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Recall and private monitoring ☆

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ABSTRACT

For a general class of games with *private monitoring* we show for any finite state strategy (or automaton strategy) with D_i states for players $i \in \{1, \ldots, N\}$, if there exists a number of periods t such that it is possible on-path to reach any joint state from any joint state in t periods, the strategy is a strict correlated equilibrium only if each player's strategy is a function only of what the player observes in the last $D_i - 1$ periods.

1. Introduction

This paper considers a general class of games with *private monitoring* and considers to what extent strategies which depend on outcomes or actions long in the past can be equilibria. The general character of our main results is that strategies which depend on long-ago outcomes or actions are either not equilibria, or "fragile" equilibria which depend on indifference. Examples of such strategies commonly used in games with *public* monitoring include the stick-and-carrot equilibrium in Abreu et al. (1986) and the grim trigger strategy in prisoners-dilemma type games.

Our model considers general games of *private monitoring* where for all actions, all possible private signal profiles occur with positive probability. Further, as in Phelan and Skrzypacz (2012), we limit our analysis to strategies that can be represented by finite automata. For instance, a stick-and-carrot strategy for a quantity setting oligopoly game could have two private states: Punish (where the player chooses a high quantity) and Reward (where the player chooses a low quantity), and where the player transits between Punish and Reward depending on his privately observed price signal. Our equilibrium notion is correlated sequential equilibrium and we ask whether equilibria can be uniformly strict (USCSE), which means that the incentive constraints are satisfied by an amount $\delta > 0$ uniformly for all possible histories. We use correlated sequential equilibrium, as opposed to simply sequential equilibrium, solely for the sake of generality, since sequential equilibria are a special case of the correlated sequential equilibria we consider here.

Our first main result, Proposition 1, states that if the set of possible beliefs a player can have about the state of his opponents while being in one state of his automaton "overlaps" with the corresponding set while being in another state

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of his automaton, then his strategy cannot be a part of a USCSE. The idea behind this result is that the optimality of a player's continuation strategy depends only on his beliefs regarding the continuation strategy of his opponents. If a player's belief sets associated with two distinct states overlap, then there must be two histories, one which puts the player in the first state and another which puts him in the second, which induce (almost) the same beliefs regarding the state of his opponents. Given this, his incentives cannot hold strictly (by an amount uniformly bounded away from zero). In the body of the paper, we show that this result alone is enough to explain why grim-trigger cannot support cooperation (with uniformly strict incentives) in a private monitoring version of the prisoner's dilemma.

In Lemma 1 we show that a strategy represented by a D_i-state automaton has either infinite recall or $D_i - 1$ period (or less) period recall (i.e. the action played after any history depends only on what player i observed and has done in the last $D_i - 1$ periods). This is a strengthening of a result in Mailath and Morris (2006).

Finally, Proposition 2 states that if the equilibrium path transition matrix over joint private states of the players is *regular*, (there exists a number of periods t such that it is possible on-path to get from any joint state to any joint state in t periods) then every player i's D_i state automaton must have at most $D_i - 1$ period recall, or the profile of strategies does not form a USCSE, regardless of starting conditions. That theorem applies for example to the stick and carrot type of strategies which describe the best (and worst) equilibria in the public monitoring game of Abreu et al. (1986).

This requirement that the transition matrix of the joint automaton is regular is not necessarily crucial. Many strategy profiles which do not induce regular transition matrices nevertheless share the property that beliefs after long histories converge at least for the relevant priors, as evidenced by the grim trigger example in Section 3.2. The grim trigger automaton does not yield a regular transition matrix since the punish state is absorbing. Yet, if a player starts with an interior belief about his opponent's state, always cooperates and observes a long history of good outcomes, his beliefs will converge to the same interior point no matter what he observed early in the game. As a result, we can find two histories after which player this player has arbitrarily close beliefs about the state of his opponent and yet his strategy calls for different actions. This more general observation is captured by Proposition 1, implying that grim trigger is not a USCSE.

1.1. *Relation to previous literature*

This paper contributes to the literature studying strategies instead of payoffs in repeated games with private monitoring. The most closely related paper to ours is Mailath and Morris (2006). In that paper they show that an infinite recall strategy that is a strict perfect public equilibrium of a public monitoring game is no longer a Nash equilibrium if the monitoring is perturbed to be almost-public and rich. There are several differences between our results and theirs. Our results are more limited because we consider uniformly strict sequential equilibria rather than Nash equilibria. On the other hand our results are stronger since, other than full support, we put no conditions on the structure of the monitoring technology (it does not, for instance, need to be almost-public) and we do not require that the profile be an equilibrium (strict or not) of any public monitoring game.

The paper is also suggestive of why existing folk theorems for games with private monitoring either use belief-free strategies that necessarily involve indifference (see for example Ely and Välimäki, 2002, and Ely et al., 2005) or finite recall strategies (see for example Hörner and Olszewski, 2009 or Mailath and Olszewski, 2011). Our results suggest that other equilibria either don't exist, or involve infinite state strategies.[1]

2. The model

The underlying model is that same as Phelan and Skrzypacz (2012). The game, Γ^∞, is defined by the infinite repetition of a stage game, Γ, with N players, $i = 1, \ldots, N$, each able to take actions $a_i \in A_i$. With probability $P(y|a)$, a vector of private outcomes $y = (y_1, \ldots, y_N)$ (each $y_i \in Y_i$) is observed conditional on the vector of private actions $a = (a_1, \ldots, a_N)$, where for all (a, y), $P(y|a) > 0$ (*full support*). The sets $A = A_1 \times \ldots \times A_N$ and $Y = Y_1 \times \ldots \times Y_N$ are both assumed to be finite sets. Let $H_i = A_i \times Y_i$.

The current period payoff to player i is denoted $u_i : H_i \to R$. If player i receives payoff stream $\{u_{i,t}\}_{t=0}^{\infty}$, his lifetime discounted payoff is $(1-\beta)\sum_{t=0}^{\infty} \beta^t u_{i,t}$ where $\beta \in (0, 1)$. As usual, players care about the expected value of lifetime discounted payoffs.

