*]$. This probability is defined by Bayes rule and by matching the average type with the reservation price of the high type v_h, i.e., condition (1a) and a modified (1b):

$$R(t, v_h) = h_{\mu^*}(v_h) - (1 - t) \tag{4}$$

The exception is period t^*, when $w = v_l$ is rejected with certainty (so that $\mu_{t^*+\Delta} = \mu_{t^*} = \mu^*$).[22]

[20] For any μ_0 there exists a range of sufficiently high q such that these conditions are satisfied.

[21] As before, what matters in equilibrium is only the probability with which *any* of the buyers offers the high price and not what the individual mixing strategies are.

[22] The argument that these strategies indeed form an equilibrium is analogous to the previous example.

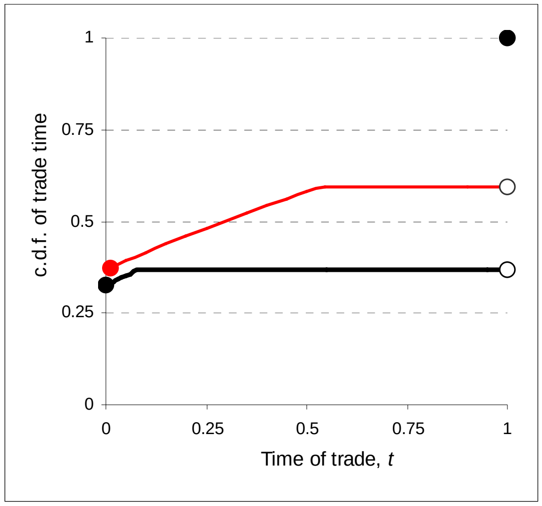

Figure 2: The c.d.f. of equilibrium trade time

As q approaches 1, the equilibrium converges to the unique equilibrium in the fully revealing grade case we constructed above. To see this note that $\mu^* \to 1$ and $t^* \to 1$ as $q \to 1$, and the conditions determining the mixing strategies converge as well to the conditions in the previous example.

To illustrate the dynamics of trade, Figure 2 shows the c.d.f. of the distribution of trade times in the constructed equilibrium when $v_h = 10$, $v_l = 5$, $q = 0.95$, $\mu_0 = \frac{1}{2}$, (and $\Delta \to 0$), (the lower curve). The c.d.f. is constant between $t^* \approx 0.08$ and $t = 1$ which means there is no trade in that time range. For a comparison, we have also drawn the c.d.f. for $q = 0.99$ (the upper line). As we can see, a more precise grade leads to more trade before graduation.

Finally, we can modify the equilibrium for the case that t^* does not happen to be on the grid defined by Δ. If t^* that solves (3) is not on the grid, we define t_1 as the largest time on the grid less than t^* and recalculate μ^* to satisfy:

$$h_{\mu^*}(v_l) - (1 - t_1) = v_l \tag{5}$$

Then we find μ_{t_1} as a solution to:

$$h_{\mu^*}(v_h) - (1 - t_1) = \mu_{t_1} v_h + (1 - \mu_{t_1}) v_l$$

These conditions guarantee that offers equal to reservation prices of the two types yield zero profit at t_1 and after t_1 a negative profit. The last modification is that at $t = t_1$ if $w_t = v_l$ is offered, it is accepted by the low type with probability $\sigma_{t_1} > 0$. That σ_{t_1} is such that the posterior drops from μ_{t_1} to exactly μ^*.

7 Productive signaling

In the model we have examined so far, any level of signaling represents a welfare loss. This assumption is too extreme in some cases. For example, consider the case of education signaling. A positive level of education may improve an employee's productivity. However, as Spence (1973, 1974) has shown, signaling leads to over-education.

To see the implication of productive delay in our model, we examine a simple modification of our previous example in Section 6.1. We modify this example by interpreting it as an education model and considering the case of productive education. We denote the productivity of type v who stays in school until time t by $\pi(v, t)$.[23] We assume that $\pi(v_l, t) < \pi(v_h, t)$ for all $t \geq 0$ and $0 \leq \pi_2(v_l, t) \leq \pi_2(v_h, t) - d$ for some $d > 0$ (where π_2 denotes the derivative with respect to the second argument). We also assume that $\pi(v, t)$ is continuously differentiable and strictly concave in t. Denote by t_v^* the efficient level of education for type v, that is[24]

$$t_v^* = \arg\max_t [\pi(v, t) - ct]$$

Note that we do not normalize the cost of delay but write it as ct. Our assumptions imply that it is efficient for the high type to stay longer at

[23] The payoff of a firm that hires a worker with productivity π at wage w is $\pi - w$.

[24] Since that game is defined over a finite time grid, we assume that t_v^* are on the grid to avoid integer problems.

school:

$$t_h^* > t_l^* > 0$$

As always, we normalize the timing of the external signal to one, and assume that the grades are fully revealing. There are two cases to consider:

First, consider the case in which it is efficient for the high type, v_h, to graduate: $t_h^* \geq 1$. In this case we argue that the equilibrium is unique and both types choose the efficient amount of education in equilibrium. Since the grade is fully revealing, the high type obtains a full return for his education, and as a result, chooses the optimal amount. This implies full separation, and therefore, the low type also obtains the efficient amount of education.

Second, suppose that it is not efficient for the high type to graduate: $t_h^* < 1$. We claim that in this case the low type will always choose either the efficient or higher-than-efficient level of education (the second with positive probability) and the high type will mix between undereducation and over-education. The intuition is that, with positive probability, trade takes place before $t = 1$ at a price equal to average productivity . Therefore the high type is sometimes pooled with the low type and is paid according to the average value. This implies less-than-full return on investment, and therefore, possible undereducation. In the Appendix, we show formally that the low type is never undereducated and the high type can be either undereducated or over-educated:

Proposition 5 *If c and Δ are sufficiently small, then in equilibrium, the high type trades with positive probability both at $t < t_h^*$ and at $t > t_h^*$. The low type always trades at $t \geq t_l^*$.*

8 Conclusions

In this paper we have shown that an external release of information has strong implications for the resulting trade patterns. It can help agents commit to costly signaling even though, in a dynamic setup, such commitment is hard to achieve. Moreover, it allows for signaling even when the direct delay costs are the same for all types. The resulting trade pattern has some discontinuity before the period when information arrives. When the public signal is fully informative, there is a discontinuity in prices and trade is continuous. When the signal is noisy, there is a market breakdown close to the announcement as the adverse selection becomes more severe and a lemons market develops endogenously.