Let $h_{i,t} = (a_{i,t}, y_{i,t})$ denote player i's private action and outcome at date $t \in \{0, 1, \ldots\}$, and $h_i^t = (h_{i,0}, \ldots, h_{i,t-1})$ denote player i's private history up to, but not including, date t. A pure (behavior) *strategy* for player i, $\sigma_i = \{\sigma_{i,t}\}_{t=0}^{\infty}$, is then, for each date t, a mapping from player i's private history h_i^t, to his action $a_i \in A_i$ in period t. Let σ denote the joint strategy $\sigma = (\sigma_1, \ldots, \sigma_N)$ and σ_{-i} denote the joint strategy of all players other than player i, or $\sigma_{-i} = (\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_N)$. (Throughout the paper we use notation $-i$ to refer to all players but player i.)

2.1. *Finite automaton strategies*

As in Phelan and Skrzypacz (2012), we describe strategies in terms of finite automata. (Representing strategies as automata is without loss. The finiteness assumption is restrictive.) Here, since we later restrict discussion to strict equilibria,

[1] See Ely (2002) and Kandori and Obara (2010) for examples of infinite state equilibria.

we also limit discussion to pure strategies. In this language, a strategy for player i is defined by four objects: 1) a private state space Ω_i (with $D_i < \infty$ *distinct* states ω_i), 2) a function $f_i(\omega_i)$ giving the (pure) action a_i for each private state $\omega_i \in \Omega_i$, 3) a deterministic transition function $\omega_i^+ : \Omega_i \times H_i \to \Omega_i$ determining next period's private state as a function of this period's private state, player i's private action a_i, and his private outcome y_i, and 4) an initial state, $\omega_{i,0}$. Two states ω_i^1 and ω_i^2 are considered distinct if there exists h_i^t such that $f_i(\omega_i^+(\omega_i^1, h_i^t)) \neq f_i(\omega_i^+(\omega_i^2, h_i^t))$, where function $\omega_i^+(\omega_i, h_i^t)$ is defined as the state reached from ω_i after history h_i^t implied by function $\omega_i^+(\omega_i, h_i)$. In words, even if the two states recommend the same action today, there must exist some subsequent history after which the recommended actions differ. Given this setup, the induced strategy $\sigma_{i,0} = f_i(\omega_{i,0})$, $\sigma_{i,1}(h_{i,0}) = f_i(\omega_i^+(\omega_{i,0}, h_{i,0}))$ and so on. Let $\psi_i = (\Omega_i, f_i, \omega_i^+)$ denote agent i's *automaton*. The collection of automata over all players $\psi \equiv \{\psi_1, \ldots, \psi_N\}$ is referred to as the *joint automaton*. Finally, let the number of joint states $D = \Pi_{i \le N} D_i$, and the number of joints states for players other than player i, $D_{-i} = \Pi_{j \neq i} D_j$.

2.2. Beliefs

Allow player i's initial beliefs over the initial state of his opponents, $\omega_{-i,0}$, to be possibly nondegenerate. In particular, let player i's beliefs about the initial state of his opponents, $\mu_{i,0}$, be a point in the $(D_{-i} - 1)$-dimensional unit-simplex, denoted $\Delta^{D_{-i}}$. Taking as given $\mu_{i,0}$, the assumption of full support ($P(y|a) > 0$ for all (a, y)) implies that the beliefs of player i regarding his opponents' private histories, h_{-i}^t, are always pinned down by Bayes' rule. For a particular initial belief, $\mu_{i,0}$, and private history, h_i^t, player i's belief over $\omega_{-i,t}$ is, like $\mu_{i,0}$, simply a point in the $(D_{-i} - 1)$-dimensional unit-simplex. Let $\mu_{i,t}(\mu_{i,0}, h_i^t)$ denote player i's belief at the beginning of period t about $\omega_{-i,t}$ after private history h_i^t given initial beliefs $\mu_{i,0}$. Let $\mu_{i,t}(\mu_{i,0}, h_i^t)(\omega_{-i})$ denote the probability assigned to the particular state ω_{-i}. Finally, let $\mu_{i,0}(x, \omega_{i,0})$ denote player i's beliefs at date $t = 0$ when the initial joint state ω_0 is determined by correlation device x and player i's initial state is $\omega_{i,0}$.

Beliefs $\mu_{i,t}(\mu_{i,0}, h_i^t)$ can be defined recursively using Bayes' rule. Let $B_i(m_i, h_i|\psi_{-i}) \in \Delta^{D_{-i}}$ denote the belief of player i over the state of his opponents at the beginning of period t, if his beliefs over his opponents' state at period $t-1$ were $m_i \in \Delta^{D_{-i}}$ and he subsequently observed $h_i = (a_i, y_i)$. To define beliefs recursively, let $B_i^s(m_i, h_i^s|\psi_{-i}) = B_i(B_i^{s-1}(m_i, h_i^{s-1}|\psi_{-i}), h_{i,s-1}|\psi_{-i})$ where $B_i^1(m_i, h_i|\psi_{-i}) = B_i(m_i, h_i|\psi_{-i})$. Then, $\mu_{i,t}(\mu_{i,0}, h_i^t) = B_i^t(\mu_{i,0}, h_i^t|\psi_{-i})$.

2.3. Equilibrium

When σ_i is expressed as an initial state-automaton pair $(\omega_{i,0}, \psi_i)$, we can write player i's lifetime payoff, conditional on ω_{-i}, as a function of his current private state ω_i (as opposed to depending directly on his private history, h_i^t). That is,

$$v_i(\omega_i, \omega_{-i}|\psi_i, \psi_{-i}) = \sum_y P(y|f_i(\omega_i), f_{-i}(\omega_{-i}))[(1-\beta)u_i(f_i(\omega_i), y_i) \\ + \beta v_i(\omega_i^+(\omega_i, f_i(\omega_i), y_i), \omega_{-i}^+(\omega_{-i}, f_{-i}(\omega_{-i}), y_{-i})|\psi_i, \psi_{-i})].$$

Then we denote player i's expected payoff, now a function of his current state, ω_i, and his beliefs over his opponents' state, ω_{-i}, as

$$Ev_i(\omega_i, m_i|\psi_i, \psi_{-i}) = \sum_{\omega_{-i}} m_i(\omega_{-i}) v_i(\omega_i, \omega_{-i}|\psi_i, \psi_{-i}).$$

Definition 1. A probability distribution over initial states, $x \in \Delta^D$, and joint automaton, ψ, form a *Uniformly Strict Correlated Sequential Equilibrium* (USCSE) of Γ^∞ if there exists $\delta > 0$ such that for all i, h_i^t, $\omega_{i,0}$ such that $\sum_{\omega_{-i,0}} x(\omega_{i,0}, \omega_{-i,0}) > 0$, and $\hat{a}_i \neq f_i(\omega_i^+(\omega_{i,0}, h_i^t))$,

$$Ev_i(\omega_i^+(\omega_{i,0}, h_i^t), \mu_{i,t}(\mu_{i,0}(x, \omega_{i,0}), h_i^t)|\psi_i, \psi_{-i}) - \delta \\ \ge \sum_{\omega_{-i}} \mu_{i,t}(\mu_{i,0}(x, \omega_{i,0}), h_i^t)(\omega_{-i})[\sum_y P(y|\hat{a}_i, f_{-i}(\omega_{-i})) \\ [(1-\beta)u_i(\hat{a}_i, y_i) + \beta v_i(\omega_i^+(\omega_{i,0}, (h_i^t, \hat{a}_i, y_i)), \omega_{-i}^+(\omega_{-i}, f_{-i}(\omega_{-i}), y_{-i})|\psi_i, \psi_{-i})]].$$

3. Analysis of equilibria

In this section we first provide necessary conditions for a correlation device over initial states-joint automaton pair, (x, ψ), to be a Uniformly Strict Correlated Sequential Equilibrium (Proposition 1). We next provide necessary conditions on the joint automaton, ψ, alone regarding whether there exists any correlation device, (x, ψ) such that (x, ψ) form a USCSE (Proposition 2).

3.1. Belief sets

In Phelan and Skrzypacz (2012), rather than considering separately the beliefs player i could have about the state of his opponents after each particular history, we showed it useful to consider *sets* of beliefs associated with each state of player i's automaton ψ_i. In particular, let $M_i(\omega_i) \subset \Delta^{D_{-i}}$ denote a closed, convex, set of beliefs, and M_i be a collection of D_i sets $M_i(\omega_i)$, one for each ω_i. Let the one-step operator $T(M_i)$ be defined as

$$T(M_i) = \{T(M_i)(\omega_i')|\omega_i' \in \Omega_i\}$$

where

$$T(M_i)(\omega_i') = \text{co}(\{m_i'|\text{ there exists } \omega_i \in \Omega_i, m_i \in M_i(\omega_i) \text{ and } h_i \in H_i \text{ such that } \omega_i^+(\omega_i, h_i) = \omega_i' \\ \text{and } m_i' = B_i(m_i, h_i|\psi_{-i})\}),$$

where co() denotes the convex hull.

Next, let the operator $T^U(M_i)$ (^{U} for union) be:

$$T^U(M_i) = \{T^U(M_i)(\omega_i)|\omega_i \in \Omega_i\} \text{ where } T^U(M_i)(\omega_i) = \text{co}(T(M_i)(\omega_i) \cup M_i(\omega_i)).$$

In words, the T operator calculates for every state ω_i', the convex hull of all the posterior beliefs player i can hold in this state if his prior beliefs, state by state, are required to belong to M_i. The T^U operator is the convex hull of the union, state by state, of the original set of priors, $M_i(\omega_i)$ and $T(M_i)(\omega_i)$.

Our results rely on properties of the fixed points of T and T^U. We write $M_i^0 \subset M_i^1$ if $M_i^0(\omega_i) \subset M_i^1(\omega_i)$ for all ω_i. Furthermore, M_i is non-empty if there exists a private state ω_i such that $M_i(\omega_i)$ is non-empty.

Both T and T^U are monotonic operators (that is, if $M_i^0 \subset M_i^1$, then $T(M_i^0) \subset T(M_i^1)$ and $T^U(M_i^0) \subset T^U(M_i^1)$). By construction, $M_i \subset T^U(M_i)$. Since $M_i \subset T^U(M_i)$, belief sets are bounded, and $T^U(M_i) \subset T^U(T^U(M_i))$ (from monotonicity), the sequence $\{M_i, T^U(M_i), T^U(T^U(M_i)), \ldots\}$ converges. The continuity of B_i implies that T^U is continuous and thus the closure of this limit is a fixed point of T^U. Call this fixed point $M_i^{*U}(M_i)$. Next note that if $M_i \subset T(M_i)$, then $T(M_i) = T^U(M_i)$. This implies that if $M_i \subset T(M_i)$, the sequence $\{M_i, T(M_i), T(T(M_i)), \ldots\}$ also converges and the closure of its limit equals $M_i^{*U}(M_i)$.

For an arbitrary correlation device, x, let the belief sets $M_{i,0}(x)$ be defined such that

$$M_{i,0}(x)(\omega_i) = \{\mu_{i,0}(x, \omega_i)\}$$

for all ω_i such that $\sum_{\omega_{-i}} x(\omega_i, \omega_{-i}) > 0$. Otherwise, let $M_{i,0}(x)(\omega_i) = \emptyset$. That is, for all ω_i, if ω_i occurs with positive probability under distribution x, $M_{i,0}(x)(\omega_i)$ is the single point belief set consisting of what player i believes about ω_{-i} when his initial state is ω_i. With some abuse of notation, we define $M_i^{*U}(x) \equiv M_i^{*U}(M_{i,0}(x))$.

The set $M_i^{*U}(x)(\omega_i)$ is the convex hull of all the beliefs player i can have regarding the state of his opponents, for all dates and on and off path histories, where his current state is ω_i. Our first main result, Proposition 1, states that if these sets for two different states "overlap", then the implied strategy for player i is not strictly optimal and thus (x, ψ) is not a USCSE.