The model we examine is a one-shot game in the sense that the release of public information occurs only once. Clearly, there are many interesting applications in which the signals arrive in a more complicated fashion. We plan to analyze such markets in future research; however, we think that the current results allow us to provide some intuition already. First, in some markets, the release of public information is gradual; signals or grades are revealed at some frequency. In this case we expect behavior similar to what we derived in every sub-interval. A more challenging setup is the case in which the timing of the release of public information is not perfectly known to all parties. Consider for example a setup in which the timing is random. In such a setup one can show that some of our basic structure carries over. For example, 1) the seller follows a reservation-price strategy, and 2) if every period a signal arrives with positive probability, then Proposition 2 holds even if $c(t, v, w)$ is constant in v. Similarly, if the signal is noisy, there will be no trade before periods with a high probability of a signal arriving.

One important case is that of a Poisson distribution. Conditional on non-arrival of a signal, agents have a stationary belief regarding the distribution of future arrival. Under this assumption (and assuming costs are the same for all types of sellers) one can show that there exists a stationary equilibrium in which both types behave the same way after the initial period. Therefore, the average type and transaction prices remain constant over time.

Another important direction of future research is to allow the good for sale to be divisible. The classic paper in this context is Leland and Pyle (1977). It shows that in a static model the seller can signal a high value by reducing the quantity he offers for sale. If the value is not realized immediately, then after the first sale the buyers will have incentives to approach the seller to buy additional units. That in turn may disturb the equilibrium, just like the lack of commitment to stay in school disturbs the education signaling.[25]

Finally, we have assumed that the offers are private, and for tractability, we also assumed that each buyer makes only one offer in the game. Making the offer public is unlikely to increase the payoffs of the buyers in our game: if rejecting an offer signals high value, then the seller is more likely to reject a public offer than a private one. Therefore, buyers have incentives to make their offers private.

Allowing each buyer to make more than one offer would be more interesting, but extending the model makes it hard to solve. The assumption that buyers are replaced after every period (borrowed from Swinkels (1999)) greatly simplifies the analysis since there is no information asymmetry among buyers. Private information may arise because buyers can use the private history of rejected offers. This would complicate significantly the equilibrium characterization. For example, the seller may no longer follow a reservation-price strategy, buyers may be able to extract information rents, and it will not be sufficient to check single-period deviations to verify that proposed strategies form an equilibrium. Still, we believe that the intuition developed here carries over, especially in the fully revealing case. For example, there still would be some signaling in any equilibrium and the high types would graduate. Also, Theorem 2 is likely to hold, since one period before graduation, the seller would follow a reservation-price strategy, even if buyers used private histories.

Admittedly, we do not capture all important features of the markets that motivate this paper, like the role of public offers, the possibility of buyers making multiple offers, divisibility of the asset, partial contractibility of the external signal, or the seller's ability to influence the timing and realization of the signal. We focus on the role of commitment and the resulting trade patterns. As such, this paper should be considered a step towards the understanding of dynamic signaling. Much more research is needed for this understanding to be complete.

[25] In some markets this problem is solved by restricting additional sales for a specified period. For example, in many IPOs the original owners are not allowed to sell shares during the six-month period after the offering.

9 Appendix

Proof of Lemma 1. The fact that $R(t,v)$ is non-decreasing (in the second parameter) follows as if $v' > v$ then v' can imitate v and get at least the same expected payoff. If $\frac{\partial^2 c(t,v,w)}{\partial t \partial v} < 0$ or v graduates with positive probability, then imitating v yields a strictly higher payoff for v' (as either the incremental costs of delay are strictly lower, or upon graduation the expected prices are strictly higher, for v' than for v). ■

Proof of Proposition 1. Suppose by contradiction that type $\bar{v}$ never graduates. First consider the case of a fully revealing grade. Type $\bar{v}$ has the option to graduate and hence his equilibrium payoff has to be at least $\bar{v} - c(1,\bar{v},\bar{v})$, so any price w that he accepts has to satisfy $w - c(t,\bar{v},w) \geq \bar{v} - c(1,\bar{v},\bar{v})$. Given that $\bar{v}$ never graduates, any type v can mimic him and earn that price for a payoff of

$$w - c(t,v,w) \geq \bar{v} - c(1,\bar{v},\bar{v}) + c(t,\bar{v},w) - c(t,v,w) > \bar{v} - 2c(1,\underline{v},\bar{v})$$

Therefore, the payoffs of all types have to be at least $\bar{v} - 2c(1,\underline{v},\bar{v})$, which is a lower bound on the average price. However, we assumed that $\bar{v} - 2c(1,\underline{v},\bar{v}) > E_{\mu_0} v$. This implies that on average buyers lose money: a contradiction.

In the case of a noisy grade, we note that Corollary 1 implies that if the highest type never graduates, then no type graduates. Hence, μ_1 is an off-equilibrium belief. "Divinity" implies that μ_1 dominates μ_0 in the first order stochastic sense, hence $E_{\mu_1} v \geq E_{\mu_0} v$.[26] The claim follows a similar argument to the first case since the highest type expects a better than average grade, and hence the expected price, conditional on him graduating, satisfies $h_{\mu_1}(\bar{v}) > E_{\mu_0} v$.

Finally, the probability that the highest type graduates is bounded away from zero, because if it is very close to zero, then all lower types can mimic him and obtain $\bar{v} - 2c(1,\underline{v},\bar{v})$ with a probability arbitrarily close to one. This

[26] The refinement argument is as follows. As the cost of waiting until $t = 1$ is weakly decreasing in v and the distribution of grades is improving in v, for any off-equilibrium beliefs that make a weaker type willing to wait, all higher types strictly prefer to wait.

still implies a contradiction, as bidders would still lose money on average. ■

Proof of Proposition 2. The first two parts follow the same line of reasoning as Swinkels (1999). Part (i) is simply implied by the buyers following a reservation-price strategy that is strictly increasing in type. Hence, if a type v weakly prefers to accept an offer, then any lower type strictly prefers to accept it and any higher type strictly prefers to decline it.