Proposition 1. *For an arbitrary correlation device over initial states x and joint automaton ψ, suppose there exist a player i and two distinct states $(\omega_i^1, \omega_i^2) \in \Omega_i^2$ such that $M_i^{*U}(x)(\omega_i^1) \cap M_i^{*U}(x)(\omega_i^2) \neq \emptyset$. Then (x, ψ) does not constitute a Uniformly Strict Correlated Sequential Equilibrium.*

Proof. Suppose for a particular correlation device x, (x, ψ) is a Correlated Sequential Equilibrium (but not necessarily uniformly strict). In Phelan and Skrzypacz (2012) (Theorem 1) we showed that a necessary condition for (x, ψ) to be a Correlated Sequential Equilibrium is that for all $\omega_i \in \Omega_i$, for all $\hat{a}_i \in A_i$ and all m_i such that m_i is an extreme point of $M_i^{*U}(x)(\omega_i)$

$$Ev_i(\omega_i, m_i|\psi_i, \psi_{-i}) \geq \sum_{\omega_{-i}} m_i(\omega_{-i}) \sum_y P(y|\hat{a}_i, f_{-i}(\omega_{-i})) \\ [(1-\beta)u_i(\hat{a}_i, y_i) + \beta v_i(\omega_i^+(\omega_i, \hat{a}_i, y_i), \omega_{-i}^+(\omega_{-i}, f_{-i}(\omega_{-i}), y_{-i})|\psi_i, \psi_{-i})]. \quad (1)$$

Since this is true for all extreme points of $M_i^{*U}(x)(\omega_i)$, it is true of all interior points as well. Thus (1) holds for both ω_i^1 and ω_i^2 for m_i equal to any element of $M_i^{*U}(x)(\omega_i^1) \cap M_i^{*U}(x)(\omega_i^2)$. This implies that given belief m_i in the non-empty intersection, action $f_i(\omega_i^1)$ is weakly preferred to action $f_i(\omega_i^2)$ and vice-versa, or that player i is indifferent at each state between playing $f_i(\omega_i^1)$ and $f_i(\omega_i^2)$ for any beliefs in $M_i^{*U}(x)(\omega_i^1) \cap M_i^{*U}(x)(\omega_i^2)$.

Next, that (1) holds for all extreme points of $M_i^{*U}(x)(\omega_i^1)$ implies player i weakly prefers playing action $f_i(\omega_i^1)$ over action $f_i(\omega_i^2)$ at these extreme points. But since payoffs for a given action are linear in beliefs, that player i is indifferent between action $f_i(\omega_i^1)$ and action $f_i(\omega_i^2)$ for an interior belief in the non-empty intersection $M_i^{*U}(x)(\omega_i^1) \cap M_i^{*U}(x)(\omega_i^2)$, he must also be indifferent for at least two extreme points of $M_i^{*U}(x)(\omega_i^1)$. (The same argument implies player i must be indifferent between $f_i(\omega_i^1)$ and $f_i(\omega_i^2)$ for at least two extreme points of $M_i^{*U}(x)(\omega_i^2)$ as well.)

In Phelan and Skrzypacz (2012) we also showed that for each extreme point of $M_i^{*U}(x)(\omega_i^1)$ and $M_i^{*U}(x)(\omega_i^2)$, there exists a private history for player i such that his beliefs are arbitrarily close to the extreme point. Since payoffs are continuous in beliefs, if $f_i(\omega_i^1) \neq f_i(\omega_i^2)$, the result is proved. (If player i is indifferent at the extreme point, there cannot exist a $\delta > 0$ such that player i's payoff from following action $f_i(\omega_i^1)$ exceeds his payoff from following action $f_i(\omega_i^2) \neq f_i(\omega_i^1)$ by at least δ for all histories.)

If $f_i(\omega_i^1) = f_i(\omega_i^2)$, then there must exist h_i^t such that $f_i(\omega_i^+(\omega_i^1, h_i^t)) \neq f_i(\omega_i^+(\omega_i^2, h_i^t))$ (or else ω_i^1 and ω_i^2 are not distinct states) which itself implies $\omega_i^+(\omega_i^1, h_i^t) \neq \omega_i^+(\omega_i^2, h_i^t)$. For this new posterior belief ($m_i' = B^t(m_i, h_i^t|\psi_{-i})$ for any $m_i \in M_i^{*U}(x)(\omega_i^1) \cap M_i^{*U}(x)(\omega_i^2)$), Eq. (1) must hold for both distinct states $\omega_i^+(\omega_i^1, h_i^t)$ and $\omega_i^+(\omega_i^2, h_i^t)$. The same argument can then be applied. □

The intuition behind Proposition 1 is that if, for a given player, his belief sets over the state of his opponents overlap, then there must exist two histories which induce (almost) the same beliefs and yet different continuation strategies (because each state is associated with a distinct continuation strategy). However, since the set of best replies depends only on the player's beliefs, and payoffs are continuous in beliefs, incentives cannot be uniformly strict. The example in the next section illustrates the need for *uniform* strictness.

3.2. An application of Proposition 1

Proposition 1 can be used to show that it is impossible for a grim-trigger strategy, as a uniformly strict correlated sequential equilibrium, to sustain cooperation in any period in a two action, two outcome private signal prisoner's dilemma game. Further, in this section we show that cooperation can be sustained as part of a strict (but not uniformly strict) correlated sequential equilibrium for this game, although in a non-generic example.

To be more specific, consider the partnership game from Phelan and Skrzypacz (2012) (itself from Mailath and Morris, 2002). Each player $i \in \{1, 2\}$ can take action $a_i \in \{C, D\}$ (cooperate or defect) and each can realize a private outcome $y_i \in \{G, B\}$ (good or bad). The $P(y|a)$ function and stage game payoffs $\{u_i(C, G), u_i(C, B), u_i(D, G), u_i(D, B)\}$ are chosen so that in terms of expected payoffs, the stage game is a prisoner's dilemma. We assume $P(G, G|C, C) + P(G, B|C, C) > P(G, G|C, D) + P(G, B|C, D)$ (or that given that player 1 cooperates, he has a higher probability of a good outcome if player 2 cooperates as well).

Let grim trigger denote a two state ($\omega_i \in \{R, P\}$) automaton where each player plays C in state R, D in state P, state P is absorbing, and a necessary condition to staying in state R is observing outcome G. We say that grim trigger supports cooperation if the initial correlation device, x, puts positive probability on the initial joint state $\omega = (R, R)$. (It is immediate that if x puts positive probability on (R, P) or (P, R) but not (R, R), then the strategy cannot be an equilibrium).

To see that cooperation is impossible as part of a USCSE, consider the particular action/outcome pair for player 1, $h_1 = (C, G)$. The Bayes updating function given (C, G) is

$$B_1(m_1, (C, G)\,|\psi_2) = \frac{m_1\ P\,(G, G|C, C)}{m_1\,(P\,(G, G|C, C) + P\,(G, B|C, C)) + (1 - m_1)\,(P\,(G, G|C, D) + P\,(G, B|C, D))}, \tag{2}$$

where m_1 is player 1's belief that player 2 is in state R. Since $B_1(0, (C, G)\,|\psi_2) = 0$, there is clearly a fixed point at zero. Next, $\frac{d^2 B_1(m_1,(C,G)|\psi_2)}{dm_1^2} < 0$ as long as $P(G, G|C, C) + P(G, B|C, C) > P(G, G|C, D) + P(G, G|C, D)$ as is assumed. Thus $B_1(m_1, (C, G)\,|\psi_2)$ is a continuous, increasing, concave function of m_1. Further, $B_1(1, (C, G)\,|\psi_2) < 1$. Thus the function has at least one fixed point (at zero) and at most two fixed points, depending on whether its first derivative at $m_1 = 0$ is greater than one.

If $B_1(m_1, (C, G)\,|\psi_2)$ has only a single fixed point at zero, then $B_1(m_1, (C, G)\,|\psi_2) < m_1$ and cooperation is never possible. In this case, if the correlation device, x, puts any probability for player 1 starting in state R, then $M_1^{*U}(x)(R)$ is non-empty and contains the point $m_1 = 0$ (since one possible private history for player 1 is (C, G) repeated indefinitely). Thus a necessary condition for the strategy to be a USCSE is that player 1 prefers to cooperate when he is certain that player 2 is in state P, which is not compatible with the stage game being a prisoner's dilemma.

If $B_1(m_1, (C, G)\,|\psi_2)$ has another fixed point at, say, $m^* > 0$, then if $0 < m_1 < m^*$, $B_1(m_1, (C, G)\,|\psi_2) > m_1$, and if $m_1 > m^*$, $B_1(m_1, (C, G)\,|\psi_2) < m_1$. Thus if x puts positive probability on the joint state (R, R), $M_1^{*U}(x)(R)$ must contain m^*. (Again since one possible private history for player 1 is (C, G) repeated indefinitely.) But $M_1^{*U}(x)(P)$ must also contain m^*, since another possible history is player 1 sees (C, B) in the first period and then (C, G) in each subsequent period. Proposition 1 then applies.

This example can also be used to illustrate our need for incentives to hold uniformly strictly (as opposed to simply strictly). We have constructed an example (for details see Appendix A) where parameters are chosen (non-generically) such that player 1 is indifferent between actions C and D at the non-zero fixed point of $B_1(m_1, (C, G) | \psi_2)$, m^*. Further, in our example, the initial correlation device x is chosen such that if player 1 starts in state R and repeatedly observes (C, G), his belief m_1 approaches this fixed point from above. Thus at each date he strictly prefers to play his path action C (but not uniformly so since he is indifferent at m^*). However, if he ever observes outcome B, and is thus in state P and he subsequently forever observes (C, G), his beliefs approach this fixed point from below. Thus $M_1^{*U}(x)(P) = [0, m^*]$ and $M_1^{*U}(x)(R) = [m^*, M_{1,0}(x)(R)]$. Incentives hold at the extreme points of each set (but with indifference at m^*) and thus (x, ψ) is a Correlated Sequential Equilibrium.

4. Recall

In this section we ask to what extent *recall* determines whether for a given joint automaton, ψ, there exists an initial correlation device, x, such that (x, ψ) is a USCSE. First we prove that if an automaton ψ_i has D_i states, it must either recall only the last $D_i - 1$ outcomes of h_i, or depend on the entire history of h_i's as well as the initial state ω_i (Lemma 1). Next, we prove two further lemmas which together with Lemma 1 imply our second main result: if the on-path transition matrix over joint states is *regular*, then (x, ψ) is a USCSE only if for all players i, ψ_i "looks back" at most $D_i - 1$ periods. (A transition matrix τ is regular if there exists an s such that τ^s has all non-zero elements. This implies there is a positive probability of transitioning from every state to every state in exactly s periods.)

Definition 2. An automaton ψ_i is said to have t period recall if t is the smallest non-negative integer such that for all private state pairs $(\omega_i^1, \omega_i^2) \in \Omega_i^2$ and all t length private histories, $h_i^t = (h_{i,0}, \ldots, h_{i,t-1})$, $\omega_i^+(\omega_i^1, h_i^t) = \omega_i^+(\omega_i^2, h_i^t)$. (After t periods, the original state is "forgotten", so the automaton has t period recall.)

Definition 3. An automaton ψ_i is said to have infinite recall if it does not exhibit t period recall for any finite t.

Lemma 1. *Suppose automaton ψ_i has $D_i < \infty$ states. Then ψ_i either exhibits $D_i - 1$ or less period recall, or exhibits infinite recall. (It cannot exhibit finite $t \geq D_i$ period recall.)*

Proof. First consider a two state automaton ψ_i with $\Omega_i = \{\omega_i^1, \omega_i^2\}$. If for any one-period private realization h_i, $\omega_i^+(\omega_i^1, h_i) = \omega_i^1$ and $\omega_i^+(\omega_i^2, h_i) = \omega_i^2$, or $\omega_i^+(\omega_i^1, h_i) = \omega_i^2$ and $\omega_i^+(\omega_i^2, h_i) = \omega_i^1$, then the automaton has infinite recall. Thus if ψ_i has finite recall, for all h_i, $\omega_i^+(\omega_i^1, h_i) = \omega_i^+(\omega_i^2, h_i)$, which implies the two state automaton has at most $t = 1$ recall.