The proof of part (ii) is the same as Lemma 2.3 in Swinkels and we outline it below. Suppose buyer 2 makes a profit $Q > 0$. He makes this profit with any price that is in support of his (mixing) strategy. Let $\underline{w}$ be the infimum over buyer 2's offers. Buyer 1 can then make a profit by offering $\underline{w}+\varepsilon$ and hence, buyer 1 must also make a positive profit in equilibrium. Since profits are bounded, every equilibrium offer would have to be accepted with positive probability. That implies that both buyers offer $\underline{w}$ with strictly positive probability. However, this implies a positive probability of a tie and hence, there is a profitable deviation, which is a contradiction. The second part of this claim follows because if an offer is accepted by more than one type and the highest accepting type is mixing, then by increasing the offer by ε, the buyer would obtain a strictly better selection of types.

To see why (iii) holds, assume that there is a positive probability of trade at t but there is no trade at $t-\Delta$. Let w^* denote the highest offer made (and accepted) with positive probability at time t, and $c^* \equiv c(t, \bar{v}, w^*) - c(t-\Delta, \bar{v}, w^*)$. Note that c^* is a lower bound on the waiting cost from $t-\Delta$ to t, assuming that the current offer is w^* and the offer accepted at t is w^*. Consider now the strategy of a buyer to offer price $w^* - c^*/2$ at time $t-\Delta$. All the seller types that in equilibrium accept w^* at time t will accept $w^* - c^*/2$ at time $t-\Delta$, and possibly even some better types will as well. As the distribution of types at $t-\Delta$ is the same as at t (and w^* yields zero expected profit), this strategy results in positive profit for the buyer; this contradicts part (ii).

To establish part (iv), we first note that if no type graduates with positive probability, then the trade has to be with probability 1 at time $t = 0$. The reasoning follows Lemma 2.4 and Theorem 2.5 in Swinkels (1999) so we just sketch the argument. Denote by $\tau > 0$ the last period (before graduation) that is reached with positive probability. In that period, trade takes place with certainty at $E_{\mu_\tau} v$. For a sufficiently small Δ we argue that $\mu_\tau = \mu_{\tau-\Delta}$. Suppose the opposite, so that with positive probability there is trade at time $\tau - \Delta$ that does not involve type $\bar{v}$. The highest price the buyers pay in such a trade is $E_{\mu_{\tau-\Delta}}[v|v < \bar{v}]$. But the reservation price for a small Δ for any type v is close to $E_{\mu_\tau} v$ (as the additional delay cost is negligible). So there is no possibility of such a trade without the buyers losing money. Second, if $\mu_{\tau-\Delta} = \mu_\tau$, then the reservation prices at time $\tau - \Delta$ are strictly less than at τ (because of the small cost of delay) so there would be trade with probability 1 at time $\tau - \Delta$, which contradicts the definition of τ. Finally, it implies that if type $\bar{v}$ does not graduate with positive probability, then (if Δ is small enough) in equilibrium his payoff is $E_{\mu_0} v$. As we argued in Proposition 1, graduating yields an expected payoff of at least $h_{\mu_0}(\bar{v}) - c(1, \bar{v}, \bar{v})$, so it is a profitable deviation. ■

Proof of Lemma 2. We prove the lemma using the direct revelation mechanism approach. Let $U(v, v')$ denote the utility of an agent of type v who pretends to be v', that is

$$U\left(v, v'\right) = Ew - Ec\left(\tau\left(v'\right), v, w\right)$$

where w is the price, $\tau(v')$ is the time that type v' trades, and $Ec(\tau(v'), v, w)$ is the cost of signaling. This implies that $U(v) = U(v, v)$. In a separating equilibrium, type v accepts a price that matches his type, that is, $w = v$. So although in general w is a function of both v and v', in a separating equilibrium it depends on v' only.[27]

This implies that the lowest type has no incentive to wait and quits at time $t = 0$. Incentive constraints imply

$$U(v, v) \geq U(v, \underline{v}) = U(\underline{v}, \underline{v}) + [Ec(\tau(\underline{v}), \underline{v}, \underline{v}) - Ec(\tau(\underline{v}), v, \underline{v})]$$

[27] w depends on v indirectly as the mechanism can assign the price conditional on the realization of the grade, which is affiliated with v.

or that

$$\frac{U(v,v)-U(\underline{v},\underline{v})}{v-\underline{v}} \geq \frac{Ec(\tau(\underline{v}),\underline{v},\underline{v})-Ec(\tau(\underline{v}),v,\underline{v})}{v-\underline{v}}$$

Switching the roles of $\underline{v}$ and v, we get the following:

$$\frac{U(v,v)-U(\underline{v},\underline{v})}{v-\underline{v}} \leq \frac{Ec(\tau(v),\underline{v},v)-Ec(\tau(v),v,v)}{v-\underline{v}}$$

This establishes differentiability of $U(v)$ since c is assumed to be continuously differentiable. Using the envelope condition we get the following derivative:

$$\frac{d}{dv}U(\underline{v}) = -E\left[\frac{\partial c(\tau(v'),v,w)}{\partial v}\right]\bigg|_{v'=v=w=\underline{v}}$$

This derivative equals zero, as $c(0,v,w)=0$ for all v and w and

$$\tau(v)_{v\downarrow\underline{v}} \to 0 \quad a.s.$$

(The last claim holds since (i) $\frac{\partial c(t,v,v)}{\partial t} > \alpha > 0$ and (ii) type v gets price $w=v$, so it must be that $\tau(v) \leq \frac{v-\underline{v}}{\alpha}$.) ■

Proof of Lemma 3. Suppose the opposite. Then for any $\varepsilon_1 > 0$ we can find a Δ such that the probability of v_{N-1} graduating, conditional on his type, is less than ε_1. By Lemma 1 the probability that any type below v_{N-1} graduates, conditional on his type, is also smaller than ε_1. Hence, for any $\varepsilon_2 > 0$ we can find a Δ such that $h_{\mu_1}(v) \geq \bar{v} - \varepsilon_2$ for all v. This holds since Proposition 1 implies that type $\bar{v}$ graduates with positive probability, uniformly bounded away from zero. As a result, for any $\varepsilon_3 > 0$ there exists a Δ such that in equilibrium all types earn at least

$$\bar{v} - c(1,\underline{v},\bar{v}) - \varepsilon_3$$

By Assumption 3, if ε_3 is small enough, it implies that buyers lose money on average. This is a contradiction. ■

Proof of Lemma 4. We take v^* as given and let $k(g) = E[v|g, v \leq v^*]$.