Next assume the result is proved for all D_i state automata and consider a $D_i + 1$ state automaton with states Ω_i with, at first, finite recall. That the $D_i + 1$ state automaton has finite recall implies there must exist a pair of states $(\omega_i^1, \omega_i^2) \in \Omega_i^2$ such that for all h_i, $\omega_i^+(\omega_i^1, h_i) = \omega_i^+(\omega_i^2, h_i)$. (If for all $(\omega_i^1, \omega_i^2) \in \Omega_i^2$ there exists an h_i such that $\omega_i^+(\omega_i^1, h_i) \neq \omega_i^+(\omega_i^2, h_i)$, then from any two initial states, one could choose an infinite sequence $h_i^\infty = (h_i, h_{i,1}, \ldots)$ such that $\omega_i^+(\omega_i^1, h_i^t) \neq \omega_i^+(\omega_i^2, h_i^t)$ for all t, and thus the automaton would have infinite recall.) Consider a new automaton that has D_i states $\hat{\Omega}_i$ by combining states $\omega_i^1 \in \Omega_i$ and $\omega_i^2 \in \Omega_i$ into state $\hat{\omega}_i \in \hat{\Omega}_i$. That is, in the new automaton, $\omega_i^+(\hat{\omega}_i, h_i) = \omega_i^+(\omega_i^1, h_i) = \omega_i^+(\omega_i^2, h_i)$ for all h_i, and for all $\omega_i \in \Omega_i \cap \hat{\Omega}_i$ and all h_i, if $\omega_i^+(\omega_i, h_i) = \omega_i^1$ or $\omega_i^+(\omega_i, h_i) = \omega_i^2$ then $\omega_i^+(\omega_i, h_i) = \hat{\omega}_i$.

Since this new automaton has D_i states, it must have $t \leq D_i - 1$ recall. (Since the original had finite recall, the new automaton cannot have infinite recall.) Thus for all $h_i^{D_i} = (h_i, \ldots, h_{i,D_i-1})$ and all $(\omega_i^1, \omega_i^2) \in \hat{\Omega}_i^2$, $\omega_i^+(\omega_i^1, h_i^{D_i}) = \omega_i^+(\omega_i^2, h_i^{D_i})$ for the new D_i state automaton. Thus for all $(\omega_i^1, \omega_i^2, h_i^{D_i})$ combinations such that $\omega_i^+(\omega_i^1, h_i^{D_i}) = \omega_i^+(\omega_i^2, h^{D_i}) \neq \hat{\omega}_i$, after D_i periods, both the original $D_i + 1$ state automaton and the new D_i state automaton have "forgotten" the original state. If under the new D_i state automaton $\omega_i^+(\omega_i^1, h_i^{D_i}) = \omega_i^+(\omega_i^2, h_i^{D_i}) = \hat{\omega}_i$ for a particular $(\omega_i^1, \omega_i^2, h_i^{D_i})$ combination, then the original $D_i + 1$ state automaton forgets the original state one period later by construction. Thus the original $D_i + 1$ state automaton, if it has finite recall, must forget the original state after at most $D_i + 1$ periods, or has at most D_i period recall.[2] □

Next we prove a lemma characterizing two properties of automata ψ_i with infinite recall. The second of these properties concerns the sets of possible "long run" beliefs an agent i can have about the state of his opponents, $\overline{M}_i$.

Define $\overline{M}_i$ to be the limit of the sequence $\{T^s(\overline{\Delta}_i)\}_{s=0}^\infty$, where $\overline{\Delta}_i$ is a collection of D_i unit-simplexes. Since T is a monotonic operator, the sequence $\{T^s(\overline{\Delta}_i)\}_{t=0}^\infty$ is decreasing in terms of set inclusion. The continuity of B_i implies the continuity of T and thus this limit is a fixed point of T. In words, the collection of sets $\overline{M}_i$ is a convex hull of all possible posteriors a player can have after all possible (infinitely) long histories if he starts with an arbitrary prior over the initial state of his opponents. In contrast, sets $M_i^{*U}(x)$ take a convex hull over histories of all lengths but only for priors induced by x.

Lemma 2. *Suppose finite D_i state automaton ψ_i has infinite recall.*

[2] We thank Ofer Zeitouni for helpful suggestions regarding this proof.

1. *Then (and only then) there exist two distinct states* (ω_i^1, ω_i^2) *and a finite* $s \le D_i(D_i - 1)$ *length history* $h_i^s = (h_i, \ldots, h_{i,s-1})$ *such that* $\omega_i^+(\omega_i^1, h_i^s) = \omega_i^1$ *and* $\omega_i^+(\omega_i^2, h_i^s) = \omega_i^2$.
2. *Then, for all* $m_i \in \Delta_i^{D_{-i}}$, *all accumulation points of the sequence* $\{B_i^{st}(m_i, (h_i^s)^t | \psi_{-i})\}_{t=1}^{\infty}$ *(of which there is at least one) are elements of* $\overline{M}_i(\omega_i^1) \cap \overline{M}_i(\omega_i^2)$ *(where* $(h_i^s)^t$ *denotes the sequence* h_i^s *from part 1 of this lemma repeated t times).*

Proof. 1. (Then) If ψ_i has infinite recall, there exists a pair of distinct states $(\omega_{i,0}^1, \omega_{i,0}^2)$ and an infinite sequence $h_i^\infty = (h_{i,0}, h_{i,1}, \ldots)$ such that for all $t \ge 1$, $\omega_{i,t}^1 \equiv \omega^+(\omega_{i,0}^1, h_i^t) \ne \omega^+(\omega_{i,0}^2, h_i^t) \equiv \omega_{i,t}^2$. Consider the sequence of ordered pairs $\{(\omega_{i,t}^1, \omega_{i,t}^2)\}_{t=1}^{D_i(D_i-1)}$. Since this sequence has $D_i(D_i - 1)$ elements, and there are $D_i(D_i - 1)$ pairs of distinct states, at least one element of the sequence must equal $(\omega_{i,0}^1, \omega_{i,0}^2)$ or one ordered pair must occur in the sequence more than once. (Only then) If there exist two distinct states (ω_i^1, ω_i^2) and a finite $s \le D_i(D_i - 1)$ length history $h_i^s = (h_i, \ldots, h_{i,s-1})$ such that $\omega_i^+(\omega_i^1, h_i^s) = \omega_i^1$ and $\omega_i^+(\omega_i^2, h_i^s) = \omega_i^2$, then, if one starts in ω_i^1 and repeats history h_i^s, the path of states for player i will never coincide with the path of states for this same history starting from state ω_i^2.