By Lemma 3, the probability that some types below $\bar{v}$ graduate is bounded away from zero. Hence, we can find an $\eta > 0$ so that $k(g) + \eta < w^{final}(g)$. The proof follows using

$$E\left[h_{\mu_1}(v)\,|v \leq v^*\right] = E\left[w^{final}(g)\,|v \leq v^*\right]$$

and

$$E\left[v|v \leq v^*\right] = E\left[E\left[v|g, v \leq v^*\right]|v \leq v^*\right] = E\left[k(g)\,|v \leq v^*\right]$$

■

Proof of Theorem 1. Denote by $\bar{t}$ the last period before graduation where there is some positive probability of trade, that is

$$\bar{t} = \max\left\{t < 1|\text{there is positive probability of trade at } t\right\}$$

Since we assume time to be discrete, this is well defined. Our goal is to show that even when Δ goes to zero, $\bar{t}$ is bounded away from one. Denote by v^* the highest type that trades with positive probability at time $\bar{t}$.

At $\bar{t}$, the reservation prices are strictly increasing in type since upon rejection the seller graduates with certainty. As a result, when there is trade, buyers attract all types $v \leq v^*$ by offering the reservation price of type v^*, $R(\bar{t}, v^*)$. We argue that if $\bar{t}$ would be too close to one, then any such offer results in losses to the buyer who makes it; this is a contradiction.

The reservation price of type v^* is

$$R(\bar{t}, v^*) = h_{\mu_1}(v^*) - \left(c\left(1, v^*, h_{\mu_1}(v^*)\right) - c\left(\bar{t}, v^*, R(t, v^*)\right)\right)$$

From Lemma 8 (which we prove below) we can bound the reservation price:

$$R(\bar{t}, v^*) > h_{\mu_1}(v^*) - M(1 - \bar{t}) \tag{6}$$

Suppose first that $v^* < \bar{v}$. Lemma 4 implies that the average price upon graduation for types not higher than v^* is strictly higher than their average value:

$$E_{\mu_1}\left[h_{\mu_1}(v)\,|v \leq v^*\right] > E_{\mu_1}\left[v|v \leq v^*\right] + \eta$$

for some $\eta > 0$. Affiliation implies that upon graduation type v^* expects a price that is even higher:

$$h_{\mu_1}(v^*) \geq E_{\mu_1}\left[h_{\mu_1}(v)\,|v \leq v^*\right]$$

Combining these inequalities with the bound (6), we obtain

$$R(\bar{t}, v^*) > E_{\mu_1}\left[v|v \leq v^*\right] + \eta - M(1-\bar{t}) \tag{7}$$

Hence, the result follows when we set $t^* \geq 1 - \frac{\eta}{M}$, as for smaller t a trade would result in buyers losing money.

Similarly, consider $v^* = \bar{v}$. Now, because the distribution μ_1 is bounded away from being degenerate,

$$h_{\mu_1}(v^*) > E\left[{}_{\mu_1} h_{\mu_1}(v)\,|v \leq v^*\right] + \eta = E_{\mu_1}\left[v|v \leq v^*\right] + \eta$$

which implies (7), and hence, again $t^* \geq 1 - \frac{\eta}{M}$. ■

Lemma 8 *Suppose that after a rejection at time t, there is no trade until graduation. Then there exists M such that $R(t, v^*) > h_{\mu_1}(v^*) - M(1-t)$.*

Proof. Denote $x \equiv R(t^*, v^*)$ and $y \equiv h_{\mu_1}(v^*)$. Upon rejecting the current offer, the seller graduates with certainty, so his reservation price x solves

$$y - x - (c(1, v, y) - c(t^*, v^*, x)) = 0 \tag{8}$$

Note that the left hand side is (continuously) strictly decreasing in x, and at $x = y$ it is equal to $c(1, v, y) - c(t, v, x) \in (-\delta(1-t), -\alpha(1-t))$, where α and δ are bounds on $\delta > \frac{\partial c(t,v,w)}{\partial t} > \alpha$ from Assumption 1. Therefore the equation has a unique solution which is strictly less than y. Also by Assumption 1, $w - c(t, v, w)$ is strictly increasing: $\frac{\partial(w - c(t,v,w))}{\partial w} \geq \phi > 0$. Therefore, for $x < y - \frac{\delta}{\alpha}(1-t)$ the left hand side of (8) is positive. Hence, the unique solution satisfies the following:

$$h_{\mu_1}(v^*) \geq R(t^*, v^*) \geq h_{\mu_1}(v^*) - \underbrace{\frac{\delta}{\phi}}_{M}(1-t)$$

■

Proof of Lemma 5. The proof is identical to that of Proposition 1: If type v_{N-1} does not graduate, then he must obtain at least $v_{N-1} - c(1, v_{N-1}, v_{N-1})$. But then every lower type has to obtain at least $v_{N-1} - c(1, v_{N-1}, v_{N-1}) - c(1, v, \bar{v})$, which, using the assumption, would make the buyers lose money on average. This is a contradiction. If type v_{N-1} graduates, then type v_N graduates as well. Finally, note that even as $\Delta \to 0$, the probability of graduation by type v_{N-1} is strictly higher than 0. ■

Proof of Lemma 6. At $t = 1 - \Delta$, for any type v, $R(t, v) < v$. Consider the lowest type that with some probability has not quit yet, that is, $v^* = \min\{v|\mu_t(v) > 0\}$. We argue that the lowest price offered at t is not lower than v^*. If the highest price is strictly less than v^* with positive probability, then there exists a price below v^* that would be accepted by v^* and result in a profit for the buyers. Therefore, by the standard Bertrand-competition argument, prices are always at least v^* and type v^* trades with certainty. ■

Proof of Lemma 7. Denote by W_1 the average price offered at $t = 1$. It is equal to the average value of types that graduate

$$W_1 = E_{\mu_1} v$$

By Lemma 5, at $t = 1 - \Delta$ the beliefs are non-degenerate, so at any $t < 1$ they are also non-degenerate. By Corollary 1, μ_1 dominates μ_t. In order for the buyers not to lose money, the highest offer at $t < 1$ can be

$$w_t \leq E_{\mu_t} v \leq E_{\mu_1} v = W_1$$

Note that, in case $R(t, v)$ is constant over some types (that never graduate), an offer can attract only the best type in this pool. However, as the worst type that graduates (with positive probability) is strictly better than any type that does not, the claim still holds. ■

Proof of Proposition 4. The following lemma helps in constructing the equilibrium:

Lemma 9 *In equilibrium only prices equal to v_l or $E_{\mu_t}v$ are accepted with positive probability.*

Using argument like in Proposition 1 we can show that type v_h graduates with positive probability in any equilibrium. Therefore, $R(t, v_l) < R(t, v_h)$. From that it can be easily argued that buyers make zero profits in any equilibrium.

Now, suppose that some price w other than the ones postulated is accepted with positive probability. Clearly, any price $w < v_l$ if accepted yields positive profit, a contradiction. Any price $w > E_t v$ if accepted yields negative profit, again a contradiction. We are only left with prices $w \in (v_l, E_t v)$. Take any w and suppose it yields zero profit. That implies $w = R(t, v_h)$ as type v_h is not accepting with probability 1. But then offering $w + \varepsilon$ makes the price accepted at least as often and yields a positive expected profit, as type v_h will accept this price with probability 1, a contradiction. ■

Now we derive the equilibrium probabilities postulated in (2). First, we have the Bayes rule:

$$\mu_{t+\Delta} = \frac{\mu_t}{\mu_t + (1-\mu_t)(1-\sigma_t)}$$

Second, the buyers are indifferent between offering v_l and the higher price, so as we argued in the lemma above for any $t > 0$,

$$E_{\mu_t}v = v_h - (1-t)$$

Third, the v_l type seller is indifferent between accepting v_l now and waiting for another period:

$$\Delta = \rho_t (E_{\mu_t}v - v_l)$$

The boundary condition is $\mu_1 = 1$ so that $\sigma_{1-\Delta} = 1$. One can verify that (2) are the unique solutions to these equations.

Clearly the proposed strategies form an equilibrium for small Δ: given the strategies of the buyers, the type v_l is indifferent in every period (other than $1-\Delta$) between accepting and rejecting $w = v_l$, so the randomization is optimal. Also, given the buyers' strategy the reservation price of the v_h type is $R(t, v_h) = v_h - (1-t)$, so indeed when the higher price is offered, both types are willing to accept it.[28] Given the seller's strategy, the two prices v_l and $v_h - (1-t)$ yield zero expected profits and any other price does not yield positive profits, so all buyers are also playing a best response.

To show uniqueness we now construct the equilibrium starting from the last period and move backwards and argue that in every step the equilibrium is essentially unique. In period $t = 1$, after the signal g arrives, prices are $w = v$. Therefore, the reservation prices at time $t = 1-\Delta$ are $R(1-\Delta, v) = v - \Delta$. If $E_{\mu_{1-\Delta}} v < R(1-\Delta, v_h)$ then the unique equilibrium outcome is for the buyers to offer $w = v_l$ and type v_l accept and type v_h reject and graduate. If $E_{\mu_{1-\Delta}} v > R(1-\Delta, v_h)$, then the unique equilibrium is for the buyers to offer $w = E_{\mu_{1-\Delta}} v$ and both types to accept. Finally, if $E_{1-\Delta} v = R(1-\Delta, v_h)$, then the buyers can mix between $w = v_l$ and $w = E_{\mu_{1-\Delta}} v$. In any such equilibrium we need to have that if the highest offer is v_l then the type v_l accepts and if it is $E_{\mu_{1-\Delta}} v$ then both types accept (we need type v_h to accept with probability 1 as otherwise the buyers would be losing money). So in this case there can be many equilibria that differ with the probability of offering $E_{\mu_{1-\Delta}} v$.

Now consider period $t = 1-2\Delta$ and assume $E_{\mu_t} v < v_h - 2\Delta$. We argue that the equilibrium strategies in that period have to be such that next period $E_{\mu_{1-\Delta}} v = R(1-\Delta, v_h)$. Suppose $E_{\mu_{1-\Delta}} v < R(1-\Delta, v_h)$. Then $R(1-2\Delta, v_l) = v_l - \Delta$ and hence type v_l accepts for sure at time $1-2\Delta$, inducing $E_{\mu_{1-\Delta}} v = v_h$, a contradiction. Suppose $E_{\mu_{1-\Delta}} v > R(1-\Delta, v_h)$. Then next period price will be higher than $v_h - \Delta$, while the type v_l accepts this period price v_l with positive probability. Clearly, for small Δ, type v_l is

[28] We need Δ to be small, to guarantee that the probabilities ρ_t defined in (2) satisfy $\rho_t \in [0,1]$.

strictly better off rejecting v_l, making $E_{\mu_{1-\Delta}}v = E_{\mu_{1-2\Delta}}v < R(1-\Delta, v_h)$, a contradiction. So if $E_{\mu_{1-2\Delta}}v < v_h - 2\Delta$ the unique equilibrium is for the buyers to offer $w = v_l$ and the seller type v_l accept with a probability such that $E_{\mu_{1-\Delta}}v = R(1-\Delta, v_h)$. Next period the buyers randomize between v_l and $R(1-\Delta, v_h)$. As we mentioned in the text, this randomization creates a trivial multiplicity of equilibria: for the payoffs of the seller only the highest price matters, so any randomization strategies by the buyers that give the same probability distribution of the highest price induce the same best responses by the seller. Therefore, we will focus on the distribution of the highest price.