2. That $\{B_i^{st}(m_i, (h_i^s)^t | \psi_{-i})\}_{t=1}^{\infty}$ has at least one accumulation point comes from the compactness of $\Delta_i^{D_{-i}}$. Next, since $\omega_i^+(\omega_i^1, h_i^s) = \omega_i^1$ and $\omega_i^+(\omega_i^2, h_i^s) = \omega_i^2$, for all t and $m_i \in \Delta_i^{D_{-i}}$, $B_i^{st}(m_i, (h_i^s)^t | \psi_{-i}) \in T^{st}(\overline{\Delta}_i)(\omega_i^1)$ and $B_i^{st}(m_i, (h_i^s)^t | \psi_{-i}) \in T^{st}(\overline{\Delta}_i)(\omega_i^2)$. Further, since $\{T^{st}(\overline{\Delta}_i)(\omega_i^1)\}_{t=0}^{\infty}$ and $\{T^{st}(\overline{\Delta}_i)(\omega_i^2)\}_{t=0}^{\infty}$ are each decreasing sequences (in terms of set inclusion) of nonempty compact sets (with non-empty limits $\overline{M}_i(\omega_i^1)$ and $\overline{M}_i(\omega_i^2)$), if $B_i^{st}(m_i, (h_i^s)^t | \psi_{-i}) \in T^{st}(\overline{\Delta}_i)(\omega_i^1)$ and $B_i^{st}(m_i, (h_i^s)^t | \psi_{-i}) \in T^{st}(\overline{\Delta}_i)(\omega_i^2)$ for all t, then any accumulation point of the sequence $\{B_i^{st}(m_i, (h_i^s)^t | \psi_{-i})\}_{t=0}^{\infty}$ is an element of both $\overline{M}_i(\omega_i^1)$ and $\overline{M}_i(\omega_i^2)$. □

The intuition for part 1 of Lemma 2 is that if an automaton has infinite recall, then for at least one infinite history and at least two states, the automaton must forever keep separate where the player is if he starts in the first state versus where the player is if he starts in the second. Otherwise, the automaton would have finite memory. But if the automaton has a finite number of states, then it must also be the case of this history that from each starting point, at last one state is visited an infinite number of times. Thus the only way for a finite automaton to have infinite recall is to cycle. Now consider infinitely repeating the finite history from part 1 which causes this cycling. Since posterior beliefs depend only on priors and histories, if the player starts with the same prior and observes this infinite history, then his beliefs at all dates will be the same regardless of his starting state. In particular, this is true of all accumulation points of this belief sequence.

Lemma 3. *Suppose for a player i and automaton* ψ_i, *there exist two distinct states* (ω_i^1, ω_i^2) *such that* $\overline{M}_i(\omega_i^1) \cap \overline{M}_i(\omega_i^2) \ne \emptyset$. *Further suppose the joint automaton* ψ *is such that its on path Markov transition matrix* $\tau(\psi)$ *is a regular matrix. Then* (x, ψ) *cannot be a USCSE for any correlation device x.*

Proof. In Phelan and Skrzypacz (2012) (Lemma 5) we showed that if ψ is such that its on path Markov transition matrix $\tau(\psi)$ is a regular matrix, then for each player i, the operator T has a unique non-empty fixed point, $\overline{M}_i$ and for any initial belief sets M_i, iterating using the T operator converges to $\overline{M}_i$. As a result, the intersection $M_i^{*U}(x)(\omega_i^1) \cap M_i^{*U}(x)(\omega_i^2) \ne \emptyset$ since $\overline{M}_i(\omega_i) \subset M_i^{*U}(x)(\omega_i)$ for all x and all ω_i. To see this, start with the belief sets induced by any initial correlation device x and iterate using the T operator. That $T(M_i) \subset T^U(M_i)$ then implies $\overline{M}_i \subset M_i^{*U}(x)$. The result then follows from Proposition 1. □

Proposition 1 states that if belief sets over *all* dates overlap (for given starting conditions) then uniformly strict equilibrium is impossible. If *long run* beliefs overlap for all starting conditions, then Proposition 1 also implies uniformly strict equilibrium is impossible for all starting conditions. The idea behind Lemma 3 is that if the automaton of a player's opponents communicates, then his long run beliefs about the joint state of his opponents are independent of his prior. Therefore, the sets of valid long run beliefs from each state are independent of how the game is started and Proposition 1 applies.

The following result follows immediately from Lemmas 1 through 3.

Proposition 2. *Suppose finite automaton* ψ *is such that* $\tau(\psi)$ *is regular. Then for all starting conditions x,* (x, ψ) *is a USCSE only if for all players i,* ψ_i *(with* D_i *states) has* $D_i - 1$ *or less period recall.*

Proposition 2 can be applied directly also to a case where the states of automata are partitioned into a set of states that are reached on-path and a set of states that are reached only off-path. In that case, $\tau(\psi)$ needs to be regular only for the on-path subset of states (the recall condition applies to all states).

5. Applications

5.1. Abreu et al. (1986)

In this section we consider the classic repeated Cournot game and show how to use our methods to prove that the stick-and-carrot class of strategies, which are optimal in the public monitoring case, cannot be uniformly strict correlated sequential equilibria with any private monitoring.