Denote by ρ_t the probability that the highest price at time t is equal to $E_{\mu_t}v$. For the type v_l to be indifferent at time $1-2\Delta$ we need:

$$\Delta = \left(E_{\mu_{1-\Delta}}v - v_l\right)\rho_{1-\Delta}$$

If $E_{\mu_{1-2\Delta}}v > v_h - 2\Delta$ the unique equilibrium is for buyers to offer $w = E_{\mu_{1-2\Delta}}v$ and both types to accept. If $E_{\mu_{1-2\Delta}}v = R(1-2\Delta, v_h)$ then we have a multiplicity of equilibria, in which in period $1-2\Delta$ the buyers randomize in any way between v_l and $E_{\mu_{1-2\Delta}}v$. The higher offer ends the game. Conditional on offer v_l the game proceeds the same way as in case $E_{\mu_{1-2\Delta}}v < R(1-2\Delta, v_h)$.

Now consider any period $t \leq 1-3\Delta$. If $E_{\mu_t}v \leq v_h - (1-t)$ then by induction we argue that the strategies have to satisfy $E_{\mu_{t+\Delta}}v = R(t+\Delta, v_h)$. Suppose not.

First, suppose $E_{\mu_{t+\Delta}}v > R(t+\Delta, v_h)$. Then next period buyers offer $w_{t+\Delta} = E_{\mu_{t+\Delta}}v$ and both types accept. That implies:

$$R(t, v_l) = E_{\mu_{t+\Delta}}v - \Delta$$

Because v_h type always has the option to graduate, we have

$$E_{\mu_{t+\Delta}}v > R(t+\Delta, v_h) \geq v_h - (1-(t+\Delta))$$

Combining:

$$R(t, v_l) > v_h - 1 > v_l$$

Therefore v_l would reject $w_t = v_l$ for sure and so $E_{\mu_{t+\Delta}} v = E_{\mu_t} v < R(t + \Delta, v_h)$, a contradiction.

Second, consider $E_{\mu_{t+\Delta}} v < R(t + \Delta, v_h)$. It implies that the next period buyers offer $w_{t+\Delta} = v_l$. But then $R(t, v_l) = v_l - \Delta$ and given the competition between buyers, type v_l would for sure accept an offer at time t, making $E_{\mu_{t+\Delta}} v = v_h$, a contradiction.

By induction we see also that in equilibrium as long as $E_{\mu_t} v < v_h - (1 - t)$, then in all future periods $t' > t$, $R(t', v_h) = v_h - (1 - t') = E_{\mu_{t'}} v$.

Combining, we get that if at $t = 0$, $E_{\mu_0} v < v_h - 1$, then the strategies have to be such that in any period $t > 0$, $E_{\mu_t} v = v_h - (1 - t)$. Also, in any period $t > 0$ the buyers have to randomize between prices v_l and $R(t, v_h)$ in a way that makes type v_l indifferent between accepting and rejecting $w = v_l$ in the previous period. Finally, the seller if offered $w = R(t, v_h)$ accepts and if offered $w = v_l$ the type v_h rejects and type v_l accepts with such probability that the average value evolves exactly the same way as $R(t, v_h)$. These are exactly the conditions of the constructed equilibrium. ■

Proof of Proposition 5. We first note a few properties of any equilibrium that still hold in the modified example: (i) the seller follows a reservation-price strategy, (ii) if c is small, then the high type graduates with positive probability, (iii) the low type reaches period $t = 1 - \Delta$ with positive probability, and (iv) in any equilibrium, trade takes place only at $w_t^l = \pi(v_l, t)$ or at $w_t^h = \mu_t \pi(v_h, t) + (1 - \mu_t) \pi(v_l, t)$. The proofs are analogous to the proofs in the unproductive signaling model. Based on the same reasoning as in Proposition 1, a sufficient condition on c for (ii) to hold is

$$c\frac{2 - \mu_0}{1 - \mu_0} < \pi(v_h, 1) - \pi(v_l, 1)$$

We can also conclude that there is no trade until t_l^* since it is commonly known that it is inefficient to trade until then. Formally, at any time t, type v_l obtains a gross payoff at least $\pi(v_l, t)$ (otherwise he would trade, violating (iii)). Therefore, if only this type was to trade, at time $t < t_l^*$ the firms would have to offer him more than $\pi(v_l, t)$ and would lose money. Also, there cannot be trade with both types, since the reservation price of the v_h type in any period is at least as high as current average productivity (otherwise this type would trade, violating (ii)). As the average productivity grows faster than at rate c before t_l^*, the reservation price of type v_h is strictly above the average productivity, and again trade would imply a negative payoff for the firms.

The only part left to show is that the high type underinvests with positive probability. First, one can show that for $t \in [t_h^*, 1]$, the game has a unique outcome; the argument is the same as for the unproductive delay (the argument is the same as in Proposition 4 for non-productive delay). Firms randomize between w_t^h and w_t^l with probabilities such that the low type is indifferent between accepting and rejecting w_t^l. When w_t^l is offered, the low type accepts it with positive probability and the high type rejects it. If w_t^h is offered, both types accept. The mixing probabilities are such that the average type increases over time to stay equal to the reservation price of the high type, which is equal to

$$R(v_h, t) = \pi(v_h, 1) - (1 - t)c$$

Now consider period $t = t_h^* - \Delta$. We claim that for a small Δ it must be the case that $\mu_{t_h^* - \Delta} < \mu_{t_h^*}$. The reservation price of the high type is

$$R(v_h, t) = R(v_h, t_h^*) - \Delta c = \mu_{t+\Delta}\pi(v_h, t + \Delta) + (1 - \mu_{t+\Delta})\pi(v_l, t + \Delta) - \Delta c$$

If $\mu_t = \mu_{t+\Delta}$, then for a small Δ the average productivity is

$$\mu_t\pi(v_h, t) + (1 - \mu_t)\pi(v_l, t) > R(v_h, t)$$

(This is true since $\pi_2(v_h, t)$ converges continuously to c as t converges to t_h^*, so that $\pi(v_h, t)$ grows approximately at a rate c, while $\pi(v_l, t)$ grows at a strictly lower rate and $\mu_t < 1$.) As a result, there would be trade at w_t^h, contradicting (ii). Therefore, in any equilibrium the beliefs have to satisfy

$\mu_{t_h^* - \Delta} < \mu_{t_h^*}$.