Consider the repeated oligopoly quantity setting game of Abreu et al. (1986) (APS), where two firms set private quantities which affect the probability distribution over a publicly observed price. When prices and quantities have continuous support, they show that the best and worst strongly symmetric public equilibria are each implemented by a simple two-state (or bang–bang) strategy: In state R (Reward), both firms choose a low quantity (lower than the Cournot–Nash level) and in state P (Punish), both firms choose a high quantity (higher than the Cournot–Nash level). In the best equilibrium, both firms start out in state R. If the observed price is in some strict subset, Y_R, of the set of all prices, they stay in state R, otherwise they transit to state P (starting from which is the worst public equilibrium). In state P, they move to state R if the observed price is in some other strict subset, Y_P, otherwise they stay in state P. Further, $Y_R \neq Y_P$. That is, the set of prices this period which causes the players to be in state R next period depends on whether the players are in state R or P this period. (Generally, reaching state R from state R requires a sufficiently *high* price, while reaching state R from state P requires a sufficiently *low* price.) When prices and quantities have finite support, as we assume in this paper, such a bang–bang strategy can arbitrarily well approximate the best and worst public equilibria as one allows a finer and finer finite grid of prices and quantities.

Now instead of assuming that there is a single public price, as in APS, suppose each firm draws a privately observed price and the joint distribution of prices is affected by both firms' quantity decisions. Further, allow this joint distribution to be arbitrary except for an assumption of full support. This allows for, but does not require, almost perfect correlation between the prices seen by the two firms and makes the above example consistent with the assumptions of this paper. With this change from public to private monitoring, is the APS bang–bang strategy a USCSE? Here, we show it is not.

First, note that it is possible to transit from each joint state to each joint state with positive probability. Thus the transition matrix on the joint state is regular. Next, note that for each player, the strategy has infinite recall. That is, since $Y_R \neq Y_P$, there must exist a price y_i such that $y_i \in Y_R$ and $y_i \notin Y_P$, or vice-versa. If $y_i \in Y_R$ and $y_i \notin Y_P$, then, by definition, $\omega_i^+(R,(a_i,y_i)) = R$ and $\omega_i^+(P,(a_i,y_i)) = P$ for all a_i. If $y_i \notin Y_R$ and $y_i \in Y_P$, then $\omega_i^+(R,(a_i,y_i)) = P$ and $\omega_i^+(P,(a_i,y_i)) = R$ for all a_i. Thus Lemma 2 implies infinite recall. Proposition 2 then implies for all starting conditions x, such a bang–bang strategy cannot be a uniformly strict equilibrium.

5.2. On the need for on-path communication between states

In this section we consider an example game and strategy where the transition matrix over joint states is not regular and construct a uniformly strict equilibrium with infinite recall by deterministically alternating between two pure strategy strict Nash equilibria of the stage game.

Consider a two player coordination game with two locations, say, *Ballet* and *Hockey*. If both players choose the same location, each player receives an outcome G with (high) probability $p < 1$ and an outcome B with probability $1 - p < 1$. If they choose different locations, then each player receives an outcome B with (high) probability $p < 1$ and an outcome G with probability $1 - p < 1$. Assume $u_i(Ballet, G) = u_i(Hockey, G) > u_i(Ballet, B) = u_i(Hockey, B)$ for $i \in \{1, 2\}$. With $p > \frac{1}{2}$, the stage game has two pure strategy Nash equilibria: Both players playing *Ballet*, and both players playing *Hockey*.

Next, consider the following strategy represented by a two-state automaton with $\Omega_i = \{PlayBallet, PlayHockey\}$. Let the probability distribution over initial states, $x \in \Delta^D$, be such that x puts full probability on each player starting in the same state, and let ψ_i be such that each player switches states, regardless of his own action and outcome, with probability one. (The action from each state is, of course, implied by the name of the state.)

First, note that this strategy has infinite recall. The state each player is in at each date forever depends on his starting state. However, the transition matrix over joint states is not regular. Second, note that it is immediate that if matching the action of the other player is strictly preferred in the stage game, it is strictly preferred after all histories, since in this example strategy, a player's history does not affect his beliefs regarding his opponent's state.

6. Conclusion

Constructing equilibria in games with private monitoring is difficult. We show that one reason for this, when representing strategies as automata, is that strategies must be constructed such that each player's beliefs regarding the state of his opponents must be carefully segregated state by state. That is, what a player can believe after any private history that puts him in one state of his automaton cannot overlap with what he can believe after any private history that puts him in another state of his automaton. We also show that for strategies where every joint state is visited on path, if the strategy for any player looks back forever, such segregation of beliefs is impossible, and without this segregation, any resulting equilibria will rely on special properties such as indifference.

Appendix A. Grim trigger example from Section 3.2

The $P(y|a)$ function is such that if n players cooperate, then with probability $p_n(1-\epsilon)^2 + (1-p_n)\epsilon^2$, both players realize the good private outcome. With probability $(1-\epsilon)\epsilon > 0$, player 1 realizes the good outcome while player 2 realizes the bad. (Likewise, with this same probability, player 2 realizes the good outcome and player 1 the bad.) Finally, with probability $p_n\epsilon^2 + (1-p_n)(1-\epsilon)^2$, both players realize the bad outcome. In the example, $p_0 = .3$, $p_1 = .45$, $p_2 = .9$ and $\epsilon = .025$.

Each player's automaton ψ is defined as $\omega_i = \{R, P\}$, $f_i(R) = C$, $f_i(P) = D$, $\omega_i^+(R, (C, G)) = R$, and $\omega_i^+(\omega_i, (a_i, y_i)) = P$ for all $(\omega_i, a_i, y_i) \neq (R, C, G)$. This strategy under these parameters implies that $m^* = .943$. Payoffs are such that $u(C, G) = .75$, $u(C, B) = -1.0$, $u(D, G) = 1.4$, and $u(D, B) = 0$. With $\beta = .9116$, these payoffs ensure indifference between actions C and D (and following the strategy from then on) at m^*.

If $x(R, R) = .88$, $x(R, P) = .01$, $x(P, R) = .01$, and $x(P, P) = .1$, then if a player starts in state R, he believes his opponent starts in state R with probability $.989 > m^*$. If a player in state R deviates in any period (including period 1) and plays action D, then, under these parameters his new posterior is below m^* regardless of $y_i \in \{G, B\}$. This ensures that if the player subsequently observes an indefinite sequence of (C, G) realizations, his posterior approaches m^* from below, and thus, along the sequence, strictly prefers to play D.

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