Now, suppose that the probability that type v_h trades at $t = t_h^* - \Delta$ is zero. Consider period $t = t_h^* - 2\Delta$ and type v_l. Because next period this type accepts $w_{t+\Delta}^l$ when offered (and higher prices are not offered), the reservation price of this type is

$$R(v_l, t) \leq w_{t+\Delta}^l - \Delta c < \pi(v_l, t)$$

where the last inequality holds for a small Δ. Hence, this type would certainly trade at t, contradicting point (iii) above. Therefore, it must be the case that type v_h trades with positive probability at time $t = t_h^* - \Delta$, that is, he underinvests with positive probability. ■

9.1 Additional examples

We now provide two additional examples. First, we generalize the simple game in Section 6.1 to allow for $\widetilde{c}(v_h) < \widetilde{c}(v_l)$. Second, we construct an equilibrium in a three-period model with a continuum of types and argue that it is in general not unique.

General binary example Suppose that like in Section 6.1 there are two types $V = \{v_l, v_h\}$ and assume that the costs are $c(t, v, w) = t\widetilde{c}(v)$, with $0 < \widetilde{c}(v_h) < \widetilde{c}(v_l) < v_h - E_{\mu_0} v$. The grade is fully revealing.

In this modified game the following strategies form an equilibrium, similar to the equilibrium described in Section 6.1: at time 0 both buyers offer $w^l = v_l$ and the low type seller accepts with a positive probability (and the high type rejects). After the initial period the buyers randomize between w^l and $w_t^h = \mu_t v_h + (1 - \mu_t) v_l$. The probability of the highest wage offered being w_t^h is ρ_t. When w_l is offered the low type mixes between accepting and rejecting, while the high types rejects. When w_t^h is offered, both types accept.

Denote the probability of type v_l accepting offer w^l by σ_t. It is such that the average type, w_t^h after the initial period equals the reservation price of the v_h type, which in turn equals the payoff from graduating:

$$w_t^h = v_h - (1-t)\,\widetilde{c}\,(v_h) \text{ for } t > 0$$

The mixing probabilities by the buyers are such that the v_l type is indeed indifferent between accepting and rejecting v_l :

$$\Delta\widetilde{c}\,(v_l) = \rho_t \left(w_t^h - w^l\right)$$

Finally, the posterior evolves according to the Bayes rule:

$$\mu_{t+\Delta} = \frac{\mu_t}{\mu_t + (1-\mu_t)(1-\sigma_t)}$$

The boundary condition is $\mu_1 = 1$ and hence $\sigma_{1-\Delta} = 1$. These three conditions have a unique solution

$$\begin{aligned} &\text{for } t > 0 : \mu_t = 1 - \frac{(1-t)\,\widetilde{c}\,(v_h)}{v_h - v_l},\ \rho_t = \frac{\widetilde{c}\,(v_l)\,\Delta}{v_h - v_l - \widetilde{c}\,(v_h)\,(1-t)} \\ &\text{for } t < 1 : \sigma_t = 1 - \frac{\mu_t\,(1-\mu_{t+\Delta})}{\mu_{t+\Delta}\,(1-\mu_t)} \end{aligned}$$

Furthermore, following arguments like in Proposition 4 one can show that these strategies form the unique equilibrium for small Δ (of course, as before, up to the different randomizations of the buyers that yield the same distribution of the highest price).

As we focus on a model in which the seller has little commitment, we can look at the limit as $\Delta \to 0$. For that limit we need to consider limits of $\frac{\sigma_t}{\Delta}$ and $\frac{\rho_t}{\Delta}$ as the per-period probabilities converge to zero (for $t \in (0,1)$). Taking the limit $\Delta \to 0$ of the equilibrium strategies and beliefs we obtain:

$$\mu_t = 1 - \frac{(1-t)\,\widetilde{c}\,(v_h)}{v_h - v_l}$$
$$\frac{\sigma_t}{\Delta} \to \frac{\dot{\mu}_t}{(1-\mu_t)\,\mu_t} = \frac{v_h - v_l}{(1-t)\,(v_h - v_l - \widetilde{c}\,(v_h)\,(1-t))}$$
$$\frac{\rho_t}{\Delta} \to \frac{\widetilde{c}\,(v_l)}{v_h - v_l - \widetilde{c}\,(v_h)\,(1-t)}$$

Additionally, at $t = 0$ the probability of acceptance of $w = v_l$ by the v_l type, σ_0, is such that given μ_0 we get $\mu_{0+} = 1 - \frac{\widetilde{c}(v_h)}{v_h - v_l}$.

Example with continuum of types We now turn to construction of an equilibrium in a model with a continuum of types. It turns out that these equilibria are quite complicated, so we focus on an example of a three-period game $\left(\Delta = \frac{1}{2}\right)$ with costs $c\,(v, w, t) = \widetilde{c}t = 0.2t$ (so that the cost per period is 0.1) and the types distributed uniformly over a range $[1, 2]$. The grade is fully revealing. We abuse the notation and talk about periods $1, 2, 3$ that denote times $0, \frac{1}{2}, 1$.

Denote by F the posterior belief that the firms have in the beginning of period 2. Before we construct an equilibrium we make a few claims about every equilibrium:

1. *In any equilibrium all periods are reached with positive probability.* This is a standard claim, like in Proposition 1.

2. *In period 2 the buyers cannot play a pure strategy.* Suppose that they do and the highest price is w_2. First, it has to be the case that in equilibrium some types accept w_2 : otherwise offering $w_2' = \inf\{v : F\,(v) > 0\} - \frac{\widetilde{c}}{2}$ earns strictly positive profits. Second, consider w_2. The types that accept it are $v \leq w_2 + 0.1$. For all those types the reservation price at period 1 is $w_2 - 0.1$ and for all higher types it is $v - 0.2$. Offering $w_1 = w_2 - 0.1$ in the first period the buyers attract average type of $\frac{1 + w_2 + 0.1}{2}$. For that to be not profitable (otherwise nobody would accept w_2) we get $w_1 \geq 1.3$. If $w_1 > 1.3$ then any price loses money in the first

period, so F is uniform $[0,1]$ and therefore the unique equilibrium in the sub-game starting in the second period is for both buyers to offer $w_2 = 1.1$, a contradiction.

Finally, consider $w_2 = 1.3$ Now, in the first period any price below 1.2 does not attract any seller. Any price above 1.2 loses money. So the only candidate for price in period 1 is $w_1 = 1.2$. For that to be an equilibrium the average type accepting w_1 has to be at least 1.2. But that leads to the final contradiction: The average type accepting price 1.3 can be at most as high as 1.2, so the buyers would lose money in the second period.

3. *The posterior belief F cannot be a truncated uniform.* Suppose it is, with the lowest type that the buyers assign positive density to denoted by $\underline{v}$. Then the unique equilibrium in the second stage is for the buyers to offer price $w_2 = \underline{v} + 0.1$. But that contradicts claim 2. (Directly, the average price in period 1 would have to be $w_1 = \frac{1+\underline{v}}{2}$, but then all types below $\underline{v}$ would prefer to wait for $w_2 = \underline{v} + 0.1$ as $\underline{v} > \frac{1+\underline{v}}{2}$. Of course $\underline{v} = 1$ does not work either, as then the buyers could make profit by offering $w_1 = 1.1 - \varepsilon$ in the first period).

4. *In the first period the buyers cannot play a pure strategy.* This claim is the main difference from the equilibria in the two-type case: the claims 1 and 2 held there as well and claim 3 did not have an analogue. But this claim states that unlike the two-type game, with continuum of types the buyers need to randomize already in the first period.

   The proof goes as follows: Denote the price in period 1 by w_1. Clearly in equilibrium it has to be equal to the reservation price for at least one type. If it is equal to the reservation price of exactly one type, then we get F being a truncated uniform, which leads to a contradiction by claim 3. Suppose that there is a mass of types for which w is equal to the reservation price. These types would exit with probability 1 in period 2 if they stayed till then (otherwise their reservation prices would be strictly increasing in type). So it has to be a mass of types in a range $[1, \underline{v}]$. Clearly, by offering w_1 the buyers have to obtain at least average selection of those types (otherwise they could deviate to $w_1 + \varepsilon$ and get strictly higher profits). Suppose that all those types accept. Then F is a truncated uniform, a contradiction.

Now consider type $\underline{v}$. Given that this is the highest type that accepts at time 2 with probability 1, with positive probability there must be trade at $w_2^{\min} = \underline{v} - 0.1$. Now consider if this price can indeed be offered in equilibrium. We can write the expected type that accepts price $w_2 > w_2^{\min}$ at time 2 as:

$$E(v|w_2) = \frac{\int_1^{\underline{v}} f(v)\, vdv + \int_{\underline{v}}^{w_2+0.1} \frac{1-F(\underline{v})}{2-\underline{v}} vdv}{F(w_2+0.1)}$$

Taking the derivative with respect to w_2 at $w_2^{\min}$ get:

$$\frac{\partial E\left(v|w_2 = w_2^{\min}\right)}{\partial w_2} = \frac{\frac{1-F(\underline{v})}{2-\underline{v}}(\underline{v})}{F(\underline{v})} > 1$$

Therefore, increasing price slightly above $w_2^{\min}$ gives strictly higher profits. That contradicts trade at $w_2^{\min}$ in equilibrium.

Summing up, we need to have the buyers randomizing in every period and the posterior F not to be a truncated uniform distribution. Once we know that the equilibrium cannot be 'simple' we are ready to construct an example of an equilibrium.

Suppose that at period 2 buyers randomize with equal probabilities between prices $w_2 \in \{1.1, 1.3\}$ (i.e. that the probability that the highest price is 1.3 is $\frac{1}{2}$). Then the reservation prices at time 0 are:

$$\begin{aligned} R(0,v) &= 1.1 \text{ for types } v \in [1, 1.2] \\ &= \frac{1}{2}(1.3) + \frac{1}{2}(v - 0.1) - 0.1 = \frac{1}{2}(v+1) \text{ for types } v \in [1.2, 1.4] \\ &= v - 0.2 \text{ for types } v \in [1.4, 2] \end{aligned}$$

That implies that a price $w_1 = 1.1$ if accepted by average quality of types with $R(v) = 1.1$ gives zero profit. Similarly, any $w_1 \in (1.1, 1.2]$ gives zero profit. Finally, any price above 1.2 loses money. Therefore, we can have the buyers mix in period 1 with any probabilities over prices in the range $[1.1, 1.2]$ and if they offer $w = 1.1$ we can have the mass of types $v \in [1.1, 1.2]$ randomize between accepting and rejecting.

In this way we can construct $F(v)$ that has density that is constant over a range $[1, 1.2]$, increasing over the range $[1.2, 1.4]$ and then constant (and higher than for any smaller v) for $v \in [1.4, 2]$.

Now, the key is to consider the behavior at period 2 : in particular we need to verify if given $F(v)$ it is indeed optimal for the buyers to randomize between the postulated prices. The crucial element of the analysis is the function describing the average type accepting offer w_2 given the posterior F :

$$E_F(v|w_2) = \int_1^{w_2+0.1} \frac{v}{F(w_2+0.1)} dF(v)$$

As $F(v)$ is uniform on [1.1.2] we have $E_F(v|w_2 = 1.1) = 1.1$ so the price 1.1 makes zero profit. We also have to check that the posterior F satisfies:

$$E_F(v|w_2) \leq w_2$$

for all $w_2 \geq 1.1$ and

$$E_F(v|w_2 = 1.3) = 1.3$$

It turns out that given the flexibility we have in F, we can construct F that satisfies these two conditions. An example is:

$$
\begin{aligned}
f(v) &= \alpha \text{ for } v < 1.2 \\
f(v) &= \beta\left(v - \frac{12}{10}\right)^2 + \alpha \text{ for } v \in (1.2, 1.3) \\
f(v) &= \left(av^2 + bv + c\right) \text{ for } v \in (1.3, 1.4) \\
f(v) &= \frac{1}{20}\gamma + \frac{7}{50}\beta - \frac{3}{25} + \alpha \text{ for } v > 1.4
\end{aligned}
$$

with

$$\alpha = \frac{1749}{14\,780},\ \beta = \frac{6630}{739},\ a = -\frac{77\,322}{739},\ b = \frac{1082\,508}{3695},\ \gamma = \frac{16\,026}{11\,135}$$

Of course, there are many other examples, and they correspond to different average prices in the first period.

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